Analysis of the Loading Factor Behavior in a LSMI Beamformer

Filipe de C. B. da Silveira, Ricardo Zelenovsky and Mylène C. Q. de Farias

***Abstract*— Diagonal loading imparts robustness to adaptive beamformers. With it, the algorithm improves both its capacity of sidelobes attenuation and the signal-to-interference-plus-noise-ratio (SINR). However, setting the appropriate loading factor γ value is not a straightforward task as it is dependent on various parameters. Therefore, this paper aims to establish its behavior related to changes in certain parameters - i.e., number of antennas, number of interferers, noise power and direction-of-arrival (DOA) uncertainty. This way, the value that gives suitable SINR results can be established.**

***Keywords*— Adaptive beamforming, uniform linear array (ULA), sample-matrix inversion (SMI), loaded SMI (LSMI), minimum variance distortionless response (MVDR) beamformer and loading factor.**

I. Introduction

Throughout the years, antenna arrays have been a relevant research topic in various applications [1]: radar; sonar; wireless communications; etc. Several approaches have been developed to increase its performance in various scenarios [2]. Different methods for DOA estimation - DS, Capon and MUSIC [3], [4] - have been proposed. When the direction-of-interest (DOI) of the signal-of-interest (SOI) is known, the simplest method is to aim the array via delay-and-sum (DS) method.

Adaptive beamforming is a versatile approach to detect and estimate the SOI [1], [5], [6]. It is used when the environment variables - e.g., noise, direction-of-arrival (DOA), array imperfections, etc. - are unknown or constantly changing. The main approach in adaptive beamforming is to maximize the beamformer output SINR. However, typical applications include the SOI in their training snapshots, which can severely degrade the SINR performance [1] as the SOI component can be mistakenly interpreted as an interferer by the algorithm. Another problem that may degrade the SINR performance is the DOA estimation error of the SOI. For this problem, many solutions have been proposed: convex quadratic constraints [7]; Bayesian approach [8]; and uncertainty set based method [9]. All the previous methods belong to the class of diagonal loading method in robust beamforming.

One popular approach when the DOI is known is the minimum variance distortionless response (MVDR) [5], [10], [11]. Two methods based on MVDR and on the samples captured by the array are used: the sample matrix inversion (SMI); and the loaded SMI (LSMI). The former uses the inverted interference-plus-noise (INR) covariance matrix; the latter just loads this matrix diagonal with a constant loading factor. This factor improves the algorithm robustness and its value has been empirically suggested in previous works [1], [12]. In order to understand its influence on the beamformer performance, a further analysis is presented.

This paper is divided as follows: Section II gives a synthesis of concepts involving adaptive beamforming; Section III analyzes the loading factor (γ) for the LSMI algorithm; Section IV compares the DS, SMI and LSMI approaches; SectionV presents the conclusions.

Filipe de C. B. da Silveira, Ricardo Zelenovsky and Mylène C. Q. de Farias. School of Technology, University of Brasília (UnB), Brasília-DF, Brazil, E-mails: filipebcs8@gmail.com, zele@unb.br, mylene@ene.unb.br. This work was partially supported by CAPES.

II. Background

An uniform linear array (ULA) with M omni-directional antenna elements is considered. The narrowband signal received by this ULA at the time instant k is represented by equation (1), where $\mathbf{s}(k)$, $\mathbf{i}(k)$ and $\mathbf{n}(k)$ are the signal, interferer and noise $M \times 1$ vectors respectively. The signal is assumed to be uncorrelated with the interferers and the noise.

$$\mathbf{x}(k) = \mathbf{s}(k) + \mathbf{i}(k) + \mathbf{n}(k) \tag{1}$$

The noise in equation (1) is typically isotropic and it can be accurately modeled as spatially white Gaussian noise (AWGN).

