

# Classical Information Transmission via Majorization Relations in Quantum Channels

Andresso da Silva and Francisco M. de Assis

**Abstract**— In this work, we investigate a communication framework based on majorization relations between quantum states. Instead of encoding information directly into quantum states, classical symbols are associated with relational structures such as comparability and incomparability under majorization. We focus on qutrit systems and analyze how these relations behave under the action of quantum channels, in particular depolarizing and dephasing noise models. Through explicit examples, we show how classical information can be transmitted by exploiting the ordering structure of state ensembles and how quantum noise affects the preservation of these relations. We further discuss a simple protocol in which decoding is performed by identifying majorization relations between received states, highlighting the role of spectral properties rather than state distinguishability. Our results indicate that, although this relational encoding provides an alternative perspective for information transmission in quantum systems, its performance is strongly dependent on the channel dynamics.

**Keywords**— Majorization, Quantum channel, Quantum states, Classical information, Partial order.

## I. INTRODUCTION

Permutation codes represent information through the relative ordering of symbols, exploiting combinatorial and metric properties of permutations to achieve robustness against certain types of errors. This approach is particularly interesting because it allows information to be encoded in the ordering structure of the data, rather than relying solely on individual symbols, and has been applied in communication, storage, and ranking modulation problems [1], [2], [3].

Motivated by these characteristics, this work aims to investigate extensions of these ideas to the context of quantum channels. In particular, we explore how properties related to ordering, permutation, and distinguishability may be preserved or modified by the action of quantum channels, with the goal of developing new strategies for encoding and transmitting quantum information based on relational structures.

Unlike the classical case, where the notion of order between symbols or probability distributions can be defined in a straightforward manner, in the quantum setting the definition of order between states is significantly more subtle due to the intrinsic properties of quantum systems. Quantum states do not, in general, admit a unique natural ordering, and different comparison criteria may lead to distinct interpretations of what it means for one state to be more ordered than another.

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Moreover, the action of a quantum channel may simultaneously affect both classical and genuinely quantum features of the system, making the extension of classical notions of permutation and ordering to the quantum regime nontrivial.

A possible approach to defining ordering relations between quantum states is based on majorization theory [4]. In general, majorization provides a criterion for comparing probability distributions, allowing one to determine when a state can be regarded as more “ordered” or less dispersed than another [5]. This relation is deeply connected to concepts such as mixing, entropy, and physically admissible transformations, making it a natural tool for investigating order structures in quantum systems and the effect of quantum channels on these relations [6].

Motivated by these ideas, we investigate the possibility of encoding classical information through majorization relations between quantum states. Rather than associating information solely with individual states, the proposed approach exploits relational structures based on the partial order induced by majorization between the eigenvalue spectra of density operators. In this way, information is represented by comparability and incomparability relations between states, allowing one to analyze how these structures are preserved, degraded, or transformed under the action of quantum channels.

Several works have studied the role of majorization in quantum information, both as a tool to characterize state transformations under quantum channels [6], [7] and as a notion of disorder in quantum processes [8]. In particular, it has been shown that certain classes of quantum channels exhibit majorization-preserving properties [9], [10], establishing a direct link between channel dynamics and spectral ordering. A comprehensive overview of majorization-based resource theories and their applications in quantum information and quantum thermodynamics is presented in [11], highlighting majorization as a unifying framework for characterizing resources and state transformations across different quantum settings. Recent work [12] investigates the use of catalytic pre-processing to enhance coherence under noisy channels. In that setting, the input state is suitably transformed before the action of a quantum channel, with the goal of increasing the coherence fraction of the output state compared to the direct channel action. However, these approaches do not consider the use of majorization relations themselves as carriers of classical information.

The remainder of this paper is organized as follows. Section II introduces the fundamental concepts of majorization and quantum channels, providing the theoretical background necessary for the development of the proposed framework. Section III investigates the possibility of transmitting classical information through majorization relations between quantum

states, discussing the encoding strategy and the effects of quantum channels on the preservation of these relations. Finally, Section IV presents the conclusions and discusses possible directions for future research.

