On OFDM Systems under WSS-US Channels

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Abstract— **Using the assumption that (a) the subcarrier symbols are i.i.d., (b) the interference between OFDM symbols is negligible, and (c) the WSS-US channel model holds, we (i) show that the subcarrier response consists of an average of the frequency domain channel samples, (ii) derive the subcarrier correlation and its power spectral density, implying that each subcarrier has same power, (iii) calculate an exact expression for the ICI power, which is found to have equal value for all subcarriers, and (iv) show that the two-path spectrum has the largest ICI for channels with the same maximum Doppler frequency.**

Keywords— **Orthogonal frequency division multiplexing (OFDM), wide-sense stationary–uncorrelated scattering (WSS-US) channels, intercarrier interference (ICI).**

I. Introduction

Whenever we consider a multicarrier system, the design of the subcarrier number and spacing is crucial for the avoidance of interference among the adjacent frequency tones. However, when the system is time-varying, the desired property of orthogonality among the different carriers is no longer ensured.

In orthogonal frequency division multiplexing (OFDM) over time-varying channels, the subcarriers consist of the average of N_c (the number of sub-carriers) channel samples in the frequency-domain. Many papers do not consider this assumption, which have various implications. For example, in subcarrier estimation problems, the subcarrier correlations are generally required. In this letter, we make efforts in finding the subcarrier correlations. As consequence of this average property, the subcarriers experience a power loss with respect to the channel power. The remaining power appears as a term of the intercarrier interference (ICI) power.

For independent and identically distributed (i.i.d.) symbols, and wide-sense stationary–uncorrelated scattering (WSS-US) channels, we will show in the sequel the subcarriers experience the same ICI power level.

Hence, with those implications in mind, we derive an expression for the ICI power which is more accurate than the one proposed in [1], [2], [3]. In this sense, our result is an exact expression for the computation of ICI power in time-varying OFDM systems. We have also found an upper bound for the ICI power, and we perform a comparison of the ICI power for different Doppler spectra.

The rest of the paper is organized as follows. In Section II, we describe the OFDM system and find the average property. The computation of the subcarrier correlation is carried out in Section III. Section IV provides the expression for the ICI power, which is derived in the Appendix, and an upper bound is found for the ICI power. Our conclusions are stated in Section V.

Notation. In this paper, vectors appear as bold italic letters and matrices as bold letters. Special matrices which are constructed with data from other matrices are represented by uppercase calligraphic letters. Ensemble averages are represented by an overline on the variable of interest. The (l,m)-th element of a matrix is represented by $(\cdot)_{l,m}$ and $\operatorname{diag}(\cdot)$ is a diagonal matrix with the elements of the argument. The operator $(\cdot)_n$ returns the n-th element of a vector or its entry module n, depending on the entry type. Finally, $(\cdot)^T$ and $(\cdot)^H$ stand for the transpose and transpose conjugation of the argument, respectively.

II. OFDM Description

Let $\boldsymbol{a}[n] = (a[n,0],\dots,a[n,N_c-1])^T$ be the vector containing the frequency-domain symbols allocated at the N_c subcarriers. We assume each symbol $a[n,k]$ has unitary mean power $\sigma_a^2 = 1$. The OFDM symbol is obtained by a normalized IDFT (Inverse Discrete Fourier Transform) application to $\boldsymbol{a}[n]$, and, in the sequel, its cyclic prefix (CP) is added.

Let $\mathbf{F} = 1/\sqrt{N_c}\cdot\mathbf{W}$ be the normalized version of the Fourier matrix $\mathbf{W}$, whose (k,l)-th entry is given by $\omega^{kl} = \exp(-j2\pi kl/N_c)$. The resulting OFDM symbol is then serialized and transmitted through the channel.

The received OFDM symbol is constituted by the parallel version of the received signal. The demodulation procedure consists of the CP removal and posterior translation to the frequency-domain by a normalized DFT. With the constraint that the channel length L is smaller than the CP length N_{cp}, i.e, $L \leq N_{\mathrm{cp}}+1$, there is no interference between adjacent OFDM symbols, and the received signal at the n-th subcarrier can be written as

$$\begin{aligned}\boldsymbol{x}[n] &= \mathbf{F}\mathbf{H}_v[n]\mathbf{F}^H\boldsymbol{a}[n] + \boldsymbol{n}[n]\\ &= \mathcal{H}_v[n]\boldsymbol{a}[n] + \boldsymbol{n}[n], \end{aligned} \tag{1}$$

where $\boldsymbol{n}[n]$ is the frequency-domain noise with power σ_n^2, $\mathbf{H}_v[n]$ is the time-varying channel matrix, and $\mathcal{H}_v[n] = \mathbf{F}\mathbf{H}_v[n]\mathbf{F}^H$. Since the matrix $\mathbf{H}_v[n]$ is not circulant, $\mathcal{H}_v[n]$ is no longer diagonal and there is intercarrier interference.

