Rewriting the Partial Order Permutation Entropy Using Partially Commutative Monoids

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Abstract— **Permutation entropy is a popular complexity measure for time series based on the distribution of ordinal patterns defined over a totally ordered alphabet. The extension to partially ordered alphabets, known as Partially Ordered Permutation Entropy (POPE), allows analysis of data where only a partial order between symbols exists, broadening the applicability of the method. However, the lack of a known formula to enumerate equivalence classes under partial order has prevented the definition of a normalized entropy in this setting. In this work, we reinterpret POPE through the algebraic framework of partially commutative monoids, which naturally model commutativity relations among symbols via a graph structure. This approach enables the explicit calculation of the number of equivalence classes (generalized ordinal patterns) of length L as a function of the commutativity graph. Leveraging these results, we introduce a normalized version of the partially ordered permutation entropy, allowing for a meaningful complexity measure comparable across different systems.**

Keywords— **Entropy, Partial Order, Partially Commutative Monoid, Complexity.**

I. Introduction

Permutation entropy, originally introduced by Bandt and Pompe [1], is a widely used complexity measure for time series that captures the temporal structure of signals by analyzing the distribution of ordinal patterns. One of its key strengths is its simplicity and robustness to noise, making it suitable for a variety of practical applications [2], [3], [4]. However, the classical definition assumes a totally ordered alphabet, which limits its applicability in contexts where such an order is not naturally defined or is overly restrictive.

To address this limitation, Haruna [5] proposed a generalization known as Partially Ordered Permutation Entropy (POPE), which extends the concept of permutation entropy to settings where elements of the alphabet are partially ordered. While this extension allows a more flexible analysis of symbolic data, it introduces significant combinatorial challenges, particularly in enumerating the number of distinct ordinal patterns under a partial order. As a consequence, unlike in the totally ordered case, a normalized entropy has not been defined in this generalized setting.

This paper addresses that gap by reinterpreting partially ordered permutation entropy using the algebraic framework of partially commutative monoids. These monoids naturally encode concurrent behavior and partial order through a commutativity graph, where symbols that can be permuted without affecting equivalence are connected. By establishing a correspondence between equivalence classes of words and ordinal patterns under partial order, we show that each subsequence in a time series can be associated with a class in the monoid. It is worth noting that Li [6] has recently presented results concerning Shannon entropy in the context of ordered monoids. Although these results are related to the present work, ordered monoids are distinct from partially commutative monoids, and POPE is not considered in Li's analysis.

The main contributions of this article are twofold: (i) we show that POPE can be reinterpreted in terms of partially commutative monoids, and (ii) we define a normalized entropy measure for partially ordered alphabets, made possible by the combinatorial enumeration of equivalence classes. This formalization not only provides a clearer understanding of the underlying structures involved in POPE but also extends its applicability to a broader class of systems, including those with concurrency and partial temporal relationships.

The remainder of the article is organized as follows: Section II presents the necessary background on permutation entropy and its generalization to partial orders, introduces the theory of partially commutative monoids and describes how these structures model partial orderings and equivalence classes. In Section III, we establish the connection between POPE and the partially commutative monoid framework and derive the formula for normalized POPE. Finally, Section IV presents concluding remarks and outlines directions for future research.

II. Definitions and known results

In this section, we introduce the notation and preliminary results that form the foundation for the main developments of this work. We begin by defining fundamental concepts related to finite alphabets, partially commutative monoids, and commutativity graphs, which are essential for the reinterpretation of partial order permutation entropy. We then revisit classical results concerning the number of equivalence classes in partially commutative monoids, highlighting their connection with dependency polynomials and combinatorial structures such as heaps of pieces. These elements will provide the necessary support for the contributions presented in the following sections.

Let Σ be a finite alphabet. The set Σ^*, called the free monoid over Σ, consists of all finite words formed from the elements of Σ, including the empty word ϵ. The subset Σ^n contains all words of length n over the alphabet Σ.

