Robust ℓ_p Deconvolution for Sparse Reflectivity Estimation in Impulsive Noise

Pedro A. O. Santos, Tales V. R. O. Câmara, Kenji Nose-Filho, Tiago T. L. Barros, Luiz F. Q. Silveira

***Abstract*— Studies have shown that seismic data may contain non-Gaussian noise in certain geological environments, thereby compromising the validity of the Gaussian assumption commonly adopted by classical signal processing methods. This work presents a robust deconvolution method for seismic data affected by impulsive noise, modeled using α-stable distributions. The method relies on the minimization of the ℓ_p quasi-norm, which offers increased flexibility in error modeling and improved robustness to outliers when compared to conventional ℓ_1- or ℓ_2-norm-based techniques. The seismic signal is modeled using a classical convolutional formulation, and an ℓ_1-norm regularization term is incorporated into the cost function to promote sparse reflectivity. The resulting optimization problem is solved iteratively via a gradient descent algorithm, and its effectiveness is evaluated using synthetic seismic traces contaminated with α-stable noise. The results demonstrate that an appropriate choice of the parameter p can overcome limitations of traditional methods and enhance the quality of reflectivity estimation in impulsive noise scenarios.**

***Keywords*— Seismic signals deconvolution, Reflectivity estimation, ℓ_p Objective function minimization, Non-Gaussian noise, α-stable Distribution.**

I. INTRODUCTION

The adequate estimation of subsurface reflectivity from seismic data is crucial in oil and gas exploration applications and for the geological characterization of subsurface formations. An approach based on signal deconvolution can be used to estimate this reflectivity from the acquired signals, since such signals can be modeled as a convolution between the impulsive response of the medium, which characterizes the desired reflectivity, and the signal from the seismic source. The quality of this estimation is strongly associated with the processing methods' ability to deal adequately with the noise present in the data.

Traditionally, deconvolution has been formulated as an estimation problem based on the minimization of the mean squared error (ℓ_2-norm), assuming that the noise in the data is Gaussian [1]. However, in some complex geological environments, there may be contamination in the seismic data by impulsive non-Gaussian noise (*spikes*) during seismic acquisition, significantly compromising the validity of the Gaussian hypothesis [2], [3]. Using the ℓ_2-norm in these situations becomes unfeasible due to its sensitivity to *outliers*. Non-Gaussian noise directly affects the seismic resolution and may mask real reflectivity events.

An effective way to model the presence of non-Gaussian noise in seismic data is through α-stable distributions. These distributions generalize the Gaussian distribution, allowing heavier tails that adequately capture the statistical behavior of data that deviates from the Gaussian hypothesis [4]. In particular, experimental results have corroborated the suitability of α-stable distributions to model the statistical behavior of impulsive noise in seismic data [2].

In seismic reflectivity estimation problems, the choice of the cost function used to measure the data adjustment error (error between data and model) is a determining aspect in the method's robustness against noise. Although robust approaches based on the minimization of the ℓ_1-norm are recognized for their greater robustness against *outliers* when compared to the ℓ_2-norm [5], in more severe scenarios, characterized by α-stable noise with $\alpha < 1$, the minimization criterion based on the ℓ_1-norm loses statistical support, since the moment of order one of the adjustment errors (residuals) becomes undefined in this scenario [4]. In this context, cost functions associated with the quasi-norm ℓ_p with $p < 1$ become an attractive alternative, providing greater tolerance to large residuals and favoring the robustness of reflectivity estimation in more impulsive environments [4]. This work proposes applying this strategy for seismic deconvolution, investigating its performance in data contaminated by α-stable noise, a scenario of practical interest and still little explored in the specific literature of the area.

Furthermore, considering that seismic reflectivity is assumed to be sparse, this work adopts a regularization term based on the ℓ_1-norm to favor more adequate sparse solutions. The combination of a robust cost function, suitable for environments with impulsive noise, with an ℓ_1-norm-based regularization that promotes sparsity, configures a desirable strategy for the problem of seismic deconvolution in environments with impulsive noise modeled by α-stable distributions. Additionally, this work investigates the impact of the parameter p on the robustness of methods based on the ℓ_p quasi-norm, considering different levels of noise impulsivity. This analysis seeks to fill a gap in the literature, where the adjustment of p is typically performed in a fixed or empirical manner, without considering the statistical characteristics of the noise in the data.