In what follows, we will write the diagonal elements of $\mathcal{H}_v[n]$ as a function of the discrete-time channel impulse response $h[m,l]$, where m and l index time and delay w.r.t. sample instants. We can easily check

$$\mathbf{H}_v[n] = \begin{pmatrix} h^n_{0,0} & h^n_{0,N_c-1} & \cdots & h^n_{0,1}\\ h^n_{1,1} & h^n_{1,0} & \cdots & h^n_{1,2}\\ \vdots & \vdots & \ddots & \vdots\\ h^n_{N_c-1,N_c-1} & h^n_{N_c-1,N_c-2} & \cdots & h^n_{N_c-1,0}\end{pmatrix},$$

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where we denote

$$h^n_{p,l} = \begin{cases} h[(n-1)N_s + N_{\text{cp}} + p, l], & \text{for } 0 \le l < L, \\ 0, & \text{otherwise,} \end{cases}$$

and $N_s = N_{\text{cp}} + N_c$ is the total OFDM symbol duration. Now let $\boldsymbol{h}[m] = (h[m,0], \dots, h[m, L-1], 0, \dots, 0)^T$ be the $N_c \times 1$ vector tailed by $N_c - L$ zeros. With the stated definitions, the diagonal elements of $\mathcal{H}_v[n]$ can be written as

$$\begin{aligned}(\mathcal{H}_v[n])_{m,m} &= \frac{1}{N_c}\sum_{p=0}^{N_c-1}\sum_{k=0}^{N_c-1} h^n_{p,(p-k)_{N_c}}\omega^{m(p-k)} \\ &= \frac{1}{N_c}\sum_{p=0}^{N_c-1}(\mathbf{W}\boldsymbol{h}[(n-1)N_s + N_{\text{cp}} + p])_m \\ &= (\mathbf{W}\overline{\boldsymbol{h}}[n])_m,\end{aligned}$$

where

$$\overline{\boldsymbol{h}}[n] = \frac{1}{N_c}\sum_{m=0}^{N_c-1}\boldsymbol{h}[(n-1)N_s + N_{\text{cp}} + m].$$

Let $\mathcal{C}(\overline{\boldsymbol{h}}[n])$ denote the circulant matrix whose entries are the elements of $\overline{\boldsymbol{h}}[n]$. Then we can decompose $\mathbf{H}_v[n]$ as

$$\mathbf{H}_v[n] = \mathcal{C}(\overline{\boldsymbol{h}}[n]) + \mathbf{H}_e[n],$$

where $\mathbf{H}_e$ is an error matrix and such that $\mathcal{H}_v[n]$ can be written as

$$\mathcal{H}_v[n] = \text{diag}(\mathbf{W}\overline{\boldsymbol{h}}[n]) + \mathcal{H}_e[n],$$

where $\mathcal{H}_e[n] = \mathbf{F}\mathbf{H}_e[n]\mathbf{F}^H$. The matrix $\mathcal{H}_e[n]$ constitutes the pure interference term, since $\text{diag}(\mathcal{H}_e[n]) = \mathbf{0}$.

Now we can insert in Eq. (1) the ICI term $\boldsymbol{u}[n] = \mathcal{H}_e[n]\boldsymbol{a}[n]$, which results in

$$\boldsymbol{x}[n] = \text{diag}(\boldsymbol{H}[n])\boldsymbol{a}[n] + \boldsymbol{u}[n] + \boldsymbol{n}[n], \tag{2}$$

where we select $\boldsymbol{H}[n] = \mathbf{W}\overline{\boldsymbol{h}}[n]$.

In time-varying channels, we can show that the subcarriers consist of the average of N_c channel samples in frequency-domain. In the existing literature, this assumption is not considered in the subcarrier correlation, which causes a considerable impact. This is the task of the next section.