## II. QUANTUM CHANNELS AND MAJORIZATION

Quantum channels describe the physical evolution of quantum states under noise, decoherence, and information transmission processes. Mathematically, they are represented by completely positive and trace-preserving (CPTP) maps acting on density operators. Since quantum channels modify the eigenvalue spectra of quantum states, they are naturally connected to majorization theory, which provides a partial ordering based on spectral distributions. In this context, majorization can be used to analyze how a channel affects the disorder or mixedness of transmitted states, as well as whether ordering relations between states are preserved after the transmission process.

*Definition 1 (Quantum Channel [6, Definition 2.13]):* A quantum channel is a linear map

$$\Phi : L(\mathcal{X}) \rightarrow L(\mathcal{Y}) \quad (1)$$

for some choice of complex Euclidean spaces  $\mathcal{X}$  and  $\mathcal{Y}$ , satisfying two properties:

- 1)  $\Phi$  is completely positive (CP)
- 2)  $\Phi$  is trace preserving (TP).

*Definition 2 (Unital Quantum Channel [6, Definition 4.1]):* Let  $\mathcal{X}$  be a complex Euclidean space. A channel  $\Phi \in C(\mathcal{X})$  is a unital channel if  $\Phi(\mathbb{1}_{\mathcal{X}}) = \mathbb{1}_{\mathcal{X}}$ .

The notion of ordering plays a fundamental role in classical information theory, where symbols and messages are often organized according to numerical or probabilistic structures. In quantum systems, however, defining an ordering between states is substantially more challenging. Majorization defines a partial ordering between quantum states through the comparison of the eigenvalues of their density operators. This concept has several applications in quantum information theory and quantum channels. For further details on majorization and its applications in quantum information theory, see [5] and [6, Chap. 4], where the geometric and algebraic aspects of quantum states and quantum channels are discussed in greater detail.

*Definition 3 (Majorization [5]):* Let  $u = (u_1, \dots, u_N)$  and  $v = (v_1, \dots, v_N)$  be vectors in  $\mathbb{R}^N$ . Denote by  $u^\downarrow$  and  $v^\downarrow$  the vectors obtained by arranging the components of  $u$  and  $v$  in non-increasing order, respectively. Then,  $u$  is said to be majorized by  $v$ , denoted by

$$u \prec v,$$

if the following conditions hold:

$$\sum_{i=1}^k u_i^\downarrow \leq \sum_{i=1}^k v_i^\downarrow, \quad \forall k = 1, \dots, N-1, \quad (2)$$

and

$$\sum_{i=1}^N u_i^\downarrow = \sum_{i=1}^N v_i^\downarrow. \quad (3)$$

For probability vectors, majorization defines a partial ordering ranging from the maximally mixed distribution to the

maximally ordered one. In particular, any probability vector  $u$  satisfies

$$u_{(N)} \equiv \left( \frac{1}{N}, \frac{1}{N}, \dots, \frac{1}{N} \right) \prec u \prec (1, 0, \dots, 0) \equiv u_{(1)}. \quad (4)$$

The vector  $u_{(N)}$  represents the maximally mixed distribution, while  $u_{(1)}$  corresponds to the maximally ordered distribution. Therefore, every probability vector  $u$  lies between these two extremal cases under the majorization relation.

*Theorem 1 ([6, Theorem 4.32]):* Let  $X, Y \in \text{Herm}(\mathcal{X})$  be Hermitian operators, for  $X$  a complex Euclidean space. The following statements are equivalent:

- 1)  $Y \prec X$ .
- 2) There exists a unital channel  $\Phi \in C(\mathcal{X})$  such that  $Y = \Phi(X)$ ;
- 3) There exists a positive, trace-preserving, and unital map  $\Phi \in T(\mathcal{X})$  such that  $Y = \Phi(X)$ .
- 4)  $\lambda(Y) \prec \lambda(X)$ .