A point source - far-field consideration - is assumed so the signal and the interferers arrive at the ULA as a plane wave. The contribution of the signal $s(k)$ is expressed by equation (2).

$$\mathbf{x}_s(k) = s(k)\,\mathbf{a}(\theta_s) \tag{2}$$

In equation (2), the $\mathbf{a}(\theta_s)$ represents the steering vector for the signal arriving from the direction θ_s [3] and it can be modeled by equation (3), where $\phi_s = 2\pi\frac{d}{\lambda}\cos(\theta_s)$.

$$\mathbf{a}(\theta_s) = \begin{bmatrix}1 & \exp^{-j\phi_s} & \cdots & \exp^{-j(M-1)\phi_s}\end{bmatrix}^T \tag{3}$$

The beamformer output is given by equation (4), where the vector $\mathbf{w}$ is a $M \times 1$ complex weight vector and $(\cdot)^H$ denotes the Hermitian transpose operation.

$$y(k) = \mathbf{w}^H\mathbf{x}(k) \tag{4}$$

The vector $\mathbf{w}$ can be determined by solving the optimization problem established in equation (5), where $\mathbf{R}_{i+n}$ is the autocorrelation matrix for the interferers and the noise.

$$\min_{\mathbf{w}} \mathbf{w}^H\mathbf{R}_{i+n}\mathbf{w} \quad s.t. \quad \mathbf{w}^H\mathbf{a}(\theta_s) = 1 \tag{5}$$

The solution to equation (5) is known as the MVDR beamformer [5] and it can be expressed by equation (6), where α is a scaling factor.

$$\mathbf{w}_{MVDR} = \alpha \mathbf{R}_{i+n}^{-1} \mathbf{a}(\theta_s) \tag{6}$$

In practice, it is not possible to know the actual $\mathbf{R}_{i+n}$ as this would imply having all past and future samples available. So, this matrix is replaced by a data sample covariance matrix $\hat{\mathbf{R}}$ given by equation (7), where K represents the number of samples [13].

$$\hat{\mathbf{R}} \triangleq \frac{1}{K} \sum_{k=1}^{K} \mathbf{x}(k) \mathbf{x}^H(k) \tag{7}$$

The SMI beamformer [14] is obtained by replacing the interference-plus-noise covariance matrix $\mathbf{R}_{i+n}$ in the MVDR beamformer in equation (6) by the sample estimate of the data covariance matrix (7).

The LSMI beamformer is a robust approach to the SMI beamformer and it is based on the diagonal loading factor γ of the sample covariance matrix [15] as it is shown in (8). So, in equation (6), the matrix $\mathbf{R}_{i+n}$ is replaced by the matrix $\mathbf{R}_{DL}$.

$$\hat{\mathbf{R}}_{DL} = \hat{\mathbf{R}} + \gamma \mathbf{I} \tag{8}$$

Equation (9) resumes the LSMI method. A problem persists: how to choose the loading factor γ?

$$\mathbf{w}_{LSMI} = \hat{\mathbf{R}}_{DL}^{-1} \mathbf{a}(\theta_s) = \left(\hat{\mathbf{R}} + \gamma \mathbf{I}\right)^{-1} \mathbf{a}(\theta_s) \tag{9}$$

It is worth noticing that, as the loading factor increases, the LSMI solution moves towards the DS one, presenting a similar spatial spectrum. A loaded matrix R_{DL} with a very large loading factor is similar to the identity matrix in its behavior, which makes the LSMI solution work like the DS one. So some study concerning the loading factor in the LSMI solution should be made to understand its role.

III. Analysis of the Loading Factor γ

Here, the loading factor γ influence on the LSMI performance is studied for different parameters: number of antennas (M); number of interferers (N); noise power (σ_n^2); and DOI uncertainty (σ_{DOI}^2). For each case, different values of γ will be tested and the corresponding SINR will be calculated based on the adaptation after 1,000 samples.

Unless otherwise stated, all simulations have the following standard configuration: $1,000$ samples; 2 users - SOI coming from 30° and interferer, from 70°; both SOI and interferer have the same power, $\sigma_s^2 = 1$ and $\sigma_{int}^2 = 1$; 16 antennas ULA spaced by 0.5λ; AWGN with variance $\sigma_n^2 = 1$.

A. Loading Factor x Number of antennas

Figure 1 presents LSMI performance as the γ value is varied for different number of antennas: 8; 16; 32; 64; and 128.

As expected, the SINR increases as the number of antennas is increased. It is clear that the performance depends on the γ value: for 8 antennas, a good choice is $\gamma = 1$; for 128 antennas, this value should be increased to $\gamma \approx 100$. In this case, there is no penalty for bigger values of γ, which means that the SINR will not improve after a certain value for the loading factor γ. So, the simplest suggestion is $\gamma = M\sigma_n^2$.