III. Subcarrier Correlation

The considered channel model is the WSS-US one with a constant number of paths. In this case, the base-band impulse response is given by

$$h[m,l] = \sum_{i=0}^{K-1}\gamma_i[m]g_i[l], \tag{3}$$

where $\gamma_i[m]$ is the complex amplitude of the i-th path, and $g_i[l] = g(lT_s - \tau_i)$, where T_s is the sample period and $g(\tau)$ is the shaping filter impulse response that satisfies the Nyquist criterion. Let $\mathbb{E}\{\cdot\}$ be the expectation operator. The WSS-US assumption tell us that

$$\mathbb{E}\{\gamma^*_{i'}[m']\gamma_i[m'+m]\} = \begin{cases}\rho_i r_i[m], & \text{if } i' = i, \\ 0, & \text{if } i' \ne i,\end{cases}$$

where ρ_i and $r_i[m]$ denote the mean power and normalized correlation of the i-th path, respectively.

From Eq. (3), the channel response in the frequency domain (subcarriers) can be expanded as

$$H[n,k] = \sum_{l=0}^{L-1}\overline{h}[n,l]\omega^{kl} = \sum_{i=0}^{K-1}\overline{\gamma}_i[n]\left(\sum_{l=0}^{L-1}g_i[l]\omega^{kl}\right), \tag{4}$$

where we denoted

$$\overline{\gamma}_i[n] = \frac{1}{N_c}\sum_{i=0}^{N_c-1}\gamma_i[(n-1)N_s + N_{\text{cp}} + i].$$

With the assumption $g_i(lT_s - \tau_i) \approx 0$, for $l \ne 0, \dots, L-1$, and due to the Nyquist criterion, the term in parentheses in Eq. (4) can be approximated as

$$\approx \sum_{l=-\infty}^{\infty} g(lT_s - \tau_i)\exp(-j2\pi kl/N_c) = \exp(-j2\pi k\Delta f\tau_i) \tag{5}$$

where $\Delta f = 1/T_sN_c$. Then we have

$$H[n,k] = \sum_{i=0}^{K-1}\overline{\gamma}_i[n]\exp(-j2\pi k\Delta f\tau_i).$$

Now we can write the subcarrier correlation as follows

$$\begin{aligned}r_H[n,k] &= \mathbb{E}\{H^*[n',k']H[n'+n,k'+k]\} \\ &= \sum_{i=0}^{K-1}\mathbb{E}\{\overline{\gamma}^*_i[n']\overline{\gamma}_i[n'+n]\}\exp(-j2\pi k\Delta f\tau_i).\end{aligned} \tag{6}$$

From the expansion of $\overline{\gamma}_i[n]$ in the expectation above, a straightforward computation shows us

$$\begin{aligned}\mathbb{E}\{\overline{\gamma}^*_i[n']\overline{\gamma}_i[n'+n]\} &= \frac{\rho_i}{N_c^2}\sum_{i_1=0}^{N_c-1}\sum_{i_2=0}^{N_c-1}r_i[nN_s + i_2 - i_1] \\ &= \kappa_i\rho_i\overline{r}_i[n],\end{aligned} \tag{7}$$

where $\overline{r}_i[n]$ is the normalized correlation of $\overline{\gamma}_i[n]$, and we have defined the normalization factor as

$$\kappa_i = \frac{1}{N_c^2}\sum_{i_1=0}^{N_c-1}\sum_{i_2=0}^{N_c-1}r_i[i_2 - i_1].$$

Observe that this factor satisfies $0 \le \kappa_i \le 1$ and can be interpreted as the power loss ratio of the i-th path, since $\kappa_i = \mathbb{E}|\overline{\gamma}_i[n]|^2/\rho_i$. The inclusion of the above definitions in Eq. (6) results in

$$r_H[n,k] = \sum_{i=0}^{K-1}\kappa_i\rho_i\overline{r}_i[n]\exp(-j2\pi k\Delta f\tau_i).$$

Just when all paths have the same correlation function $r_t[n]$, the separability property [4] is valid:

$$r_H[n,k] = \kappa\sigma_h^2\overline{r}_t[n]r_f[k],$$

where we define the normalized frequency correlation

$$r_f[k] = \sum_{i=0}^{K-1}\frac{\rho_i}{\sigma_h^2}\exp(-j2\pi k\Delta f\tau_i), \qquad \sigma_h^2 = \sum_{i=0}^{K-1}\rho_i,$$

and observe that all paths have the same factor κ. The channel power σ_h^2 is attenuated by the factor κ, such that the sub-carrier power is given as $\sigma_H^2 = \kappa\sigma_h^2$.