The notation  $\lambda(X)$  and  $\lambda(Y)$  denotes the vector composed of the eigenvalues of the operators  $X$  and  $Y$ , respectively, and usually arranged in non-increasing order.

It is important to emphasize that the existence of incomparability relations requires systems with dimension  $N > 2$ . In the qubit case, the spectrum of a density operator is completely determined by a single independent eigenvalue due to the normalization constraint. As a consequence, the majorization relation between two qubit states becomes totally ordered. Therefore, incomparability cannot emerge in two-dimensional systems, since the eigenvalue vectors can always be ordered according to their degree of mixedness. For this reason, the definitions of the considered quantum channels will be presented for qutrit systems.

The following quantum channels will be used in the examples presented in the next sections.

*Definition 4 (Dephasing Channel [13]):* For a qutrit system, the dephasing channel can be defined as a quantum channel that preserves the diagonal elements of the density matrix while attenuating its off-diagonal coherences. A common representation is given by

$$\Phi_p(\rho) = (1-p)\rho + p \sum_{i=0}^2 |i\rangle \langle i| \rho |i\rangle \langle i|, \quad (5)$$

where  $0 \leq p \leq 1$  is the dephasing parameter and  $\{|0\rangle, |1\rangle, |2\rangle\}$  denotes the computational basis of the qutrit Hilbert space. Under this channel, the populations remain unchanged, while the coherences are reduced according to the parameter  $p$ . This channel is widely used to model phase noise and decoherence processes in quantum communication and quantum computing systems.

*Definition 5 (Depolarizing Channel [13]):* The depolarizing channel is defined as

$$\Phi_p(\rho) = (1-p)\rho + p \frac{\mathbb{1}}{d} \text{Tr}(\rho), \quad 0 \leq p \leq \frac{d^2}{d^2-1}, \quad (6)$$

where  $0 \leq p \leq 1$ ,  $d$  is the dimension of the Hilbert space, and  $\mathbb{1}/d$  represents the maximally mixed state. This channel replaces the input state with the maximally mixed state with probability  $p$ , modeling a uniform loss of information.

### III. ENCODING INFORMATION WITH STATE MAJORIZATION

Classical information can be encoded through majorization relations by associating logical symbols with ordering structures between quantum states instead of the states themselves. In this approach, the information is carried by whether a pair of density operators satisfies a majorization relation, such as  $\rho_i \prec \rho_j$ , or whether the states are incomparable under majorization. Consequently, the decoding process is based on identifying the preservation or modification of these relations after transmission through a quantum channel. This strategy differs from conventional quantum communication protocols, which typically rely on state distinguishability or orthogonality, and instead explores the spectral structure of quantum states as the fundamental carrier of information.

Higher-dimensional ( $d > 2$ ) systems are particularly relevant in this context because incomparability relations can emerge due to the partial nature of majorization. By considering both comparable and incomparable pairs of states, different logical symbols may be associated with distinct ordering relations. The transmission process can then be analyzed according to whether these relations are preserved or modified after the action of a quantum channel.

In this work, we analyze the simplest scenario, where information is encoded only through comparability and incomparability relations between pairs of quantum states. More general constructions are also possible, involving ordering relations among larger sets of ( $n$ ) states, which may lead to more complex coding structures and richer relational configurations. However, restricting the analysis to pairs of states allows a simpler and more direct investigation of how majorization relations behave under the action of quantum channels.

Some examples will now be presented to motivate the definitions and concepts introduced in this section, as well as to illustrate how majorization relations between quantum states can be affected by the action of quantum channels.

*Example 1 (Dephasing Channel):* Considering the channel defined in Eq. 5, the qutrit dephasing channel does not modify the eigenvalues of the input states when they are diagonal in the computational basis. This occurs because the channel acts only on the off-diagonal elements of the density matrix, preserving the populations of the state. Consequently, for diagonal states, the action of the channel leaves the spectrum unchanged.

