

## Rank codes – the State of Art

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**Abstract— The State of Art for rank codes is presented. The theory and applications are considered.**

**Keywords— rank metric; matrix rank code; vector rank code; random rank error; rank row erasure; rank column erasure; fast decoding algorithm; space-time code; network coding; public-key cryptography.**

### I. RANK CODES

#### A. Definition

We denote by  $\{\mathbb{L}, \mathfrak{N}\}$  a normed space, where  $\mathbb{L}$  is a space and  $\mathfrak{N}$  is a norm function defined on  $\mathbb{L}$ .

There exist two types (or, representations) of rank codes: the *matrix* representation and the *vector* representation.

In the *matrix* representation, rank codes are defined as subsets of a normed space  $\{\mathbb{F}_q^{N \times n}, \text{Rk}\}$  of  $N \times n$  matrices over a finite (base) field  $\mathbb{F}_q$ , where the norm of a matrix  $M \in \mathbb{F}_q^{N \times n}$  is defined to be the algebraic rank  $\text{Rk}(M)$  of this matrix over  $\mathbb{F}_q$ . The *rank distance* between two matrices  $M_1$  and  $M_2$  is the rank of their difference  $\text{Rk}(M_1 - M_2)$ . The *rank distance* of a matrix rank code  $\mathcal{M} \subset \mathbb{F}_q^{N \times n}$  is defined as the minimal pairwise distance:  $d(\mathcal{M}) = d = \min\{\text{Rk}(M_i - M_j) : M_i, M_j \in \mathcal{M}, i \neq j\}$ .

In the *vector* representation, rank codes are defined as subsets of a normed  $n$ -dimensional space  $\{\mathbb{F}_{q^N}^n, \text{Rk}\}$  of  $n$ -vectors over an extension field  $\mathbb{F}_{q^N}$ , where the norm of a vector  $\mathbf{v} \in \mathbb{F}_{q^N}^n$  is defined to be the *column rank*  $\text{Rk}(\mathbf{v} | \mathbb{F}_q)$  of this vector over  $\mathbb{F}_q$ , i.e., the maximal number of coordinates of  $\mathbf{v}$  which are linearly independent over the base field  $\mathbb{F}_q$ . The *rank distance* between two vectors  $\mathbf{v}_1, \mathbf{v}_2$  is the column rank of their difference  $\text{Rk}(\mathbf{v}_1 - \mathbf{v}_2 | \mathbb{F}_q)$ . The *rank distance* of a vector rank code  $\mathcal{V} \subset \mathbb{F}_{q^N}^n$  is defined as the minimal pairwise distance:  $d(\mathcal{V}) = d = \min\{\text{Rk}(\mathbf{v}_i - \mathbf{v}_j) : \mathbf{v}_i, \mathbf{v}_j \in \mathcal{V}, i \neq j\}$ .

#### B. Background

Algebraic coding theory may be considered as the theory of subsets of a certain *normed* finite-dimensional space  $\Gamma$  over the finite field equipped with a norm function  $\mathfrak{N}$ . The most known norm in coding theory is the Hamming weight of a vector. It turns out that the rank function  $\text{Rk}(A)$  of matrices  $A$  over fields can be considered as the norm function. In particular, the well-known inequalities for sums of matrices  $|\text{Rk}(A) - \text{Rk}(B)| \leq \text{Rk}(A + B) \leq \text{Rk}(A) + \text{Rk}(B)$  define implicitly the rank distance relations on the space of all matrices of identical size. Explicitly, the concept of the rank metric was introduced by Loo-Keng Hua [1] as

”Arithmetic distance”. Philippe Delsarte [2] defined the rank distance (or,  $q$ -distance) on the set of bilinear forms (equivalently, on the set of rectangular matrices) and proposed the construction of optimal codes in *bilinear form* representation but without any decoding algorithm. Ernst M. Gabidulin [3] introduced the rank distance for vector spaces over extension fields and found connections between rank codes in the *vector* representation and in the *matrix* representation. Optimal codes in *vector* representation were described. Fast coding and decoding algorithms were proposed for optimal codes.