II. SEISMIC DATA MODELING

In reflection seismic, the recorded signals can be described by the convolutional model, that considering the representation

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in discrete time, the seismic trace is given by:

$$x_n = w_n * r_n + v_n = \sum_{\ell=0}^{L} r_\ell w_{n-\ell} + v_n, \tag{1}$$

where n represents the discrete time index, w_n represents the signal emitted by the seismic source, also known as the seismic wavelet, r_n is the reflectivity function of the subsurface, and v_n corresponds to additive noise, which models environmental disturbances and uncertainties in data acquisition.

This model can also be expressed in matrix form as follows:

$$\mathbf{x} = \mathbf{W}\mathbf{r} + \mathbf{v}, \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^N$ denotes the observed seismic trace, modeled as a column vector of length N, $\mathbf{r} \in \mathbb{R}^N$ denotes the subsurface reflectivity, whose entries r_i correspond to local impedance contrasts, $\mathbf{v} \in \mathbb{R}^N$ denote the additive noise vector affecting the observed seismic data, and $\mathbf{W} \in \mathbb{R}^{N\times N}$ denote the convolution matrix constructed from the source wavelet w_n, given by:

$$\mathbf{W} = \begin{bmatrix} w_0 & 0 & 0 & \cdots & 0 \\ w_1 & w_0 & 0 & \cdots & 0 \\ w_2 & w_1 & w_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & w_0 \end{bmatrix}_{N\times N}.$$

An essential aspect in this context is the proper treatment of the noise vector $\mathbf{v}$. Traditionally, it is assumed that the noise can be modeled by a Gaussian distribution, an assumption justified by the central limit theorem and valid in several practical scenarios.

However, the seismic data acquisition can take place both onshore and offshore, and in either setting, the recorded data are often contaminated by non-Gaussian noise. There is growing evidence that the seismic environment may deviate from Gaussian behaviour [2], particularly in marine settings [6]. In such cases, more flexible models are preferable, such as the generalized Gaussian distribution or the α-stable distribution, which includes the Gaussian as a particular case. The latter model will be discussed in more detail below.

III. Alpha-Stable Distributions

α-stable distributions are often used in statistical modeling of non-Gaussian signal sources. Their theoretical foundation lies in the generalized central limit theorem and the stability property, which make them applicable in a wide range of scenarios. The generalized central limit theorem states that, under suitable normalization, the sum of a large number of independent and identically distributed (i.i.d.) random variables, with or without finite variance, converges in distribution to an α-stable law. The second defining feature is the stability property: α-stable distributions are closed under convolution, i.e., the sum of two independent random variables with the same characteristic exponent is also α-stable and preserves that exponent [4].

There are different parametrizations of α-stable distributions, each based on a distinct form of the characteristic function. In this work, it is considered the parameterization $\boldsymbol{\theta}_\alpha = (\alpha, \beta, \gamma, \delta)$ and the following characteristic function [4]:

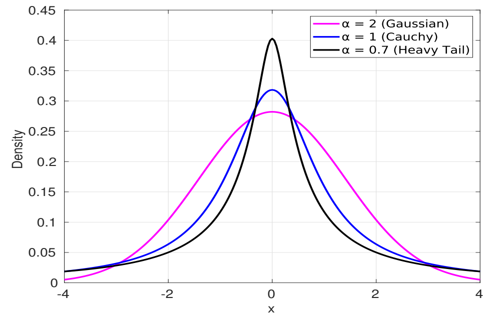


Fig. 1: Probability distribution function of symmetrical α-stable with $\beta = \delta = 0$ and $\gamma = 1$.

$$\varphi(\omega; \boldsymbol{\theta}_\alpha) = \exp(-\gamma^\alpha |\omega|^\alpha [1 - j\Theta(\omega; \alpha, \beta)] + j\delta\omega), \tag{3}$$

with

$$\Theta = \begin{cases} \beta(\tan \frac{\pi\alpha}{2})(\text{sign } \omega), & \alpha \neq 1 \\ -\beta \frac{2}{\pi}(\ln |\omega|), & \alpha = 1, \end{cases} \tag{4}$$

where α is the *characteristic exponent* such that $0 < \alpha \leq 2$, β is the symmetry parameter such that $-1 \leq \beta \leq 1$, γ is the dispersion or scale parameter such that $\gamma > 0$, δ is the location parameter such that $-\infty < \delta < \infty$.

