A Time-Varying Approach for the Blind Source Separation of Linear Quadratic Mixtures

Alexandre Miccheleti Lucena, Kenji Nose-Filho, Ricardo Suyama

Abstract—This paper evaluates nonlinear blind source separation (NLBSS) methods by interpreting the linear-quadratic (LQ) model over a time-varying local linear approximation. Two complementary approaches are explored: decomposing LQ mixtures via local linear approximations and explicitly reformulating them as time-varying system. Simulations validate the method under three scenarios: single-trial separation, Monte Carlo analysis with random matrices, and robustness tests across nonlinearity levels. Results highlight the N-EASI-R algorithm's superior performance for all quadratic terms tested. The study proposes whether combining both strategies could enhance robustness, potentially bridging theory and practical BSS applications.

Keywords—Blind source separation, linear quadratic, nonlinear mixtures, independent component analysis, nonlinear regression.

I. INTRODUCTION

Source separation is an important challenge in signal processing, involving the recovery of individual signals from mixed observations without prior knowledge of the mixing process. It arises in many contexts, when sensors capture overlapping signals due to physical constraints (e.g., microphone arrays in noisy environments [1]) or inherent signal interactions (e.g., spectral overlap in hyperspectral imaging [2]). The lack of mixing process information, whether due to sensor limitations or dynamic environments, motivates the development of robust BSS algorithms adaptable to diverse real-world conditions.

The classical approach to BSS assumes that the observed signals are linear mixtures, allowing separation methods to rely on solutions that leverage statistical independence, time correlation, or spectral properties [3]. However, many real-world systems exhibit nonlinear mixing behavior, leading to ineffective linear approaches. Due to the diversity of nonlinearities, no universal model exists, where tailored solutions for specific mixing characteristics become necessary.

There are cases where, although the standard independence criteria may fail, they can still be used in constrained nonlinear models such as the Linear-Quadratic (LQ) model [4]. It can be shown that, despite the nonlinear mixing process, the statistical independence of the sources alone remains sufficient for separation under certain conditions. It has practical significance in applications where signal interactions exhibit

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mild nonlinearities, such as scanned document processing [5] and chemical sensing [6].

While the independence criterion remains applicable to LQ, alternatives continue to be explored. A notable advancement in nonlinear BSS was proposed by Ehsandoust et al. (2017) [7], who demonstrated that time-invariant nonlinear mixtures can be reformulated as linear time-varying models through a local approximation, enabling the usage of adaptive separation algorithms (Fig. 1). Their work introduced an algorithm with promising results for certain nonlinear mappings, though the LQ case was not explicitly tested. Building on this, another work further investigated the framework, reporting improved performance under specific conditions. Simulations showed that the usage of a nonlinear regression algorithm can improve the source estimation even when applied without the local approximation when the model can be framed as time-varying [8].

These works suggest that the LQ model could be evaluated from a time-varying mixture perspective, offering new insights into its separation conditions. As a nonlinear model, the LQ mixture may benefit from two distinct yet complementary approaches: leveraging local linear approximations to decompose the mixture, and explicitly reformulating the LQ equations as a time-varying system. Each approach relies on different assumptions, as local approximation methods may require smooth nonlinearities, while the time-varying interpretation could exploit temporal structure in the mixing process and potentially lead to distinct separation strategies.

This work investigates these assumptions through simulated scenarios, assessing their effectiveness in separating LQ mixtures and is organized as follows: In Section II, we provide an overview of the nonlinear source separation problem formulation and its interpretation as a time-varying local linear approximation, presenting the algorithms studied and applied under this framework. Section III details the LQ model formulation, emphasizing its interpretation as a time-varying linear system. Section IV presents simulation results

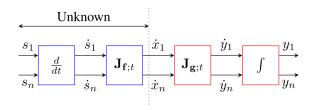


Fig. 1: Transforming the nonlinear BSS problem model to the linear time-variant one [7].

across three scenarios: (i) a single LQ separation trial, (ii) a Monte Carlo analysis with random mixing matrices, and (iii) an evaluation of algorithm robustness under varying nonlinearity levels (controlled by quadratic terms), and finally Section V presents the concluding remarks.