### II. THEORY

The normed spaces  $\{\mathbb{F}_q^{N \times n}, \text{Rk}\}$  and  $\{\mathbb{F}_{q^N}^n, \text{Rk}\}$  are isomorphic isometrically. Let a basis  $\Omega = \{\omega_1, \omega_2, \dots, \omega_N\}$  of  $\mathbb{F}_{q^N}$  over  $\mathbb{F}_q$  be chosen. Then each vector  $\mathbf{v} = [v_1 \ v_2 \ \dots \ v_n] \in \mathbb{F}_{q^N}^n$  can be mapped into the  $N \times n$  matrix  $M \in \mathbb{F}_q^{N \times n}$  by replacing each coordinate  $v_j$  with the  $N$ -column consisting of coefficients in representing  $v_j$  by the basis  $\Omega$ . This mapping is bijective and isometric.

Given a rank code  $\mathcal{M}$  in matrix representation one can construct a rank code  $\mathcal{V}$  in vector representation with the same size, code distance and pairwise distances, and vice versa. We refer to codes  $\mathcal{M}$  and  $\mathcal{V}$  as related codes.

The size  $S = |\mathcal{M}| = |\mathcal{V}|$  of related codes with code distance  $d$  satisfy the Singleton bound

$$\begin{aligned} \log_q S &= \log_q |\mathcal{M}| = \log_q |\mathcal{V}| \\ &\leq \min(N(n-d+1), n(N-d+1)) = Nn - (d-1) \max\{N, n\}. \end{aligned}$$

*Proof:* Let  $M_1, M_2, \dots, M_S$  be  $N \times n$  matrices of a code  $\mathcal{M}$ . Consider the set of submatrices consisting of the first  $N-d+1$  rows of each matrix. All these submatrices must be different. Otherwise a couple of matrices  $M_i, M_j$  will exist at rank distance  $\rho(M_i - M_j) \leq d-1$ . The number of different  $(N-d+1) \times n$  submatrices is equal to  $q^{(N-d+1)n}$ . Hence,  $S \leq q^{(N-d+1)n}$ . Replacing roles of rows and columns leads to the inequality  $S \leq q^{(n-d+1)N}$ . ■

Codes reaching this bound are called maximum rank distance codes, or, MRD codes.

A rank code  $\mathcal{M}$  in matrix representation is called  $\mathbb{F}_q$ -linear if  $\mathcal{M}$  is a subspace of  $\mathbb{F}_q^{N \times n}$ .

A rank code  $\mathcal{V}$  in vector representation is called  $\mathbb{F}_{q^N}$ -linear if  $\mathcal{V}$  is a subspace of  $\mathbb{F}_{q^N}^n$ .

Mapping a  $\mathbb{F}_{q^N}$ -linear code  $\mathcal{V}$  in vector representation into

the related code  $\mathcal{M}$  in matrix representation results in a  $\mathbb{F}_q$ -linear code.

Mapping a  $\mathbb{F}_q$ -linear code  $\mathcal{M}$  in matrix representation into the related code  $\mathcal{V}$  in vector representation results in *not necessary* a  $\mathbb{F}_{q^N}$ -linear code.

Constructions of  $\mathbb{F}_q$ -linear rank codes in the matrix representation and  $\mathbb{F}_{q^N}$ -linear rank codes in the vector representation will be considered.

#### A. Delsarte's optimal rank codes in matrix representation

Delsarte's construction of rank codes in bilinear form representation is presented here in matrix representation.

Assume that  $n \leq N$ . Let  $\text{Tr}(x) = \sum_{l=0}^{N-1} x^{q^l}$ ,  $x \in \mathbb{F}_{q^N}$ , be the Trace function from  $\mathbb{F}_{q^N}$  into  $\mathbb{F}_q$ . Let  $d$  be an integer in  $\{1, 2, \dots, n\}$ . Let  $\mathbf{u} = [u_0 \ u_1 \ \dots \ u_{n-d}] \in \mathbb{F}_{q^N}^{n-d+1}$ . Let  $\mu_1, \mu_2, \dots, \mu_n$  be elements of the field  $\mathbb{F}_{q^N}$  linearly independent over the base field  $\mathbb{F}_q$ . Let  $\Omega = \{\omega_1, \omega_2, \dots, \omega_N\}$  be a basis for  $\mathbb{F}_{q^N}$ .