Consider three diagonal qutrit states  $\rho_0$ ,  $\rho_1$ , and  $\rho_2$  satisfying the relations  $\rho_1 \prec \rho_0$  and  $\rho_1 \not\prec \rho_2$ . The relation  $\rho_1 \prec \rho_0$  may be associated with the classical bit 0, while the incomparability relation  $\rho_1 \not\prec \rho_2$  may be associated with the classical bit 1. Therefore, if the pairs  $\{\rho_1, \rho_0\}$  or  $\{\rho_1, \rho_2\}$  are transmitted through the channel and no phase modifications occur, the majorization relations are trivially preserved. Since the states are diagonal in the computational basis, the action of the dephasing channel leaves their spectra unchanged, and consequently the ordering relations between the states remain unaffected after transmission.

It is worth noting that using majorization relations in this case is equivalent to finding a way of transmitting classical

information with zero error through this channel. However, two quantum states are required to transmit a single classical bit, resulting in a transmission rate of  $1/2$  bit per channel use. For the dephasing channel, it is known that the zero-error capacity for  $d$ -dimensional states is given by  $C_0 = \log_2 d$  bits per channel use [14].

If the order in which the states are transmitted is also considered, the transmission rate can be improved, as illustrated in Table I.

TABLE I  
ENCODING OF CLASSICAL SYMBOLS USING MAJORIZATION RELATIONS BETWEEN PAIRS OF QUANTUM STATES, TAKING INTO ACCOUNT THE ORDER IN WHICH THE STATES ARE TRANSMITTED. EACH RELATION IS ASSOCIATED WITH A DISTINCT CLASSICAL SYMBOL IN THE PROPOSED ENCODING SCHEME.

| Maj. Relation                 | Classical State |
|-------------------------------|-----------------|
| $\rho_1^t \prec \rho_0^{t+1}$ | 0               |
| $\rho_0^t \succ \rho_1^{t+1}$ | 1               |
| $\rho_1 \not\prec \rho_2$     | 2               |

In this case, three distinct classical symbols can be associated with different ordering relations, leading to a transmission rate given by

$$R = \frac{\log_2 M}{n} = \frac{\log_2 3}{2} \approx 0.792 \quad (7)$$

bits per channel use.

*Example 2 (Depolarizing Channel):* In this example, the depolarizing channel defined in Eq. 6 will be considered. Let  $\mathcal{S} = \{\rho_0, \rho_1, \rho_2\}$  be the input states, where

- $\rho_0 = \text{diag}(0.7, 0.2, 0.1)$
- $\rho_1 = \text{diag}(0.5, 0.3, 0.2)$
- $\rho_2 = \text{diag}(0.45, 0.4, 0.15)$ .

Since the states are diagonal matrices, their eigenvalues are directly given by the diagonal elements  $\lambda(\rho_0) = (0.7, 0.2, 0.1)$ ,  $\lambda(\rho_1) = (0.5, 0.3, 0.2)$  and  $\lambda(\rho_2) = (0.45, 0.4, 0.15)$ . Note that all eigenvalue vectors are already written in non-increasing order. Table II presents the partial sums of the eigenvalues associated with the states in  $\mathcal{S}$ .

TABLE II  
PARTIAL SUMS OF THE COMPONENTS OF  $\lambda(\rho_0)$ ,  $\lambda(\rho_1)$  AND  $\lambda(\rho_2)$ .

| $k$ | $\sum_{i=1}^k \lambda(\rho_0)_i^\downarrow$ | $\sum_{i=1}^k \lambda(\rho_1)_i^\downarrow$ | $\sum_{i=1}^k \lambda(\rho_2)_i^\downarrow$ |
|-----|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| 1   | 0.7                                         | 0.5                                         | 0.45                                        |
| 2   | 0.9                                         | 0.8                                         | 0.85                                        |
| 3   | 1.0                                         | 1.0                                         | 1.0                                         |

By comparing the partial sums according to the definition of majorization, it follows that  $\rho_1 \prec \rho_0$ . Performing the same analysis for the remaining states also leads to  $\rho_2 \prec \rho_0$  and  $\rho_1 \not\prec \rho_2$ . The majorization relations among the elements of  $\mathcal{S}$  are illustrated in Fig. 1.