The approximation in Eq. (5) implies the sub-carrier correlation in Eq. (6) does not depend on k'. As consequence, all subcarriers have same power $\sigma_H = \mathbb{E}\,|H[n,k]|^2$. Such approximation is perfectly reasonable if the interference between OFDM symbols is negligible. In the ICI power found below, we used the same kind of approximation.

In what follows, we will find a relationship between the Fourier transforms of $r_t[n]$ and $\overline{r}_t[n]$, which we denote by $p_t[n]$ and $\overline{p}_t[n]$, respectively. Applying the Fourier transform to $\overline{r}_t[n]$ given in Eq. (7), we obtain

$$\kappa\overline{p}_t(\nu) = \frac{1}{N_c^2}\sum_{i_1,i_2=0}^{N_c-1}\sum_{n=-\infty}^{\infty} r_t[nN_s + i_2 - i_1]\exp(-j2\pi n\nu).$$

The last summation in the above equation is identified as [5]

$$\frac{1}{N_s}p_t\Big(\frac{\nu}{N_s}\Big)\exp\Big(j2\pi\frac{i_2-i_1}{N_s}\nu\Big)$$

for $-1/2 \leq \nu \leq 1/2$, whose substitution implies

$$\kappa\overline{p}_t(\nu) = \frac{1}{N_s}p_t\Big(\frac{\nu}{N_s}\Big)\Big\{\frac{1}{N_c^2}\sum_{i_1,i_2=0}^{N_c-1}\exp\Big(j2\pi\frac{i_2-i_1}{N_s}\nu\Big)\Big\}.$$

If the term in the brackets is denoted by $m_t(\nu/N_s)$, we have

$$\kappa\overline{p}_t(\nu) = \frac{1}{N_s}p_t\Big(\frac{\nu}{N_s}\Big)m_t\Big(\frac{\nu}{N_s}\Big), \tag{8}$$

and further simplifications result in

$$m_t(\nu) = \frac{\mathrm{sinc}^2(N_c\nu)}{\mathrm{sinc}^2(\nu)}. \tag{9}$$

Eqs. (8) and (9) provide the desired relationship. The function $m_t(\nu)$ is even and strictly decreasing in $[0, 1/2]$ with maximum $m_t(0) = 1$. Such property justifies the appearance of the attenuation factor $0 \leq \kappa \leq 1$, and shows how the power spectral density of $H[n,k]$ is attenuated. In addition, the factor κ can be alternatively expressed as

$$\kappa = \int_{-1/2}^{1/2} p_t(\nu)m_t(\nu)d\nu. \tag{10}$$

As we will see in the next section, the remaining power $\sigma_h^2 - \sigma_H^2$ appears as the ICI power.

IV. ICI Power

Since the transmitted symbols $a[n,k]$ are i.i.d., and so $H[n,k]a[n,k]$ is uncorrelated from its intercarrier interference, we have from Eq. (2) for the k-th subcarrier

$$\begin{aligned}\sigma_x^2[k] &= \sigma_H^2\sigma_a^2 + \sigma_{\mathrm{ICI}}^2[k]\sigma_a^2 + \sigma_n^2\\ &= \sigma_H^2 + \sigma_{\mathrm{ICI}}^2[k] + \sigma_n^2,\end{aligned}$$

where $\sigma_x^2[k]$ denotes the power of $x[n,k]$, and $\sigma_{\mathrm{ICI}}^2[k]$ is the ICI power over the k-th subcarrier.

One could ask if $\sigma_{\mathrm{ICI}}^2[k]$ is the same for all subcarriers. In fact, it does. In order to find the ICI power, we could calculate $\sigma_x^2[k]$ and see that the ICI power does not depend on k. We chose to find $\sigma_{\mathrm{ICI}}^2[k]$ first, since its calculations have shown to be less tedious and more appropriated for our purpose. As demonstrated in the Appendix for $\sigma_a^2 = 1$, we have the desired result

$$\sigma_{\mathrm{ICI}}^2 = \sigma_h^2 - \sigma_H^2, \tag{11}$$

where the index k was omitted. Such result permits to conclude that the received signal $x[n,k]$ has equal power for all k. Then, for $\sigma_a^2 = 1$, we can write

$$\sigma_x^2 = \sigma_h^2 + \sigma_n^2,$$

where the index k was omitted again.