Define a code in matrix representation as the set of  $N \times n$  matrices  $\mathcal{M} = \{M(\mathbf{u}) = [M_{ij}(\mathbf{u})] : \mathbf{u} \in \mathbb{F}_{q^N}^{n-d+1}\}$ , where

$$M_{ij}(\mathbf{u}) = \text{Tr} \left( \sum_{s=0}^{n-d} u_s \omega_i \mu_j^{q^s} \right).$$

Then  $\mathcal{M}$  is a rank code with code distance  $d$  reaching the Singleton bound  $|\mathcal{M}| = q^{N(n-d+1)}$ .

Let  $A_i(n, d)$ ,  $i = 0, 1, \dots, n$ , be the number of code matrices with rank  $i$ . The weight distribution is as follows:

$$\begin{aligned} A_0(n, d) &= 1, \quad i = 0 \\ A_i(n, d) &= 0, \quad i = 1, \dots, d-1. \end{aligned}$$

$$A_i(n, d) = \binom{n}{i} \sum_{s=0}^{i-d} (-1)^s \binom{i}{s} q^{\frac{s(s-1)}{2}} (q^{N(d-i+1-s)} - 1),$$

if  $i = d, \dots, n$ , where  $\binom{n}{i} = \prod_{j=0}^{i-1} \frac{q^n - q^j}{q^i - q^j}$  is the Gaussian binomial coefficient.

#### B. Optimal rank codes in vector representation

A  $\mathbb{F}_{q^N}$ -linear vector code  $\mathcal{V}$  is a subspace of the normed space  $\{\mathbb{F}_{q^N}^n, \text{Rk}\}$ . Denote by  $(n, k, d)$  a code  $\mathcal{V}$  of dimension  $k \leq n$  and rank distance  $d$ . Such a code can be described in terms of a full rank generator matrix  $G_k$  over the extension field  $\mathbb{F}_{q^N}$  of size  $k \times n$ . Code vectors  $\{\mathbf{v}\}$  are all linear combinations of this matrix. Thus the size of a code is equal to  $|\mathcal{V}| = q^{Nk}$ .

Equivalently, a rank code  $\mathcal{V}$  can be described in terms of a full rank parity-check matrix  $H_{n-k}$  over  $\mathbb{F}_{q^N}$  of size  $(n-k) \times n$ . It satisfies the condition  $G_k H_{n-k}^\top = \mathbf{O}$ , where  $\mathbf{O}$  is the all zero  $k \times (n-k)$  matrix. Code vectors  $\{\mathbf{v}\}$  are all solutions of the linear system of equation  $\mathbf{v} H_{n-k}^\top = \mathbf{0}$ .

For optimal (MRD) codes, it must be  $k = n - d + 1$ , or,  $n - k = d - 1$ .

General constructions of MRD codes in terms of parity-check matrices can be described as follows. Let  $h_1, h_2, \dots, h_n$  be a set of elements from the extension field  $\mathbb{F}_{q^n}$  linearly independent over the base field  $\mathbb{F}$ . Let  $s$  be a positive integer such that  $\text{gcd}(s, N) = 1$ . Then a parity matrix of the form

$$H_{d-1} = \begin{bmatrix} h_1 & h_2 & \dots & h_n \\ h_1^{q^s} & h_2^{q^s} & \dots & h_n^{q^s} \\ h_1^{q^{2s}} & h_2^{q^{2s}} & \dots & h_n^{q^{2s}} \\ \dots & \dots & \dots & \dots \\ h_1^{q^{(d-2)s}} & h_2^{q^{(d-2)s}} & \dots & h_n^{q^{(d-2)s}} \end{bmatrix}.$$

defines an MRD  $(n, k, d)$  code with code length  $n \leq N$ , dimension  $k = n - d + 1$  and rank distance  $d = n - k + 1$ .

Equivalently, general constructions of MRD codes can be described in terms of generator matrices. Let  $g_1, g_2, \dots, g_n$  be a set of elements from the extension field  $\mathbb{F}_{q^n}$  linearly independent over the base field  $\mathbb{F}$ . Then a generator matrix of the form

$$G_k = \begin{bmatrix} g_1 & g_2 & \dots & g_n \\ g_1^{q^s} & g_2^{q^s} & \dots & g_n^{q^s} \\ g_1^{q^{2s}} & g_2^{q^{2s}} & \dots & g_n^{q^{2s}} \\ \dots & \dots & \dots & \dots \\ g_1^{q^{(k-1)s}} & g_2^{q^{(k-1)s}} & \dots & g_n^{q^{(k-1)s}} \end{bmatrix}.$$

defines an MRD  $(n, k, d)$  code with code length  $n \leq N$ , dimension  $k = n - d + 1$  and rank distance  $d = n - k + 1$ . The weight distribution of vector MRD codes coincides for a given  $d$  with the weight distribution of Delsarte's codes above.