Encoding majorization relations provides a way to associate classical information with the ordering structure between quantum states in  $\mathcal{S}$ . In particular, the relation  $\rho_1 \prec \rho_0$  is used to represent the classical bit 0, while the incomparability relation  $\rho_1 \not\prec \rho_2$  is used to represent the classical bit 1.

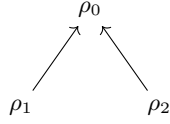


Fig. 1. Hasse diagram representing the majorization relations among the states of  $\mathcal{S}$

The transmission process can then be analyzed according to whether these relations are preserved after the action of the depolarizing channel.

The possible outputs associated with the transmission of the logical symbols through the depolarizing channel are presented in Tables III and IV. Since the depolarizing channel replaces the input state with the maximally mixed state  $\mathbb{1}/3$  with probability  $p$ , different output relations may emerge depending on which states are affected by the channel.

Table III shows the possible outcomes when transmitting the logical symbol 0, which is associated with the majorization relation  $\rho_1 \prec \rho_0$ . In most cases, the output relation is preserved, since the maximally mixed state is majorized by every state. However, when both states are mapped to  $\mathbb{1}/3$ , the resulting relation becomes degenerate, leading to a decoding error.

Similarly, Table IV presents the possible outcomes when transmitting the logical symbol 1, associated with the incomparability relation  $\rho_1 \not\prec \rho_2$ . In this case, the introduction of the maximally mixed state destroys the original relational structure, since  $\mathbb{1}/3$  is majorized by all states, forcing the received relation to become comparable and potentially leading to incorrect decoding.

Let  $p(i|j)$  denote the probability of receiving the logical symbol  $i$  given that  $j$  was transmitted. Using Tables III and IV, the probability of decoding error is given by

$$p_{err}(p) = p(1|0) + p(0|1) \quad (8)$$

$$= p^2 + 2(1-p)p \quad (9)$$

$$= -p^2 + 2p. \quad (10)$$

In this way, since  $p_{err}(p) > p$  for all values of  $p$ , except for  $p = 0$  and  $p = 1$ , where equality holds, this approach leads to a higher error probability for this channel. This indicates that, although the encoding based on majorization relations provides a valid way of transmitting classical information, it is not optimal in terms of error performance for the depolarizing channel considered here.

TABLE III

POSSIBLE ERRORS WHEN TRANSMITTING THE LOGICAL SYMBOL 0. THE INPUT RELATION CORRESPONDS TO  $\rho_1 \prec \rho_0$ , AND DIFFERENT CHANNEL OUTPUTS MAY AFFECT THE DECODED MAJORIZATION RELATION.

| Input                | Output                                           | Resulting Maj. Relation                               | Decoded Symbol |
|----------------------|--------------------------------------------------|-------------------------------------------------------|----------------|
| $\{\rho_1, \rho_0\}$ | $\{\rho_1, \rho_0\}$                             | $\rho_1 \prec \rho_0$                                 | 0              |
| $\{\rho_1, \rho_0\}$ | $\{\frac{\mathbb{1}}{3}, \rho_0\}$               | $\frac{\mathbb{1}}{3} \prec \rho_0$                   | 0              |
| $\{\rho_1, \rho_0\}$ | $\{\rho_1, \frac{\mathbb{1}}{3}\}$               | $\frac{\mathbb{1}}{3} \prec \rho_1$                   | 0              |
| $\{\rho_1, \rho_0\}$ | $\{\frac{\mathbb{1}}{3}, \frac{\mathbb{1}}{3}\}$ | $\frac{\mathbb{1}}{3} \not\prec \frac{\mathbb{1}}{3}$ | 1              |

TABLE IV

POSSIBLE ERRORS WHEN TRANSMITTING THE LOGICAL SYMBOL 1. THE INPUT RELATION CORRESPONDS TO  $\rho_1 \not\prec \rho_2$ , AND CHANNEL-INDUCED MIXING MAY DESTROY THE INCOMPARABILITY RELATION.