II. NONLINEAR BLIND SOURCE SEPARATION

The general Nonlinear Blind Source Separation (NLBSS) model can be expressed as

$$\mathbf{x}(t) = \mathbf{f}(\mathbf{s}(t)),\tag{1}$$

where $\mathbf{s}(t) = [s_1(t), ..., s_N(t)]^{\top}$ is the set of N observed source signals, $\mathbf{f}(.)$ is a nonlinear mapping, and $\mathbf{x}(t) = [x_1(t), ..., x_M(t)]^{\top}$ is resulting M observations of the nonlinear mixtures [3]. In an ideal scenario, separation involves finding $\mathbf{g}(.) = \mathbf{f}^{-1}(.)$ to recover sources estimates $\mathbf{y}(t)$ as

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t)). \tag{2}$$

Challenges arise due to the diversity of nonlinear functions, making a universal separation algorithm difficult.

A possible approach is to approximate the nonlinear model (1) as a time-varying linear model [7]. Under the assumption of time invariance in $\mathbf{f}(.)$, the mixing process reduces to a constant nonlinear mapping between sources and observed mixtures. If sources are time-varying, it is possible to consider that the nonlinear characteristics observed in the mixture are related to how the sources are being affected by different regions of the nonlinear mapping. Assuming $\mathbf{f}(.)$ is smooth and time invariant, the nonlinear model can be reinterpreted as [7]:

$$\dot{\mathbf{x}} = \mathbf{J}_{\mathbf{f};t}(\mathbf{s})\dot{\mathbf{s}},\tag{3}$$

where $\mathbf{J}_{\mathbf{f};t}(\mathbf{s})$ is the Jacobian matrix, with $\dot{\mathbf{x}}$ and $\dot{\mathbf{s}}$ as time derivatives, and separation as

$$\dot{\mathbf{y}} = \mathbf{J}_{\mathbf{g};t}(\mathbf{x})\dot{\mathbf{x}},\tag{4}$$

where $\mathbf{J}_{\mathbf{g};t}$ is the separation function Jacobian. For correct signal recovery, the following assumption must hold [7]: (i) invertibility of \mathbf{f} with time-invariance and memorylessness otherwise the Jacobian would also vary; (ii) first-order differentiability of \mathbf{f} and \mathbf{x} with continuous derivatives; and (iii) Independent Component Analysis (ICA) constraints [3] (source-observation dimension matching, independent derivatives, and non-Gaussianity of all but one derivative).

A. Equivariant Adaptive Separation via Independence

The interpretation of the nonlinear model as time-varying linear demands an adaptive approach. N-EASI (Normalized Equivariant Adaptive Separation via Independence) algorithm [9], is an adaptive BSS method that leverages source statistical independence, to estimate the time-varying separation matrix for linear mixtures. The algorithm employs adaptive serial updates for the separation matrix $\mathbf{W}(t)$, given as

$$\mathbf{W}(t+1) = \mathbf{W}(t) - \lambda_t \left[\frac{\mathbf{y}(t)\mathbf{y}(t)^{\dagger} - \mathbf{I}}{1 + \lambda_t \mathbf{y}(t)^{\dagger} \mathbf{y}(t)} + \frac{\mathbf{h}(\mathbf{y}(t))\mathbf{y}(t)^{\dagger} - \mathbf{y}(t)\mathbf{h}(\mathbf{y}(t))^{\dagger}}{1 + \lambda_t |\mathbf{y}(t)^{\dagger} \mathbf{h}(\mathbf{y}(t))|} \right] \mathbf{W}(t) \quad (5)$$

where λ_t is a sequence of positive adaptation steps and $\mathbf{h}(\cdot)$ is an arbitrary component-wise nonlinear function.

The N-EASI algorithm is used throughout this work as the main adaptive ICA algorithm for the implementation of all algorithms, since they rely on an adaptive ICA method.

B. Adaptive Algorithm for Time-Variant Linear mixtures

Following the proposed linear time-varying model (Fig. 1), a first approach to the problem is to apply an adaptive separation algorithm over the observed mixture signal derivatives to estimate $\mathbf{J}_{\mathbf{g};t}$. The estimated Jacobians are used to separate the mixed signal derivatives, leading to source derivative estimates, where a final integration step is necessary to recover the sources. The described procedure is called the Adaptive Algorithm for Time-Variant Linear (AATVL) mixtures [7] and the algorithm steps are illustrated in Fig. 2.



Fig. 2: AATVL algorithm block diagram.

AATVL assumes that the ICA algorithm will be able to follow $J_{g;t}$ variations to properly separate the sources.