The concept of a partially ordered set (POSET) plays a central role in this work and is defined as follows.

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Definition 1: POSET A partially ordered set (POSET) is a par $\mathcal{P} = (P, \preceq)$ such that P is a set and $\preceq$ is binary relation which is

1) reflexive ($x \preceq x$ for all $x \in P$)
2) anti-symmetric (if $x \preceq y$ and $y \preceq x$ then $x = y$)
3) transitive (if $x \preceq y$ and $y \preceq z$, then $x \preceq z$ for all $x, y, z \in P$)

A POSET can be represented in matrix form by constructing its adjacency matrix, which encodes the partial order relations among its elements.

Definition 2 (Adjacency Matrix): Given a finite POSET $(P, \preceq)$, with $|P| = n$, we define an $n \times n$ binary matrix M such that $M_{ij} = 1$ if and only if the element $x_i \preceq x_j$ in the POSET.

Let $\mathcal{M}_n(\{0,1\})$ denote the set of all binary matrices of order n, that is, all $n \times n$ natrices whose entries belong to the set $\{0, 1\}$. The adjacency matrices of the POSET belong to this set, $M \in \mathcal{M}_n(\{0,1\})$.

A. Permutation Entropy

The permutation entropy was introduced by Bandt and Pompe [1] as a measure of time series complexity. It is robust to noise, simple to compute, and invariant to certain transformations. Due to these properties, permutation entropy has been applied in a wide range of domains.

The permutation entropy is defined based on the statistical properties of ordinal patterns extracted from time series data. These ordinal patterns encode the temporal relationships between neighboring elements in the sequence.

Definition 3: (Ordinal Pattern [7]) Given a real valued vector $\mathbf{x} = (x_1, ..., x_L) \in \mathbb{R}^L$ where $x_i \neq x_j$ if $i = j$, then the ordinal pattern $\pi(\mathbf{x})$ is permutation of

$$(\pi_1, \pi_2, \ldots, \pi_L) \tag{1}$$

where $\{\pi_i\}_{i=1}^{L} = \{1, 2, \ldots, L\}$ and $\pi_i < \pi_j$ if and only if $x_i < x_j$.

Since no repetitions are allowed, the number of ordinal patterns is $L!$, corresponding to the number of allowed permutations. Fig.1 shows a graphical representation of the patterns for $L = 3$.

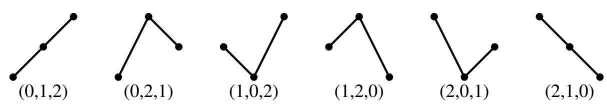

Fig. 1: Example of ordinal patterns for $L = 3$. There are $L! = 6$ patterns in total.

Given a sequence $\mathbf{x} = \{x_1, \ldots, x_n\}$, the permutation entropy is computed based on its subsequences. A sliding window of size $n - L + 1$ is defined to traverse $\mathbf{x}$, producing L overlapping subsequences of the form

$$x_{k,L} = (x_k, x_{k+1}, \ldots, x_{k+(L-1)}), \tag{2}$$

where $k = 1, 2, \ldots, n - L + 1$.

The probability of an ordinal pattern is given by its frequency among the $n - L + 1$ subsequences, that is,

$$p(\pi) = \frac{N(\pi, \mathbf{x})}{n - L + 1}, \tag{3}$$

where $N(\pi, \mathbf{x})$ enotes the number of occurrences of the ordinal pattern π among the subsequences $x_{k,L}$ of $\mathbf{x}$ for $k = 1, 2, \ldots, n - L + 1$. The probability $p(\pi)$ thus represents the best empirical estimate of the ordinal pattern distribution in the finite sequence [1]. Based on the distribution of ordinal patterns, the permutation entropy can be defined.