| Input                | Output                                           | Resulting Maj. Relation                               | Decoded Symbol |
|----------------------|--------------------------------------------------|-------------------------------------------------------|----------------|
| $\{\rho_1, \rho_2\}$ | $\{\rho_1, \rho_2\}$                             | $\rho_1 \not\prec \rho_2$                             | 1              |
| $\{\rho_1, \rho_2\}$ | $\{\frac{\mathbb{1}}{3}, \rho_2\}$               | $\frac{\mathbb{1}}{3} \prec \rho_2$                   | 0              |
| $\{\rho_1, \rho_2\}$ | $\{\rho_1, \frac{\mathbb{1}}{3}\}$               | $\frac{\mathbb{1}}{3} \prec \rho_1$                   | 0              |
| $\{\rho_1, \rho_2\}$ | $\{\frac{\mathbb{1}}{3}, \frac{\mathbb{1}}{3}\}$ | $\frac{\mathbb{1}}{3} \not\prec \frac{\mathbb{1}}{3}$ | 1              |

Based on these examples, we now outline a simple communication protocol that exploits majorization relations as information carriers. The sender encodes classical symbols into pairs of quantum states chosen from a predefined set  $\mathcal{S}$ , where each symbol is associated with a specific ordering relation (either comparability or incomparability) between the states. These states are then transmitted through a quantum channel, and the receiver infers the encoded symbol by evaluating the majorization relation between the received outputs.

The decoding rule is entirely relational: rather than identifying individual states, the receiver determines whether the observed pair satisfies a majorization relation, its reverse, or becomes comparable after the action of the channel. In this way, the protocol transforms the problem of state discrimination into the problem of detecting ordering structures. As illustrated in the previous examples, the performance of this scheme depends crucially on how the quantum channel preserves or degrades these relations, providing a direct operational link between channel noise and the robustness of the encoding strategy.

The examples show that, even in simple scenarios involving qutrit states and standard noise models, the preservation or destruction of ordering relations plays a central role in the decoding process. In particular, the dependence on the transmission order and the emergence of the maximally mixed state highlight the nontrivial interplay between channel noise and relational encoding.

Although the analysis was restricted to a small set of states and pairwise relations, the framework can be naturally extended to larger ensembles, where more complex ordering structures and richer coding alphabets may be constructed. In higher-dimensional settings, the presence of multiple incomparable pairs further increases the expressive power of the scheme, suggesting possible generalizations beyond binary or ternary encodings. These observations indicate that majorization-based encoding provides a flexible alternative perspective for classical information transmission in quantum systems, which may be further explored in more general channels and state configurations.

#### IV. CONCLUSIONS

In this work, we have explored a communication framework based on majorization relations between quantum states. Instead of encoding information in the states themselves, classical symbols were associated with relational structures, namely comparability and incomparability under majorization. We showed, through explicit examples involving qutrit systems

and standard quantum channels, how these relations can be used to construct simple encoding and decoding schemes, and how their preservation depends on the action of the channel.

The analysis highlighted that quantum noise not only affects the states individually but can also alter the relational structure between them, which is the key resource in the proposed approach. In particular, we observed that even simple noise models such as the depolarizing channel can introduce nontrivial errors in the decoding process by modifying or destroying majorization relations. This provides a complementary perspective to conventional state-based communication protocols, emphasizing the role of spectral order rather than state distinguishability.

Future work may extend this framework in several directions. One natural generalization is the study of larger sets of quantum states, leading to richer partially ordered structures and higher-dimensional encoding alphabets. It would also be interesting to investigate other classes of quantum channels, including non-unital and non-Markovian dynamics, and their impact on the preservation of majorization relations. Moreover, connections with resource theories and quantum thermodynamics may provide deeper insights into the operational meaning of relational encoding, potentially leading to more efficient or noise-resilient communication protocols.

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