C. Batch Algorithm for Time-Invariant Nonlinear mixtures

Although AATVL can be seen as an immediate approach to this model, it does not fully exploit the time-invariance and smoothness of the mixing function. As the ICA algorithm iterates, it is expected that new estimates of $\mathbf{J}_{\mathbf{g};t}$ could be used to estimate the overall mapping. This means that previous estimated $\mathbf{J}_{\mathbf{g};t}$ may benefit of a regression step to approximate the nonlinear function. In this sense the Batch algorithm for time-invariant (BATIN) mixtures adds a nonlinear regression step after the adaptive separation algorithm to refine $\mathbf{J}_{\mathbf{g};t}$ estimates. The BATIN algorithm is illustrated in Fig. 3.

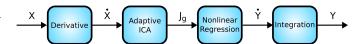


Fig. 3: BATIN algorithm block diagram.

Considering results presented in previous work [8], the General Regression Neural Networks (GRNN) [10] is chosen for the nonlinear regression step. In the context of the BATIN algorithm, the GRNN formulation can be written as

$$[\mathbf{J}_{\mathbf{g}}(\mathbf{x})]_{ij} = \frac{\sum_{k=1}^{N} [\mathbf{J}_{\mathbf{g};k}]_{ij} K(\mathbf{x}, \mathbf{x}_k)}{\sum_{k=1}^{N} K(\mathbf{x}, \mathbf{x}_k)},$$
 (6)

where, $[\mathbf{J}_{\mathbf{g};k}]_{ij}$ is the ij coefficient of the separating Jacobian matrix estimates, $K(\mathbf{x}, \mathbf{x}_k)$ is a radial basis function kernel.

D. N-EASI with a regression step

The adaptive characteristic of the N-EASI algorithm makes it applicable in a time-varying scenario assuming its capability to adapt to a new sample will be able to track the time-varying separation matrix, as already used on the AATVL and BATIN algorithms.

Interesting results over the direct application of the N-EASI to nonlinear mixtures are shown in [8] when the model can be written as a time-varying linear mixture. Differently from the model on Fig. 1, if the nonlinear model can be seen directly as time-varying, the local approximation may be overlooked. Also, the fact that the nonlinear mapping depend on the sources reveals that a nonlinear regression step can also benefit the separation matrices estimates. The work in [8] applies a nonlinear regression step (the GRNN from equation (6)) to refine the separation matrix estimates from N-EASI, similar to the BATIN algorithm but without derivative and integration steps, improving source estimation in some cases.

As to be shown in the following section, the LQ model can be written as time-varying source-dependent and may benefit from this approach, justifying the application of a nonlinear regression step over the N-EASI estimates, and the resulting algorithm is referred as N-EASI-R.

III. THE LINEAR-QUADRATIC MODEL

The general form of LQ mixture model can support any number of sources and mixtures. For the specific case of two sources/two mixtures case, the observed signals can be expressed by the following equation system as [4]:

$$\begin{cases} x_1(t) = a_{11}s_1(t) + a_{12}s_2(t) + b_1s_1(t)s_2(t), \\ x_2(t) = a_{21}s_1(t) + a_{22}s_2(t) + b_2s_1(t)s_2(t). \end{cases}$$
(7)

The quadratic term can be interpreted as a third source, allowing the model (7) to be alternatively described through the following vector notation [11] as a linear mixing model as:

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t),\tag{8}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{bmatrix}, \quad \mathbf{s}(t) = \begin{bmatrix} s_1(t) \\ s_2(t) \\ s_1(t)s_2(t) \end{bmatrix}, \quad (9)$$

where $\bf A$ is the matrix with the coefficients defined in (7) and ${\bf s}(t)$ is an augmented source vector. One might rewrite the dynamics for the mixing matrices by incorporation the quadratic terms to the coefficients of the matrix, making it source dependent as in:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} + b_1 s_1(t) \\ a_{21} & a_{22} + b_2 s_1(t) \end{bmatrix} \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix}.$$
 (10)

From equation (10), although this might not be the only way to express it, it is possible to consider that the mixing matrix can be seen as time-varying, since it depends on the sources, that are also time-varying, but its relation to the sources remain linear as in:

$$\mathbf{x}(t) = \mathbf{A}_t(\mathbf{s})\mathbf{s}(t) . \tag{11}$$

This reinterpretation naturally leads to the question of whether adaptive algorithms can effectively recover sources under such conditions (i.e. N-EASI and N-EASI-R). While the LQ model fundamentally represents a nonlinear system, and the performance of local approximation methods in this context (i.e. AATVL and BATIN) remains underexplored in existing literature.

IV. SIMULATION RESULTS

To test the different approaches described in Section II on the separation task of the LQ mixing model, a simple simulation setup of two sources and two observations (N=M=2) was designed. The source data was generated synthetically according to the sine and triangle derivative models used in in [8] considering $\omega=1/100$, with 5000 samples each.