Definition 4 (Permutation Entropy [1]):

$$H(L) = -\sum p(\pi) \log p(\pi) \tag{4}$$

The maximum value of $H(L)$ is attained when $p(\pi)$ follows a uniform distribution, where $p(\pi) = 1/L!$ and $H(L) = \log L!$, in such a way that $0 \leq H(L) \leq \log L!$. It is sometimes convenient to normalize the value of $H(L)$ so that it can be interpreted as a measure of complexity. The normalized permutation entropy is therefore defined as

$$\overline{H}(L) = \frac{H(L)}{\log L!} \tag{5}$$

Since $0 \leq \overline{H}(L) \leq 1$, a process with $\overline{H}(L) = 0$ can be interpreted as exhibiting low complexity, whereas $\overline{H}(L) = 1$ indicates maximum complexity.

B. Partially Ordered Permutation Entropy

The permutation entropy is a measure based on ordinal patterns, which can be interpreted as being defined over an alphabet with a total order. Partially ordered permutation entropy extends this concept by allowing a partial order among the elements of the alphabet. This generalization was introduced by Haruna [5], and the results presented in this section are based on that work and related developments [5], [8], [9].

A characteristic of sequences defined over partially ordered alphabets is that multiple equivalent sequences can exist, unlike in the totally ordered case where each sequence has a unique representation. To address this ambiguity and enable the definition of entropy, Haruna [5] proposed a mapping that transforms sequences into a matrix representation of the POSET induced by the sequence. This map is defined in Eq. 6.

$$\phi^s_{\Sigma,L} : \Sigma^L \to \mathcal{M}_L(\{0,1\}) \tag{6}$$

The function $\phi^s_{\Sigma,L}$ maps each sequence $x_{k,L} \in \Sigma^L$ to a square matrix $M \in \mathcal{M}_L(\{0,1\})$ representing the partial order induced by the sequence. Here $x_{k,L}$ is defined as before (see Eq. 2), but now a partial order is assumed among the elements of the sequence $\mathbf{x} = (x_1, ..., x_L)$, where each x_i belongs to a finite alphabet Σ.

In this context, the matrix M encodes the equivalent of the permutation type $\pi(x_{k,L})$ under the partial order setting. Definition 5 introduces the entropy measure. Although two entropy variants are defined in [5], we shall focus on the one given below.

Definition 5: (Partially Ordered Permutation Entropy [5, p.3]) Let $\mathbf{X}$ be a stochastic process over a partially ordered finite alphabet Σ. For any $M \in \mathcal{M}_L(\{0,1\})$, the probably of its occurrence in $\mathbf{X}$ is given by

$$p_s(M) = \sum_{x_{k,L} \in \left(\phi^s_{\Sigma,L}\right)^{-1}(M)} p(x_{k,L}) \quad (7)$$

and the square partially ordered permutation entropy of $\mathbf{X}$ is given by

$$H_s(X) = -\sum_{M \in \mathcal{M}_L(\{0,1\})} p_s(M) \log_2 p_s(M) \quad (8)$$

It is important to note that the definition of probability in Eq. 7 does not require the alphabet to be totally ordered. When the alphabet is totally ordered, the definition of partially ordered permutation entropy coincides with that of standard permutation entropy.

In contrast to permutation entropy, no normalized version of partially ordered permutation entropy has been formally defined in the literature. This absence is primarily attributed to the combinatorial complexity involved in counting the number of distinct length-L sequences under a partial order.

C. Partially Commutative Monoid

Partially commutative monoids generalize free monoids by allowing certain pairs of symbols to commute[10]. Mazurkiewicz [11] used partially commutative monoids to analyze concurrent systems through sequential representations known as traces. In these representations, symbols corresponding to processes that can be executed in parallel are allowed to commute, meaning their order can be interchanged without affecting the system's behavior. In contrast, processes that cannot be executed concurrently are represented by non-commuting symbols, preserving their relative order in all equivalent traces.