The case  $s = 1$  is used mostly.

No other constructions of MRD codes are known.

#### C. Correcting random rank errors only

The ring  $R_n[x]$  of linearized polynomials with coefficients in the extension field  $\mathbb{F}_{q^N}$  consists of polynomials  $F(x) = \sum_{i=0}^L f_i x^{q^i}$  with operations: **Addition** in  $R_n[x]$  is defined by

$$F(x) + G(x) = \sum_{i=0}^L (f_i + g_i) x^{q^i}; \quad \text{Symbolic multiplication is defined by}$$

$$F(x) \otimes G(x) = F(G(x)) = \sum_{i=0}^{L_1+L_2} \sum_{k+s=i} (f_s g_k^{q^s}) x^{q^i}.$$

Let a code vector be  $\mathbf{v} = (v_1, v_2, \dots, v_m)$ , a received vector  $\mathbf{y} = \mathbf{v} + \mathbf{e}$ , where  $\mathbf{e} = t_1 \mathbf{u}_1 + t_2 \mathbf{u}_2 + \dots + t_t \mathbf{u}_t$ . Unknown elements  $t_i \in \mathbb{F}_{q^N}$  are linearly independent over the base field  $\mathbb{F}$ . Unknown  $n$ -vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_t \in \mathbb{F}_{q^N}^n$  are **linearly independent** over  $\mathbb{F}$ . The rank  $t$  is unknown.

The syndrome vector has the form

$$(t_1 \mathbf{u}_1 + t_2 \mathbf{u}_2 + \dots + t_t \mathbf{u}_t) H^\top = \mathbf{s} = (s_0, s_1, \dots, s_{d-2}),$$

where known syndrome coordinates are  $s_i = \sum_{j=1}^t t_j \theta_j^i$ ,  $i = 0, 1, \dots, d_r - 2$ . Introduce the linearized syndrome polynomial by  $S(x) = \sum_{p=0}^{d-2} s_p x^{q^p}$ . Denote by  $T(x)$  the linearized polynomial, whose roots are all the possible linear combinations of  $t_1, t_2, \dots, t_t$  with coefficients in the base field  $\mathbb{F}$ :  $T(x) = \sum_{p=0}^t \Delta_p x^{q^p}$ ,  $\Delta_t = 1$ . Introduce the auxiliary linearized polynomial  $F(x)$  by  $F(x) = \sum_{p=0}^{t-1} \sum_{i=0}^p \Delta_i s_{p-i} x^{q^p}$ . Solve the Key equation

$$F(x) = T(x) \otimes S(x) \quad \text{mod } x^{q^{d_r-1}},$$

iteratively by the right Euclidean division algorithm with initial values  $F_{-1}(x) = x^{q^{d_r-1}}$ ,  $F_0(x) = S(x)$ . The stop rule: Continue calculations till  $i = t$  such that

$$\deg_q F_{t-1}(x) \geq \frac{d-1}{2},$$

$$\deg_q F_t(x) < \left\lfloor \frac{d-1}{2} \right\rfloor.$$

Find the polynomial  $T(x)$ . Find its linear independent roots  $t_1, t_2, \dots, t_t$ . Find  $\{\theta_i\}$ . Find  $\{\mathbf{u}_i\}$ . Find the final error:

$$\mathbf{e} = t_1 \mathbf{u}_1 + t_2 \mathbf{u}_2 + \dots + t_t \mathbf{u}_t.$$

#### D. Correcting rank errors and rank erasures

Let a MRD  $(n, k, d = n - k + 1)$  code  $\mathcal{V}$  be given. Let a transmitted signal be  $\mathbf{v}$  and received signal be  $\mathbf{y} = \mathbf{v} + \mathbf{e}_{\text{total}}$ , where  $\mathbf{e}_{\text{total}}$  is an error. The code  $\mathcal{V}$  can correct in general vector errors of the form