Figure 1 shows the variation of the α parameter versus the corresponding random variable that reflects the impulsiveness level of the distribution. Lower values of α indicate higher impulsiveness and more pronounced non-Gaussian behavior, while higher values suggest that the distribution is closer to the Gaussian case; specifically, $\alpha = 2$ corresponds to the Gaussian distribution. Another important property concerns the existence of statistical moments. A non-Gaussian α-stable random variable has finite moments of order p only when $0 \leq p < \alpha$, thus for a non-Gaussian α-stable random variable X, this property implies [7]:

$$E\{|X|^p\} = \infty, \text{ if } p \geq \alpha \tag{5}$$

and

$$E\{|X|^p\} < \infty, \text{ if } 0 \leq p < \alpha. \tag{6}$$

It is important to note that the Gaussian distribution is a particular case of an α-stable distribution with $\alpha = 2$. Unlike the general α-stable case, it has finite moments of all orders, i.e.,

$$E\{|X|^p\} < \infty, \text{ for all } p \geq 0. \tag{7}$$

This feature helps explain why certain statistical moments perform better than others as cost functions in specific environments, and highlights that the assumed probability distribution of the dataset is closely related to the choice of cost function.

In this work, we explore the parameterization of a cost function defined by the ℓ_p-norm in scenarios where $p < \alpha$, including values of p less than one, in order to demonstrate the versatility of this parameter in handling different types of noisy data, as will be illustrated in Section V.

IV. ℓ_p Norm Deconvolution

The ℓ_p-norm is a metric of size or distance, it generalizes the idea of length to spaces of arbitrary dimension and to functions, with a parameter p, which can take any positive value $1 \leq p < \infty$ [9], if it takes on a value in the interval $0 < p < 1$, this metric ceases to be a norm and becomes a quasi-norm [10], but it can still be used to describe statistical moments of lower fractional order.

The ℓ_p norm of any discrete-time signal $x[n]$ is given by the expression:

$$||x||_p = \left(\sum_{n=1}^{N} |x_n|^p \right)^{\frac{1}{p}} \tag{8}$$

This expression includes several norms used in other areas, for example when $p = 2$, it is called the Euclidean norm, often used in analytical geometry and as a cost function in various applications in Gaussian environments. In contrast, when $p = 1$, the ℓ_1-norm presents itself as a widely used cost function and exhibits robustness to various types of noise, including impulsive noise.

However, as noted above, for values of the α parameter of the $\alpha-$stable distribution smaller than one, it is possible to see that the ℓ_1-norm faces difficulties, and this is where the ℓ_p norm comes in, where the parameter p can even take on fractional values, including information on statistical moments of fractional order in the error modeling, and for this case when $p < 1$.

In the seismic context, from the convolution equation, we can define as a cost function the ℓ_p-norm of the error between the observed seismic trace $\mathbf{x}$ and the convolutional modeled seismic trace $\hat{\mathbf{x}}$. The objective is to minimize the error between the modeled data and the observed data, given by:

$$\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}. \tag{9}$$

The cost function for this problem can be written as:

$$J_0(\mathbf{r}) = \frac{1}{p}||\mathbf{e}||_p^p = \frac{1}{p}\sum_{n=0}^{N-1} |x_n - \hat{x}_n|^p. \tag{10}$$

However, since the deconvolution problem by estimation is poorly posed — that is, it can admit multiple solutions for the same observed seismic trace —, it becomes necessary to impose restrictions or incorporate a priori information about reflectivity, the parameter on which we want to optimize the model.

In this context, assuming that reflectivity is a sparse signal, whose amplitudes correspond to abrupt transitions between geological interfaces, it is possible to obtain more adequate solutions to the deconvolution problem through regularization, as done by [8].

In this case, the regularization factor can be defined by applying the norm ℓ_1 on the reflectivity vector weighted by the penalty factor λ which can adjust how sparse the estimated term will be. Thus, the regularized cost function is now defined as:

$$J(\mathbf{r}) = \frac{1}{p}||\mathbf{e}||_p^p + \lambda||\mathbf{r}||_1. \tag{11}$$

Once the cost function has been defined, the next step is to minimize it in relation to reflectivity. To do this, the gradient descent algorithm can be applied, which requires the calculation of the partial derivatives of the cost function for each reflectivity coefficient. Considering, for example, the i-th term:

$$\frac{\partial}{\partial r_i}J(\mathbf{r}) = -\sum_{n=0}^{N-1} w_{n-i}e_{p_n} + \lambda sgn(r_i), \tag{12}$$

where $e_{p_n} = |x_n - \hat{x}_n|^{p-1} sgn(x_n - \hat{x}_n)$, that can be combined to form a matrix representation of the cost function gradient:

$$\frac{\partial J(\mathbf{r})}{\partial \mathbf{r}} = -\mathbf{W}^T\mathbf{e_p} + \lambda sgn(\mathbf{r}). \tag{13}$$