The expression in Eq. (11) has broad applicability, since few premises were assumed. In the derivation of Eq. (11), only the following assumptions were used: (a) the symbols $a[n,k]$ are i.i.d., (b) the interference between OFDM symbols is negligible, and (c) the WSS-US channel model holds. Additionally, the tapped delay line (TDL) assumption could be easily suppressed.

In the case the channel paths have the same Doppler spectrum, we can introduce the factor κ:

$$\sigma_{\mathrm{ICI}}^2 = (1-\kappa)\sigma_h^2. \tag{12}$$

The above equation together with Eq. (10) is similar to that found in [3], except for the use of $\mathrm{sinc}^2(N_c\nu)$ in the place of $m_t(\nu)$. In [3], an infinite number of subcarriers was regarded, resulting in an upper bound for the ICI power. In this work, instead, the expression in Eq. (12) is exact, since we consider a finite number of interfering subcarriers.

A. *Upper Bound for* σ_{ICI}^2

In this subsection, we derive an upper bound for σ_{ICI}^2 for certain maximum Doppler frequency ν_d. From Eq. (12), we can state that σ_{ICI}^2 is maximized when κ is minimized. Then, from κ defined in Eq. (10), we can write

$$\begin{aligned}\kappa &= \int_{-\nu_d}^{\nu_d} p_t(\nu)m_t(\nu)d\nu\\ &\geq \Big(\min_{\nu_d\leq\nu\leq\nu_d} m_t(\nu)\Big)\cdot\int_{-\nu_d}^{\nu_d} p_t(\nu)d\nu = m_t(\nu_d),\end{aligned}$$

where we used the fact that $m_t(\nu)$ in $[-\nu_d, \nu_d]$ is minimized at $\nu = \nu_d$. Indeed, $m_t(\nu)$ is concave in $[-1/2, 1/2]$ with maximum at $\nu = 0$. Therefore, the maximum σ_{ICI}^2 is given by

$$\sigma_{\mathrm{ICI,max}}^2 = [1 - m_t(\nu_d)]\sigma_h^2.$$

This maximum value is attained for the two-path spectrum

$$p_t(\nu) = \frac{1}{2}[\delta(\nu+\nu_d) + \delta(\nu-\nu_d)],$$

whose corresponding correlation function is

$$r_t[n] = \cos(2\pi\nu_d n).$$

The two-path spectrum corresponds to an OFDM system with a fixed offset of $T_s\nu_d$ Hz.

We also consider the classical Jakes spectrum

$$p_t(\nu) = \begin{cases}\dfrac{1}{\pi\nu_d}\dfrac{1}{\sqrt{1-(\nu/\nu_d)^2}}, & \text{for } |\nu| < \nu_d,\\ 0, & \text{otherwise,}\end{cases}$$

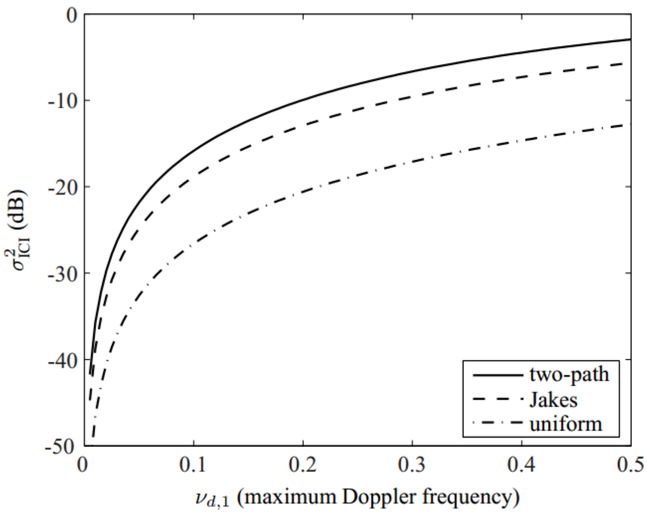

Fig. 1. Comparison of exact ICI power for two-path, Jakes and uniform spectra.

and the uniform spectrum

$$p_t(\nu) = \begin{cases} \dfrac{1}{2\nu_d}, & \text{for } |\nu| < \nu_d, \\ 0, & \text{otherwise,} \end{cases}$$

whose correlation functions are

$$r_t[n] = J_0(2\pi\nu_d n), \qquad r_t[n] = \mathrm{sinc}(2\nu_d n),$$

respectively, where $J_0(\cdot)$ is the zeroth-order Bessel function of the first kind.