$$\begin{aligned} \mathbf{e}_{\text{total}} &= \mathbf{e} + \mathbf{e}_{\text{row}} + \mathbf{e}_{\text{col}} \\ &= e_1 \mathbf{u}_1 + e_2 \mathbf{u}_2 + \dots + e_t \mathbf{u}_t + \\ &\quad + a_1 \mathbf{r}_1 + a_2 \mathbf{r}_2 + \dots + a_v \mathbf{r}_v + \\ &\quad + w_1 \mathbf{c}_1 + w_2 \mathbf{c}_2 + \dots + w_l \mathbf{c}_l \end{aligned}$$

provided that  $2t + v + l \leq d - 1$ .

The part  $\mathbf{e} = e_1 \mathbf{u}_1 + e_2 \mathbf{u}_2 + \dots + e_t \mathbf{u}_t$  is called a *random rank error* of rank  $t$  under assumption that elements  $e_i \in \mathbb{F}_{q^N}$  are linearly independent over the base field  $\mathbb{F}_q$  and *unknown* to the decoder;  $n$ -vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_t$  have coordinates in the base field  $\mathbb{F}_q$ , are linearly independent over the base field  $\mathbb{F}_q$  and also *unknown* to the decoder. The rank  $t$  is *unknown* to the decoder.

The part  $\mathbf{e}_{\text{row}} = a_1 \mathbf{r}_1 + a_2 \mathbf{r}_2 + \dots + a_v \mathbf{r}_v$  is called a *vector rank row erasure with side information* under assumption that elements  $a_i \in \mathbb{F}_{q^N}$  are linearly independent over the base field  $\mathbb{F}$  and *known* to the decoder;  $n$ -vectors  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_v$  have coordinates in the base field  $\mathbb{F}_q$ , are linearly independent over the base field  $\mathbb{F}_q$  and *unknown* to the decoder.

The part  $\mathbf{e}_{\text{col}} = w_1 \mathbf{c}_1 + w_2 \mathbf{c}_2 + \dots + w_l \mathbf{c}_l$  is called a *vector rank column erasure with side information* under assumption that elements  $w_i \in \mathbb{F}_{q^N}$  are linearly independent over

the base field  $\mathbb{F}_q$  and are *unknown* to the decoder;  $n$ -vectors  $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_l$  have coordinates in the base field  $\mathbb{F}_q$ , are linearly independent over the base field  $\mathbb{F}_q$  and *known* to the decoder.

First fast correcting random rank errors only was proposed in [3]. The algorithm is based on the extended Euclidean division algorithm for linearized polynomials. There exist several further modifications.

Algorithms for correcting random rank errors and rank erasures simultaneously are proposed in [4], [5], [7].

### III. APPLICATIONS

#### A. Rank codes as space-time codes

Space-time codes are introduced by Tarokh, Jafarkhani, and Calderbank in 1998 [10]. Codes are designed to simultaneously take advantage of two dimensions, namely the spatial diversity of antenna elements, and coding gain introduced by designed redundancy in the time dimension.

It is assumed that the base station is equipped with  $T$  transmit antennas and the terminal is equipped with  $m$  receiving antennas. A signal is transmitted in time slots  $1, 2, \dots, n$ .

The received signal can be written as a  $m \times n$  matrix

$$\mathbf{Y} = \mathbf{AC} + \mathbf{N}.$$

$\mathbf{C}$  is a  $T \times n$  signal code matrix with entries in some constellation.

$\mathbf{A}$  is a  $m \times T$  matrix of complex transfer coefficients from the  $j$ th transmit antenna to the  $i$ th receiving antenna.

$\mathbf{N}$  is an AWGN  $m \times n$  matrix.

The Full rank criterion was proposed:

**Choose a matrix code  $\mathbf{C} = \{\mathbf{C}_i\}$  in such a manner that the difference  $\mathbf{C}_{i,j} = \{\mathbf{C}_i - \mathbf{C}_j\}$  has full rank.**

MRD codes over finite fields can not be used directly as space-time codes. However, it is still possible to use MRD codes over finite fields as *templates*. Known results in this direction are as follows [11].

**Non-constructive statement.** There exist infinitely many of finite complex or real constellations such that differential MRD codes (i.e. full rank of difference matrices) can be constructed.