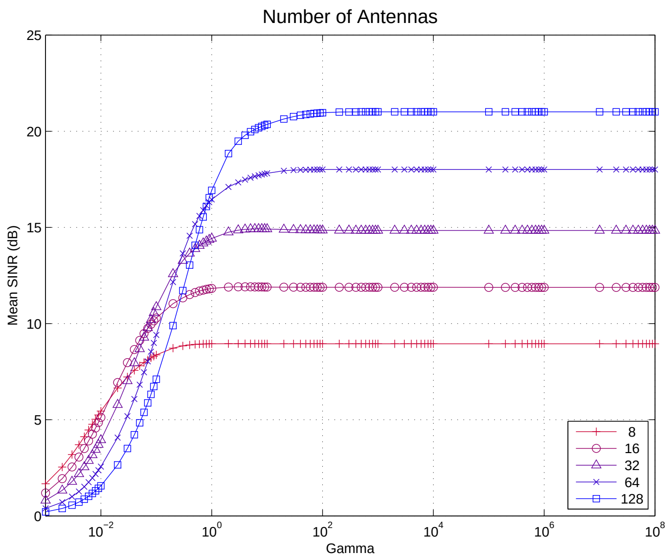

Fig. 1. SINR average according to number of antennas.

B. Loading Factor x Number of interferers

In this case, an ULA with 16 antenna elements is evaluated under different scenarios: SOI with no interferers; and SOI with a different number of interferers.

In figure 2, when there is no interferer, the SINR: increases when the loading factor γ is increased; remains constant when it assumes values greater than 1. When there is one interferer, the system behavior is similar: the SINR decreases slowly for values greater than 10^8, nonetheless this behavior is not visible in figure 2. When there are two or more interferers, an optimal value for the loading factor γ is clearly defined. In this simulation, the appropriate value should be $\gamma \approx 0.1\sigma_n^2$.

This value changes with variations in noise power and in the number of antennas as well.

C. Loading factor x Noise power

Here, the results are intuitive: as the noise power increases, the average SINR decreases. It can be seen in figure 3 that, at high noise power values, the difference between the peak SINR's are rather small compared to situations in which there is low noise power. Moreover, after a specific value for γ, the SINR reaches its peak and it does not improve similarly to what happened in figure 1 with the number of antennas variation. For this case, the simplest suggestion for the loading factor value is $\gamma = 1\sigma_n^2$.

D. Loading Factor x DOI Uncertainty

LSMI is known as a good algorithm when there is DOI uncertainty. In order to analyze such situations, a simulation

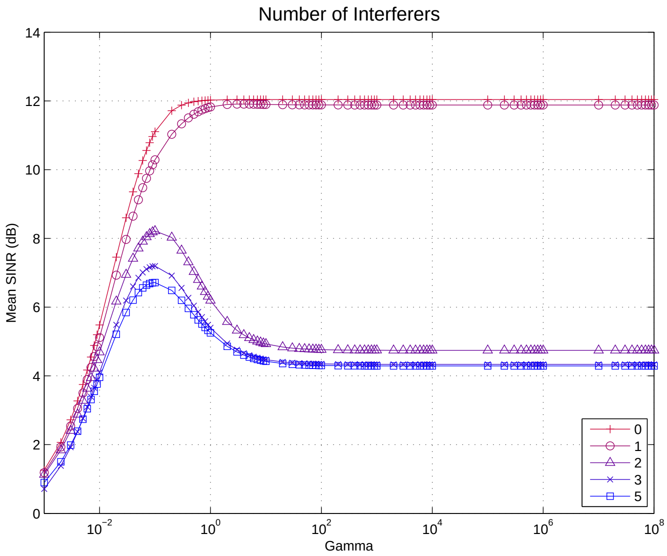

Fig. 2. SINR average according to number of interferers.