Fig. 1 shows the ICI power σ^2_{ICI} as function of the maximum Doppler frequency ν_d for the two-path, Jakes and uniform spectra. We employed the values $\sigma_h^2 = \sigma_a^2 = 1$ and $N_c = 128$. As previously indicated, the two-path spectrum presents the largest ICI power.

V. Conclusions

This letters shows an analytical result to the computation of the power of the intercarrier interference in OFDM systems under WSS-US channels. We take into account that the sub-carriers consist of the average taken over the number of sub-carriers samples of the channel in frequency-domain. As a consequence, the sub-carriers have different correlations and experience a power loss with respect to the channel power. The used assumptions to derive the result are very usual in practical systems. The achieved expression has been shown to be more accurate than the previous one reported in the literature.

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Appendix
Proof of the Result in Eq. (11)

If the symbols at the sub-carriers are i.i.d., the ICI power at the k-th subcarrier is given by

$$\sigma^2_{\mathrm{ICI}}[k] = \mathbb{E}\left|\sum_{i=0}^{N_c-1} (\mathcal{H}_e[n])_{ki} a[n,i]\right|^2 = \sum_{i=0}^{N_c-1} \mathbb{E}\,|(\mathcal{H}_e[n])_{ki}|^2.$$

Let $e_{ml} = \left(h^n_{m,(m-l)_{N_c}} - \frac{1}{N_c}\sum_{i=0}^{N_c-1} h^n_{i,(m-l)_{N_c}}\right)$ be the (m,l)-th entry of $\mathbf{H}_e[n]$. Since the (k,i)-th entry of $\mathcal{H}_e[n]$ is given by

$$(\mathbf{H}_e[n])_{ki} = \frac{1}{N_c}\sum_{m=0}^{N_c-1}\sum_{l=0}^{N_c-1} e_{ml}\omega^{km-il},$$

after some simplifications, we can write

$$\sigma^2_{\mathrm{ICI}}[k] = \frac{1}{N_c}\sum_{l,m_1,m_2=0}^{N_c-1} \mathbb{E}\{e^*_{m_1 l} e_{m_2 l}\}\omega^{k(m_2-m_1)}.$$

The expansion of the summation above in terms of $h^n_{m,l}$ and further simplifications result in

$$\sigma^2_{\mathrm{ICI}}[k] = \left(\frac{1}{N_c}\sum_{l,m_1,m_2=0}^{N_c-1} \mathbb{E}\{h^{n*}_{m_1,(m_1-l)_{N_c}} h^n_{m_2,(m_2-l)_{N_c}}\} \cdot \omega^{k(m_2-m_1)} - \mathbb{E}\,|H[n,k]|^2\right). \quad (13)$$

Writing $h^n_{m_2,(m_2-l)_{N_c}}$ as a function of $\gamma_k[m]$ and $g_k[l]$, the expectation in Eq. (13) can be expressed as

$$\sum_{i=0}^{K-1} \rho_i r_i[m_2-m_1] g_i^*[(m_1-l)_{N_c}] g_i[(m_2-l)_{N_c}].$$

Then, for the summation in Eq. (13), we have

$$\frac{1}{N_c}\sum_{i=0}^{K-1}\sum_{l,q=0}^{N_c-1} \rho_i r_i[q] \cdot \left(\sum_{(m_2-m_1)_{N_c}=q} g_i^*[(m_1-l)_{N_c}] g_i[(m_2-l)_{N_c}]\right)\cdot \omega^{kq}. \quad (14)$$

The last summation above is recognized as the q-th element of the circular convolution of $g_i[m]$ with itself. Since the Fourier transform of $g_i[m]$ is approximated by $\exp(-j2\pi k\Delta f \tau_i)$, we have that this convolution is an impulse, i.e., equal to 1, for $q=0$, and 0, otherwise. Then, Eq. (14) results in

$$\frac{1}{N_c}\sum_{i=0}^{K-1}\sum_{l=0}^{N_c-1} \rho_i r_i[0] = \sigma_h^2.$$

Finally, we obtain the desired result

$$\sigma^2_{\mathrm{ICI}} = \sigma_h^2 - \sigma_H^2,$$

where the index k was omitted in $\sigma^2_{\mathrm{ICI}}[k]$, since this term has the same value for all k.