A natural way to represent symbols and their commutativity relations is through a graph structure. The commutativity graph $G = (V, E)$ is a simple undirected graph where V is the set of vertices and E is the set of edges. Each vertex in V corresponds to a unique symbol from a finite alphabet Σ, via a bijective mapping. Therefore, without loss of generality, the sets V and Σ are treated interchangeably throughout this work.

If two symbols $x, y \in \Sigma$ commute, then the corresponding vertices $x, y \in V$ are connected by an edge $(x, y) \in E$ in the commutativity graph. This commutativity relation is denoted by $xy \equiv_G yx$, indicating that the xy is equivalent to yx with respect to the graph G. Conversely, if x and y do not commute, the relation is represented by $xy \not\equiv_G yx$.

In this context, two vertices are said to be adjacent if they are connected by an edge. The complement of the commutativity graph, denoted by $\overline{G} = (V, \overline{E})$, is called the non-commutativity graph. In this graph, an edge between two vertices indicates that the corresponding symbols do not commute.

Swapping the positions of consecutive symbols that commute results in equivalent words. Two words $\mathbf{u}, \mathbf{v} \in \Sigma^*$ are equivalent under the relation $\equiv_G$ if one can be obtained from the other by a finite sequence of such swaps, where each swap involves adjacent symbols that commute according to the commutativity graph G.

Definition 6 (Equivalence Class): Let $\mathcal{E}_G(\mathbf{u})$ be the set of words equivalent to a word $\mathbf{u} \in \Sigma^n$, according to the relation $\equiv_G$. The set $\mathcal{E}_G(\mathbf{u})$ is called the equivalence class of $\mathbf{u}$.

The partially commutative monoid $\mathcal{M}(\Sigma, G)$ is the set of all equivalence classes defined by the alphabet Σ and the commutativity relations represented by the commutativity graph G. If two words belong to the same equivalence class, they are said to be congruent.

The value $\tau_G(n)$ represents the total number of distinct equivalence classes $\mathcal{E}(\mathbf{u}_i)$ of words $\mathbf{u}_i, i = 1, 2, \ldots, \tau_G(n)$ of length of n, where no two words in the same class are congruent under the relation $\equiv_G$. We can then define

$$\mathcal{M}^n(\Sigma, G) = \mathcal{E}_G(\mathbf{u}_1) \cup \mathcal{E}_G(\mathbf{u}_2) \cup \cdots \cup \mathcal{E}_G(\mathbf{u}_{\tau_G(n)}) \quad (9)$$

the subset of the monoid $\mathcal{M}(\Sigma, G)$ consisting of these equivalence classes formed exclusively by words of length n and where $\mathcal{E}_G(\mathbf{u}_i) \cap \mathcal{E}_G(\mathbf{u}_j) = \emptyset$ for $i \neq j$, $|\mathbf{u}_i| = n, i = 1, 2, \ldots, \tau_G(n)$ and $\tau_G(n) = |\mathcal{M}^n(\Sigma, G)|$.

Fisher [12] has developed methods for determining the number $\tau_G(n)$ of equivalence classes of length n. The main tool for determining $\tau_G(n)$ is the dependence polynomial of a commutativity graph G [13].

Definition 7: (Dependence Polynomial [13]) The dependence polynomial of the commutativity graph G is defined by

$$D(G, z) = \sum_{k=0}^{\omega} (-1)^k c_k z^k, \quad (10)$$

where c_k denotes the number cliques of size k in the graph G and ω is the clique number of G.

It should be emphasized that computing the dependence polynomial $D(G, z)$ is NP-complete, as this task depends on determining the coefficients c_k, which is itself an NP-complete problem. From the dependence polynomial, it is possible to compute the value of $\tau_G(n)$ by using

$$\frac{1}{D(G, z)} = \sum_{n=0}^{\infty} \tau_G(n) z^n. \quad (11)$$

The $1/D(G, z)$ polynomial captures the growth behavior of the number of non-equivalent words and provides a compact representation for $\tau_G(n)$ across different values of n.