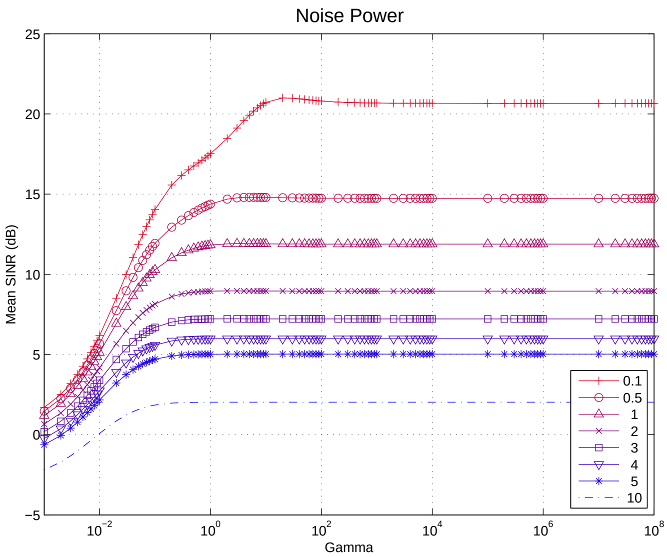

Fig. 3. SINR average according to noise power σ_n^2.

with the DOI corrupted by noise was done. Here, each snapshot uses a DOI with a random deviation from its nominal direction.

Figure 4 shows the results. The bigger is the noise variance, the bigger is the displacement of the nominal DOI. With small uncertainty in the DOI ($\sigma_{DOI}^2 = 0.1$), a good choice is $\gamma > 1\sigma_n^2$; with greater uncertainty in the DOI, a good choice is $\gamma = 0.1\sigma_n^2$. In fact, there is a specific optimal value for each case.

Comparing figures 2 and 4, a similar behavior is noticeable for the same loading factor γ.

IV. DS, SMI E LSMI COMPARISON

The three methods - DS, SMI and LSMI - are strongly related.

The DS solution, represented by equation (10), points the mainlobe towards the DOI and ignores the interferers. When

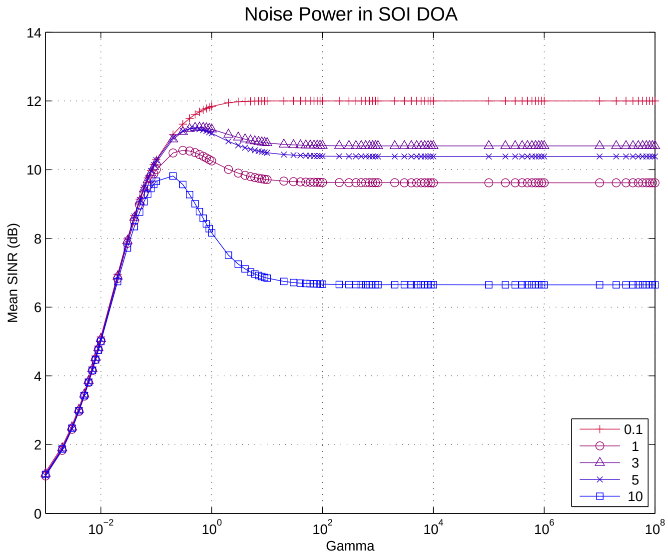

Fig. 4. SINR average according to noise power σ_n^2 in SOI DOA.

the number of antennas is increased, the amplitude of the sidelobes decreases.

$$\mathbf{w}_{DS} = \mathbf{a}\left(\theta_s\right) \tag{10}$$

The SMI solution places the mainlobe on the DOI and the nulls on the directions of the interferers. By doing this, the SMI solution creates several sidelobes with great amplitudes and this lets a lot of noise to be perceived by the beamformer from other directions.

Examining the LSMI equation given by equation (9), it is possible to see that, as the loading factor γ is increased, $\mathbf{w}_{LSMI}$ becomes similar to $\mathbf{w}_{DS}$. Choosing a very large loading factor γ value for the LSMI makes it practically identical to the DS solution. So, the LSMI just treads the reduction of the sidelobes with the ability to refuse interferers. As the γ value increases, the sidelobe reduction happens faster than the ability to refuse interferers. Therefore, an optimal value for γ may be expected.