**Constructive statements.**

- 1) MRD codes over the binary field can be transformed to differential MRD codes over the complex or real constellations of size 2.
- 2) For  $T = 2, 4, 8$  and  $q = 2^s$ , MRD codes over  $\mathbb{F}_q$  can be transformed to differential MRD codes over complex constellations of size  $q$ .
- 3) MRD codes over  $\mathbb{F}_p$ ,  $p \equiv 1 \pmod{4}$ , can be transformed to differential MRD codes over the complex Gaussian field of size  $p$ .

Decoding differential MRD codes is reduced to decoding in finite field by hard-decision algorithms.

### B. Rank codes in network coding

Consider a communication network, where a single source transmits information to a single destination. The model of a network was proposed and investigated in [4]. The source formats the information to be transmitted into  $N$  packets  $X(1), \dots, X(N)$  of length  $N+n$  over the finite field  $\mathbb{F}_q$  and constructs a  $(N \times (N+n))$  matrix  $X$  with these packets as rows. The source choose a code  $\mathcal{X}$  consisting of matrices  $X$  which can be transmitted.

Each intermediate node calculates random linear combinations of ingoing packets, where a packet is represented as an element of a finite field  $\mathbb{F}_{q^{N+n}}$ . The node retransmits randomly calculated packets. Therefore, the destination collects a random number  $N_r$  of packets  $Y(1), \dots, Y(N_r)$  of length  $N+n$  and creates a  $N_r \times (N+n)$  matrix  $Y$ .

The problem is to recover the original packets  $X(1), \dots, X(N)$ , or the matrix  $X$  from the received matrix  $Y$ .

Koetter and Kschischang [6] introduce the concept of subspace codes when an alphabet of transmitted messages consists of subspaces, not symbols. They constructed a family of subspace codes. Silva, Kschischang, Koetter, proposed a rank-metric approach to error control in random network coding [7]. Silva and Kschischang proposed their algorithms of fast encoding and decoding of Gabidulin codes [8].

The basic model of a channel induced by random network coding is described as follows. The transmitted matrix  $X$  and the received matrix  $Y$  are connected by the relation  $Y = AX + BZ$ , where  $A$  is an  $N_r \times N$  matrix corresponding to the overall linear transformation applied by intermediate nodes of the network;  $Z$  is an  $l \times (N+n)$  matrix whose rows are the error packets  $z_1, \dots, z_l$ ;  $B$  is an  $N_r \times l$  matrix corresponding to the overall linear transformation applied to  $z_1, \dots, z_l$  on route to the destination. The number of nonzero rows of  $Z$  gives the total number of corrupt packets injected in the network. Random matrices  $A, B, Z$  are unknown to the destination.

It is proposed to apply so called lifting construction for constructing a code  $\mathcal{X}$ . Each matrix  $X \in \mathcal{X}$  has the form  $X = [I_N \ M]$ , where  $I_N$  is the identity matrix of order  $N$  while  $M \in \mathcal{M}$  is a code matrix of some matrix code  $\mathcal{M}$  consisting of  $N \times n$  matrices over the field  $\mathbb{F}_q$ . A code  $\mathcal{M}$  is assumed to be a MRD *rank code* with rank distance  $d$ , if  $n \leq N$ , or a transposed MRD rank code, if  $N < n$ . The corresponding code  $\mathcal{X}$  that was analyzed in the basic model of a channel induced by random network coding is described as follows. The transmitted matrix  $X$  and the received matrix  $Y$  are connected by the relation  $Y = AX + BZ$ , where  $A$  is an  $N_r \times N$  matrix corresponding to the overall linear transformation applied by intermediate nodes of the network;  $Z$  is an  $l \times (N+n)$  matrix whose rows are the error packets  $z_1, \dots, z_l$ ;  $B$  is an  $N_r \times l$  matrix corresponding to the overall linear transformation applied to