The concept of heap of pieces, introduced by Viennot [14], provides a combinatorial and geometric representation of words in partially commutative monoids. In this model, a heap is a configuration of labeled pieces stacked according to certain constraints that reflect the non-commutativity between symbols: two pieces can be stacked independently if their corresponding labels commute, otherwise one must be placed above the other. This construction naturally gives rise to a POSET, where the elements correspond to the pieces and the order relation reflects the required precedence imposed by non-commutativity.

Definition 8 (Heap of Pieces [14]): Let $(\Sigma, \preceq)$ be a POSET and let $G = (V, E)$ be a commutativity graph. A labeled heap of pieces is a triple $(\Sigma, \preceq, \lambda)$, where $(\Sigma, \preceq)$ is the POSET and λ is a function that maps Σ to the elements of V, such that

1) For every $x, y \in \Sigma$ such that $\lambda(x)\lambda(y) \not\equiv_G \lambda(y)\lambda(x)$, it holds that either $x \preceq y$ or $y \preceq x$; and
2) The relation $\preceq$ is the transitive closure of the relations defined in item 1.

In this way, each word can be naturally associated with a POSET derived from its heap representation. A characteristic of equivalence classes is that all elements of the same class have the same POSET representation [14, Lemma 4]. In the next section, this correspondence will be leveraged to reformulate the partially ordered permutation entropy in the framework of partially commutative monoids. By exploiting the combinatorial structure and algebraic properties of these monoids, new insights and results concerning the entropy of sequences with partial order constraints will be established.

III. Rewriting partially ordered permutation entropy

The fundamental principle that allows the definition of partially ordered permutation entropy on a subsequence $x_{k,L}$ is the use of the mapping $\phi^s_{\Sigma,L}$ (see Eq. 6), which associates the subsequence with a matrix representing the corresponding POSET. This matrix encodes the order relations among the elements of the subsequence. As a result, all subsequences that are mapped to the same matrix by $\phi^s_{\Sigma,L}$ are considered equivalent and belong to the same partition class within the sequence space.

Similarly, the elements of an equivalence class are mapped to the same POSET when the sequence is represented as a heap of pieces $(\Sigma, \preceq, \lambda)$. Therefore, it becomes evident that the mapping function $\phi^s_{\Sigma,L}$ effectively captures this structure by assigning each sequence to the POSET associated with its corresponding heap of pieces representation. Therefore, the subsequence $x_{k,L} = \{x_k, x_{k+1}, \ldots, x_{k+L-1}\}$ defined over an alphabet Σ equipped with a partial order, can be interpreted as a word in the partially commutative monoid $\mathcal{M}(\Sigma, G)$, where G encodes the commutativity relations between x_i and x_j.

This interpretation provides a natural framework for incorporating partial order constraints into the symbolic representation of sequences. In this way, the equivalence classes in the partially commutative monoid naturally generalize the concept of ordinal patterns, being equivalent to the image of the map $\phi^s_{\Sigma,L}$, which associates each sequence to its corresponding poset structure.

Now that it is established that the equivalence classes in the partially commutative monoid $\mathcal{M}(\Sigma, G)$ generalize ordinal patterns, it becomes possible to determine the number of distinct classes of a given length L using Eq. 11. Since $D(G, z)$ depends directly on the commutativity graph G, the topological and combinatorial properties of G play a fundamental role in determining the combinatorial complexity of the system.