One simulation considering different values for the loading factor: $\gamma = 2\sigma_n^2$ and $\gamma = 10\sigma_n^2$, as suggested in previous works; plus an extreme case ($\gamma = 1,000,000\sigma_n^2$) was developed. This last case will be evaluated in order to analyze the beamformer behavior as the loading factor is raised to extreme values. These three cases for the LSMI will be treated as LSMI 1, LSMI 2 and LSMI 3 respectively. The system and the environment is configured as follows: a 64 antennas ULA with a 0.5λ space between them; 2 signals, the SOI coming from $30°$ and the interferer, from $70°$; both signals have unitary variance; and the noise is considered AWGN with variance $\sigma_n^2 = 1$. Moreover, both signals do not change their directions during all the simulation.

Figure 5 presents the spatial spectrum. The SMI solution has: the mainlobe at $30°$; the null at $70°$; and the sidelobes with relatively big amplitudes. These sidelobes also explain why the SMI has the worst performance as it perceives a lot of noise and interferers from other directions than the DOI.

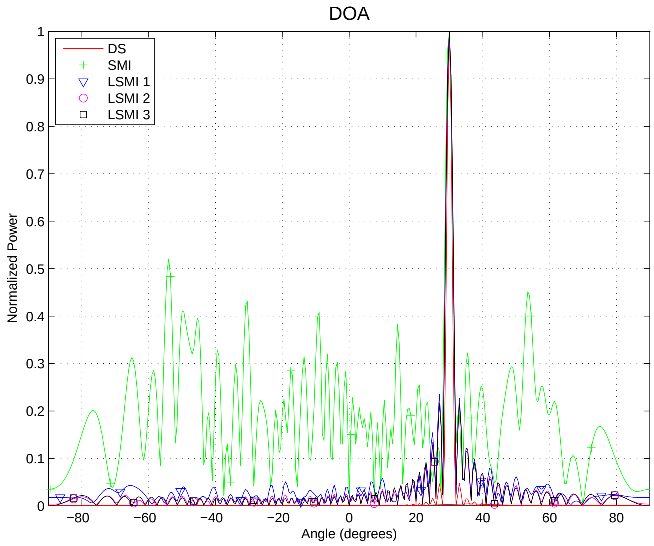

Fig. 5. Spatial spectrum for the five cases.

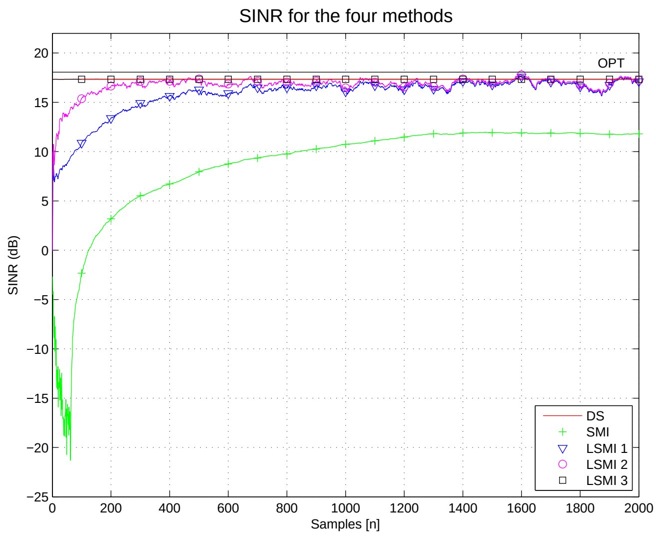

Fig. 6. SINR behavior for the five cases.

From figure 6, it is noticeable that the DS solution presents the best SINR overall, only being surpassed in brief moments by the LSMI 1 and LSMI 2 solutions. Moreover, this figure makes it clear that, as the loading factor increases, the SMI spatial spectrum becomes more and more similar to the DS one. Here, the most interesting behavior is in the SINR for the LSMI 3: it collapses to the DS solution. This behavior was expected as raising the loading factor γ to extreme values makes the autocorrelation matrix $\hat{\mathbf{R}}$ work like an identity matrix $\mathbf{I}$ as the diagonal elements are way bigger than the others.

V. Conclusions

This paper analyzed the influence of the loading factor γ on the LSMI performance. Four different parameters variations were studied: (i) number of antennas; (ii) number of interferers; (iii) noise power; and (iv) DOA uncertainty.