Regarding the LQ mixing process, three simulation scenarios were considered: the first one examining a single experiment with an arbitrary mixing matrix; a second scenario considering the average separation performance for randomly selected mixing matrices; and a third, examining the influence of the quadratic term over the algorithm performance.

Recovered source signals quality is calculated using the Signal-to-Interference Ratio (SIR) as performance measurement. In summary, the SIR evaluates the ratio between the energy of the target source s_i and the energy of the residual interference signal (i.e. difference between estimated and true signal) $e_i[k] = \hat{s}_i[k] - s_i[k]$ as in

$$SIR_i := 10 \log_{10} \frac{\sum_k s_i^2[k]}{\sum_k e_i^2[k]}$$
 (12)

with permutation, scale and offsets between source and estimates compensated before the calculations.

A. Scenario 1: Arbitrary Mixing Matrix

The first scenario was based on testing the separating algorithms over a single realization of the LQ mixture model. To evaluate this case, an arbitrary mixing matrix was employed to simulate the observed signals through the process specified in Equation (8), with the matrix

$$\mathbf{A} = \begin{bmatrix} 0.78 & 0.20 & 0.58 \\ -0.31 & 0.72 & 0.62 \end{bmatrix}. \tag{13}$$

where each row of the matrix has a unitary norm. Fig. 4 illustrate the sources and observed signals obtained.

Another way to assess the observed signals and the nonlinear effects caused by the LQ model is to visualize the source and mixture spaces. Fig. 5 illustrates the distortions caused by the LQ model to a regular grid on the [-1,+1] interval. The colors of the grid are chosen in a way to allow the viewer to perceive the quadratic characteristic over the observation that would be otherwise difficult to note in Fig. 4.

Given the observed signals, each algorithm was tested on the recovery of the sources and then had their SIR evaluated. Fig. 6 shows the estimated signals for each one of the tested algorithms. It is possible to note that for N-EASI and AATVL algorithms that don't rely on the nonlinear step had poor estimation for the first samples and consequently overall poorer performance. As for the BATIN algorithm it had a clear dificulty to estimate the second source, while the N-EASI-R had the best performance among the algorithms. The SIR values obtained for the N-EASI, AATVL, BATIN and N-EASI-R for $s_1(t)$ were, respectively, 11.9 dB, 9.9 dB, 20.2 dB, 21.0 dB and for $s_2(t)$, 5.7 dB, 2.2 dB, 8.9 dB, 8.8 dB compared to the observed signal SIR of SIR₁ = 13.9 dB and SIR₂ = 3.5 dB.

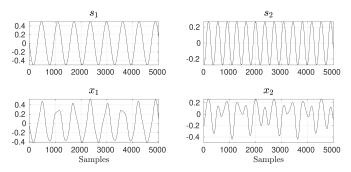


Fig. 4: The sources $s_1(t)$ and $s_2(t)$ on the top row, and the observations $x_1(t)$ and $x_2(t)$ for the simulation with the LQ model on the bottom

TABLE I: Mean SIR values with standard deviations obtained from 100 simulations (N=M=2).

Method	Source 1 (dB)	Source 2 (dB)
Mixed	-0.7 ± 6.4	-2.0 ± 3.8
N-EASI	9.6 ± 4.8	5.8 ± 3.5
AATVL	8.7 ± 4.2	1.9 ± 2.6
BATIN	14.2 ± 8.6	9.4 ± 7.6
N-EASI-R	15.9 ± 9.1	10.8 ± 8.4

B. Monte Carlo

The second scenario assesses the average algorithm performance across multiple Monte Carlo realizations with randomized mixing conditions. For each realization, the observed signals are generated using a mixing matrix with all coefficients drawn uniformly from [-1, +1], followed by rowwise normalization to ensure unit norm, but evaluating the algorithm in a less controlled environment. The mixed signals were then separated using each algorithm, and then evaluated with SIR values for each recovered source. Table I shows the results of mean SIR values (and standard deviation) obtained for each algorithm. Results show that the overall behavior observed for the single experiment remains over the Monte Carlo simulations. Although the overall mean SIR reduce drastically and present a high standard deviation as there is no guarantee that the draw matrix is solvable, for the simulated scenario, the algorithms that implement the nonlinear regression step presented a performance advantage, with a slight advantage for the N-EASI-R algorithm.