Furthermore, the definition of partially ordered permutation entropy can be reformulated using the notation of partially commutative monoids. In this context, each subsequence $x_{k,L}$ interpreted as a representative of an equivalence class $\mathcal{E}_i$ in the monoid $\mathcal{M}(\Sigma, G)$. In this way we rewrite Eq. 7 and Eq. 8 as

$$p_G(\mathcal{E}_i) = \sum_{x_{k,L} \in \mathcal{E}_i} p(x_{k,L}) \tag{12}$$

and

$$H_G(L) = -\sum_{i=1}^{\tau_G(L)} p_G(\mathcal{E}_i) \log_2 p_G(\mathcal{E}_i). \tag{13}$$

The entropy $H_G(L)$ is then computed over the distribution of these equivalence classes, rather than over individual ordinal patterns, capturing the combinatorial structure imposed by the partial commutativity of the alphabet.

As expected, the minimum value of $H_G(L)$ is zero, obtained when the sequence is completely deterministic. However, unlike the definition in Eq. 8, the current formulation allows for the explicit computation of the maximum value of $H_G(L)$, which is achieved when there is a uniform distribution over the equivalence classes, i.e.,

$$p_G(\mathcal{E}_i) = \frac{1}{\tau_G(L)} \tag{14}$$

Using Eq. 14 and Eq. 13, we can write that the maximum entropy is given by

$$\begin{aligned} H_{G_{max}}(L) &= -\sum_{i=1}^{\tau_G(L)} \frac{1}{\tau_G(L)} \log_2 \frac{1}{\tau_G(L)} && (15)\\ &= \log_2 \tau_G(L), && (16) \end{aligned}$$

where $\tau_G(L)$ denotes the number of equivalence classes of words of length L in the partially commutative monoid $\mathcal{M}(\Sigma, G)$.

Now that it is possible to determine the total number of distinct classes $\tau_G(L)$ we can define the normalized entropy as

$$\overline{H}_G(L) = \frac{H_G(L)}{\log_2 \tau_G(L)}. \tag{17}$$

This normalization ensures that $0 \leq \overline{H}_G(L) \leq 1$, allowing the values of to be interpreted as a relative measure of the combinatorial complexity of the process.

Example 1 (Total Order): For the case of an alphabet with total order, the commutativity graph has no edges, so $c_1 = |\Sigma|$, $c_2 = c_3 = ... = 0$. Then the dependence polynomial will be $D(G, z) = 1 - |\Sigma|z$ and $\tau_G(L) = |\Sigma|^L$.

Example 2 (Complete Partial Order): If all the symbols in the alphabet commute, the commutativity graph G is called complete, meaning that every pair of distinct vertices is connected by an edge. In this case, all symbols can be freely reordered, and the partially commutative monoid becomes isomorphic to the free commutative monoid over Σ. For a complete graph with the dependence polynomial is

$$D(G, z) = \sum_{k=0}^{|\Sigma|} (-1)^k \binom{|\Sigma|}{k} z^k = \frac{1}{(1-z)^m} \tag{18}$$

and $\tau_G(L)$ is

$$\tau_G(L) = \binom{|\Sigma| + L - 1}{L}, \tag{19}$$

corresponding to the number of multisets of size $|L|$ over an $|\Sigma|$-element set.

IV. Conclusions

In this work, we revisited the definition of partially ordered permutation entropy by interpreting it through the algebraic framework of partially commutative monoids. By establishing a formal correspondence between the ordinal patterns induced by partial order and the equivalence classes in the monoid $\mathcal{M}(\Sigma, G)$, we provided a new structural understanding of the entropy associated with such patterns.

This interpretation enabled the explicit computation of the number of equivalence classes of a given length, which was not directly addressed in previous definitions. As a result, we introduced a normalized version of partially ordered permutation entropy, overcoming a known limitation in the literature related to the absence of normalization due to the difficulty of enumerating distinct patterns under partial order.

These contributions provide a solid combinatorial foundation for the analysis of systems with partial order and open new directions for applying permutation-based complexity measures in domains where total order assumptions are not suitable. Future work may explore computational optimizations for counting classes and investigate applications of the normalized entropy in real-world systems involving concurrency or symbolic representations with inherent partial order.

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