In this way, at each iteration, the application of the gradient descent method allows the batch update of the entire reflectivity vector:

$$\mathbf{r}^{k+1} = \mathbf{r}^k - \mu(-\mathbf{W}^T\mathbf{e_p} + \lambda sgn(\mathbf{r}^k)). \tag{14}$$

Equation (14) seeks, therefore, to iteratively adjust the reflectivity coefficients, balancing the minimization of the reconstruction error with the imposition of sparsity, controlled by the parameter λ. Furthermore, the update step — also known as learning rate — is regulated by μ.

V. Results and discussion

In this section, we investigate the performance of the proposed deconvolution method, formulated as an optimization problem based on the minimization of the ℓ_p quasi-norm. The goal is to assess how well the method performs in estimating sparse reflectivity when the seismic trace is contaminated with impulsive noise modeled by α-stable distributions, and to analyze how its effectiveness varies with different values of the parameter p.

A. *Experiment 1*

The first experiment considers a reflectivity signal composed of a very sparse sequence with non-zero coefficients at specific positions, as shown in Fig. 2a. The synthetic seismic trace is generated via the convolution of this reflectivity signal with a Ricker wavelet of central frequency $f_0 = 25$ Hz and length $L = 51$ samples, which models the source signal (Fig. 2b). The additive noise generated from an α-stable distribution (with $\alpha = 0.8$, $\beta = 0 = \delta$ and $\gamma = 0.05$) is then added to the trace (Fig. 2c).

The deconvolution is carried out by solving the proposed optimization problem for three representative values of the parameter p: 2, 1, and 0.6, assuming that the wavelet is known. These values were selected to illustrate the behavior of the method under impulsive noise conditions. The corresponding estimated reflectivity results are visually compared to the original reflectivity in Figs. 3a–3c.

For $p = 2$ (i.e., the ℓ_2-norm) the method fails to estimate any reflector accurately and exhibits instability with oscillatory artifacts, highlighting the inadequacy of the ℓ_2-norm in impulsive noise environments. For $p = 1$, the estimated reflectivity is considerably improved and aligns with the true

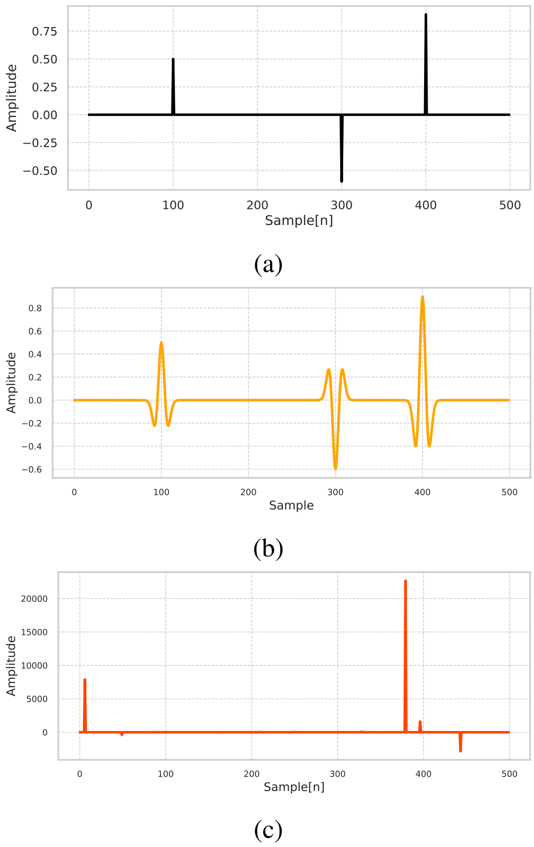


Fig. 2: (a) Reflectivity, (b) seismic trace, (c) and seismic trace added with α-stable noise with $\alpha = 0.8$, $\beta = 0 = \delta$ and $\gamma = 0.05$.