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Other network codes are known generalizing constructions above (see, [9]). Decoding codes  $\mathcal{X}$  can be reduced to decoding embedded rank codes  $\mathcal{M}$ . If the matrix  $X = [I_N \ M]$  is transmitted, then the matrix  $Y = AX + BZ = [A + BZ_1 \ AM + BZ_2]$  is received with unknown to the destination matrices  $A, B, Z_1, Z_2$ . By a linear transformation of rows and injecting all zero rows, the part  $[A + BZ_1]$  can be reduced to the upper triangular matrix of order  $N$ . Elements of the main diagonal are 0's or 1's. The number of 1's is equal to the rank of  $[A + BZ_1]$ . The same operations over the matrix  $[AM + BZ_2]$  allows to extract the submatrix of the form  $R = M + LM + DC$ , where  $R, L$  and  $C$  are known matrices. Thus, the result is a matrix  $M$  of the rank code  $\mathcal{M}$  corrupted by a *row* rank erasure  $LM$  and a *column* rank erasure  $DC$ . The unknown matrix  $M$  can be uniquely recovered from  $R$  provided that  $\text{Rk}(L) + \text{Rk}(C) \leq d - 1$ . Other network codes are known generalizing constructions above [9].

It is shown that decoding this subspace code is equivalent to correcting random errors and generalized erasures in the rank code [5].

### C. Rank codes in cryptography

The McEliece like public key cryptosystem but based on *rank* error correcting codes was proposed by Gabidulin, Paramonov, Tretjakov in 1991 [12]. The cryptosystem is

described as follows. The public key  $\mathbf{G}_{\text{pub}}$  is a left and right scrambled generator matrix of a rank code:

$$\mathbf{G}_{\text{pub}} = \mathbf{S}\mathbf{G}_k\mathbf{P}.$$

The matrix  $\mathbf{G}_k$  is used to correct rank errors of rank not greater than  $t = \lfloor \frac{n-k}{2} \rfloor$ .

A matrix  $\mathbf{S}$  over the extension field  $\mathbb{F}_{q^n}$  is called the row scrambling matrix. It is used to destroy any visible structure of the matrix  $\mathbf{G}_k$  by mixing its rows.

A matrix  $\mathbf{P} = [p_{ij}]$  is called the column scrambler. This matrix is a non singular square matrix of order  $n$ . It is used to mix columns of  $\mathbf{G}_k$ .

Another generator matrix has the form

$$\mathbf{G}_{\text{pub}} = \mathbf{S}[\mathbf{X} \ \mathbf{G}_k]\mathbf{P}.$$

Plaintext is any  $k$ -vector

$$\mathbf{m} = (m_1, m_2, \dots, m_k), m_s \in \mathbb{F}_{q^n}, s = 1, 2, \dots, k.$$

Private keys are matrices  $\mathbf{S}, \mathbf{G}_k, \mathbf{X}, \mathbf{P}$  separately and (explicitly) a fast decoding algorithm of an MRD code.

Encryption. The ciphertext is given by

$$\mathbf{c} = \mathbf{m}\mathbf{G}_{\text{pub}} + \mathbf{e} = \mathbf{m}\mathbf{S}[\mathbf{X}\mathbf{G}_k]\mathbf{P} + \mathbf{e},$$

where  $\mathbf{e}$  is an artificial vector of errors of rank  $t_2$ .

Decryption. The legitimate receiver upon receiving  $\mathbf{c}$  calculates

$$\mathbf{c}' = \mathbf{c}\mathbf{P}^{-1} = \mathbf{m}\mathbf{S}[\mathbf{X}\mathbf{G}_k] + \mathbf{e}\mathbf{P}^{-1}.$$

Then he extracts from  $\mathbf{c}'$  the plaintext  $\mathbf{m}$  using decoding algorithms and properties of public keys.

Attacks and counter-attacks. Rank codes are well structured. It makes easier creation of attacks.

Subsequently in a series of works, Gibson [13], [14], [15] developed attacks that break the system for practical instances.

Several variants of PKC were introduced to withstand Gibson's attacks [16].

Recently, R. Overbeck [17], [18] proposed a new attack which is more effective than Gibson's attacks. His method is based on the fact that a column scrambler  $\mathbf{P}$  is defined over the *base field*.

It was found [19], [20] that a cryptographer can define a proper column scrambler over the *extension field without violation* of the standard mode of the PKC. Overbeck's attack fails in this case.

#### IV. CONCLUSIONS

Theory of rank codes is of great interest for many researchers. Fast decoding algorithms are developed. Applications in many areas are possible and recommended. Space-time coding, random network coding, public key cryptosystems are areas of such applications.

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