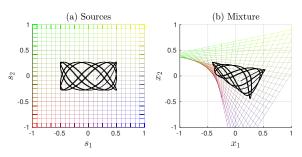


Fig. 5: Illustration of the effects of the LQ mixture model. Left figure (a) represents the sources and a grid in the domain $[-1,+1] \times [-1,+1]$, where right figure (b) represents the observed signals and the regular grid transformed by the LQ mixture model.

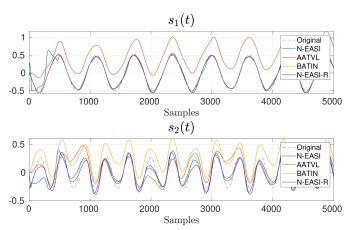


Fig. 6: Result comparison of the N-EASI, AATVL, BATIN and N-EASI-R algorithms for the LQ model.

C. Impacts of the quadratic term

In the LO mixing model, the magnitude of quadratic terms directly affects the system's nonlinearity: when these terms approach zero, the model reduces to a linear mixing process, whereas larger coefficients amplify nonlinear characteristics. To evaluate how quadratic terms influence both the mixing dynamics and source recovery performance, a simulation scenario was designed under the N=M=2 configuration. The linear part of the mixing matrix was first randomly obtained (uniform coefficients between [-1, +1]) as the previous scenario, after which the quadratic coefficients for each source were independently varied within the range [-1, +1] creating a grid of the quadratic term combinations. For each point on this grid, the SIR of the recovered signals was computed to quantify separation accuracy. This approach not only isolates the contribution of quadratic terms to the mixing nonlinearity but also benchmarks the robustness of separation algorithms across a range of nonlinear conditions.

The impact of quadratic terms on separation performance is visualized in Fig. 7, which presents a surface plot of mean SIR values (calculated as $(SIR_1 + SIR_2)/2$) for different quadratic coefficients. Each point on the surface represents the average of 50 Monte Carlo simulations, ensuring statistical reliability by accounting for variability in the linear mixing matrix initialization and stochastic algorithm behavior.

It is possible to note from Fig. 7, that as expected, all algorithm have a decrease in performance as any of the nonlinear coefficients are closer to one, meaning all of them presented peak performance as the mixture becomes linear $(b_1 = b_2 = 0)$. However the overall behavior of each algorithm changes, as the peaks and decays around zero are different. The AATVL and N-EASI algorithm once again presented overall poorer results, however the N-EASI presented more reliable results when compared to the irregular surface of the AATVL. The best performance across the quadratic terms variation were for the BATIN and N-EASI-R, that include the nonlinear regression step, although besides a higher peak for the near linear condition for the BATIN, its result are comparable to the performance of the N-EASI algorithm (without regression). The

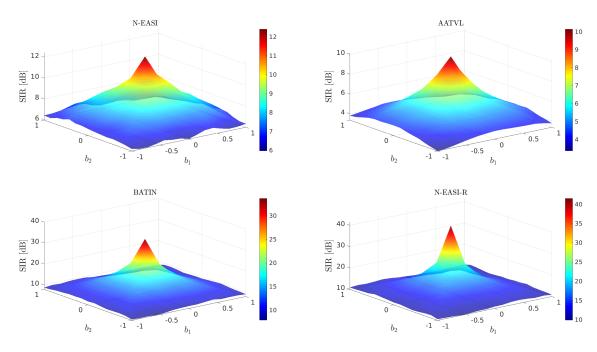


Fig. 7: Mean SIR values in decibels of the recovered sources for the LQ model over 50 simulations of the N-EASI (top-lef), AATVL (top-right), BATIN (bottom-left) and N-EASI-R (bottom-right) algorithms in a two-source-two-mixture scenario (N = M = 2).

highlights are for the performance of the N-EASI-R algorithm that presented better results for all the tested quadratic terms tested. Not only the peaks for the near linear conditions are higher, but also the minima region in dark blue is above 10 dB, which means its performance is above any other algorithm minima. In all scenarios, the combination of an adaptive algorithm combined to a nonlinear regression based on the interpretation of the LQ model as time-varying presented best performance in terms of measured SIR.

V. CONCLUSIONS

This work evaluated blind source separation methods for LQ mixtures under a time-varying local linear approximation framework. Among the considered algorithm, the N-EASI-R consistently demonstrated superior performance, showing robust and accurate source recovery even under high levels of nonlinearity. Results suggest that combining adaptive ICA techniques with nonlinear regression provides a promising approach for enhancing BSS performance in nonlinear settings. Future research may focus on hybrid strategies that integrate both local linear approximations and time-varying modeling to increase algorithm robustness and extend applicability to more complex real-world scenarios.

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