Cases (i) and (iii) indicate that there is a minimum value for the loading factor γ and no signifcant penalty in the SINR performance for bigger values; cases (ii) and (iv) show that the loading factor γ should be chosen carefully as it presents higher SINR performance for certain values. In these two situations, choosing $\gamma = 2\sigma_n^2$ or $\gamma = 10\sigma_n^2$, as suggested by previous works, is not necessarily an ideal solution.

Overall, it is relevant to note that, as the loading factor γ increases, the shape of the LSMI spatial spectrum shifts to the DS spatial spectrum and so the capacity to reject interferers is decreased. Therefore, it is necessary to balance these two properties: the reduction of the sidelobes amplitudes; and the capacity to reject interferers. In situations with a great number of antennas and a small number of interferers, it seems that there is no appreciable advantage in using the LSMI solution. So, using the DS approach may be a good solution as it has the advantage of minimal computational cost.

References

[1] H. Cox, R. Zeskind, and M. Owen, "Robust adaptive beamforming," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 35, pp. 1365–1376, Oct 1987.

[2] S. A. Vorobyov, "Principles of minimum variance robust adaptive beamforming design," *SIGNAL PROCESSING*, vol. 93, no. 12, pp. 3264–3277, 2013. VK: (Invited Paper).

[3] J. C. Liberti and T. S. Rappaport, *Smart Antennas for Wireless Communications: IS-95 and Third Generation CDMA Applications*. Upper Saddle River, NJ, USA: Prentice Hall PTR, 1999.

[4] S. S. Haykin, *Adaptive filter theory*. Pearson Education India, 2008.

[5] S. Theodoridis and R. Chellappa, *Academic Press Library in Signal Processing, Volume 3: Array and Statistical Signal Processing*. Academic Press, 1st ed., 2013.

[6] S. Chen, S. Tan, and L. Hanzo, "Adaptive beamforming for binary phase shift keying communication systems," *Signal Processing*, vol. 87, no. 1, pp. 68–78, 2007.

[7] B. D. Van Veen, "Minimum variance beamforming with soft response constraints," *IEEE Transactions on Signal Processing*, vol. 39, no. 9, pp. 1964–1972, 1991.

[8] K. L. Bell, Y. Ephraim, and H. L. Van Trees, "A bayesian approach to robust adaptive beamforming," *IEEE Transactions on Signal Processing*, vol. 48, no. 2, pp. 386–398, 2000.

[9] C.-Y. Chen and P. P. Vaidyanathan, "Quadratically constrained beamforming robust against direction-of-arrival mismatch," *IEEE Transactions on signal processing*, vol. 55, no. 8, pp. 4139–4150, 2007.

[10] Y. Cai, B. Qin, and H. Zhang, "An improved adaptive constrained constant modulus reduced-rank algorithm with sparse updates for beamforming," *Multidimensional Systems and Signal Processing*, vol. 27, no. 2, pp. 321–340, 2016.

[11] S. A. Vorobyov, "Principles of minimum variance robust adaptive beamforming design," *Signal Processing*, vol. 93, no. 12, pp. 3264–3277, 2013.

[12] S. A. Vorobyov, A. B. Gershman, and Z.-Q. Luo, "Robust adaptive beamforming using worst-case performance optimization: A solution to the signal mismatch problem," *IEEE Transactions on Signal Processing*, vol. 51, no. 2, pp. 313–324, 2003.

[13] O. Ledoit and M. Wolf, "Improved estimation of the covariance matrix of stock returns with an application to portfolio selection," *Journal of Empirical Finance*, vol. 10, no. 5, pp. 603–621, 2003.

[14] I. S. Reed, J. D. Mallett, and L. E. Brennan, "Rapid convergence rate in adaptive arrays," *IEEE Transactions on Aerospace and Electronic Systems*, vol. AES-10, pp. 853–863, Nov 1974.

[15] A. B. Gershman, Z.-Q. Luo, S. Shahbazpanahi, and S. A. Vorobyov, "Robust adaptive beamforming using worst-case performance optimization," in *Signals, Systems and Computers, 2004. Conference Record of the Thirty-Seventh Asilomar Conference on*, vol. 2, pp. 1353–1357, IEEE, 2003.