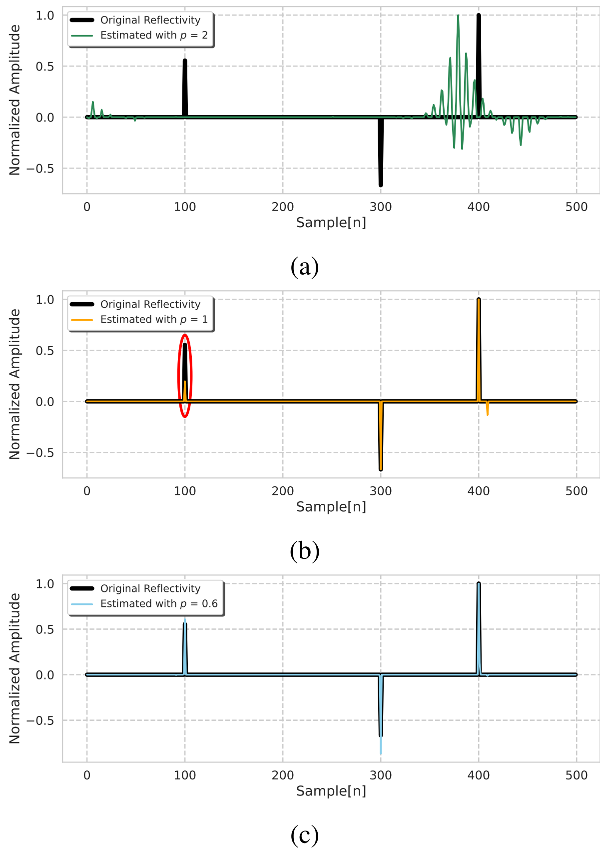


Fig. 3: (a) Estimated reflectivity for $p = 2$ ($\lambda = 5$), (b) estimated reflectivity for $p = 1$ ($\lambda = 5$), and estimated reflectivity for $p = 0.6$ ($\lambda = 8$). For all simulations were considered $n = 1000$, and $\mu = 0.01$.

signal, although one of the main reflectors, highlighted by the red ellipse, is underestimated in amplitude. In contrast, the result obtained with $p = 0.6$ provides the best match to the original reflectivity, accurately capturing both the location and amplitude of all reflectors. Notably, the reflector that was underestimated with the ℓ_1-based estimation is fully recovered when using $p = 0.6$.

B. Experiment 2

To further evaluate the proposed method, we consider a second synthetic reflectivity model representing a more complex subsurface, composed of multiple reflectors, some of which are closely spaced (Fig. 4a). In this simulation, the seismic trace was generated by convolving the reflectivity with a Ricker wavelet of central frequency $f_0 = 40$ Hz (Fig. 4b), followed by the addition of noise generated from an α-stable distribution with $\alpha = 0.6$ and $\gamma = 0.02$ (Fig. 4c). This scenario poses a significantly greater challenge for accurate reflectivity estimation.

As shown in Fig. 5, the result obtained with the ℓ_2 norm fails to identify any reflectors, once again demonstrating its sensitivity to impulsive noise. The estimation with the ℓ_1 norm is partially successful: approximately half of the reflectors are correctly localized, but only two are estimated with accurate amplitude. Additionally, several spurious peaks are detected, resulting in false positives.

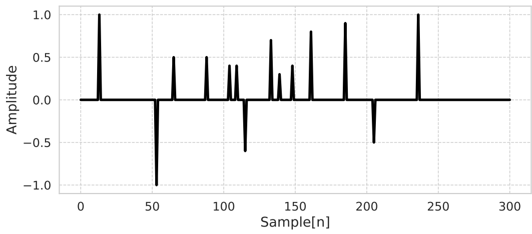
In contrast, the result for $p = 0.4$ achieves a considerably better reconstruction. Most of the true reflectors are correctly identified in both position and amplitude. Out of 14 actual reflection events, 11 are correctly estimated, and few false positives are observed.

C. Statistical Analysis

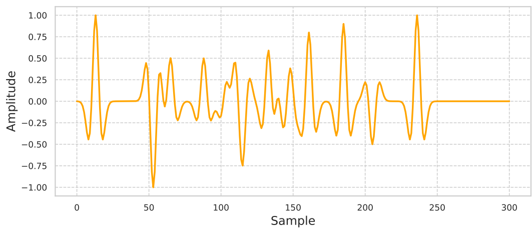
To show the robustness of this method, a statistical analysis (Monte Carlo simulations) was developed using the Pearson correlation coefficient ρ as a performance metric to compare the deconvolution of the seismic trace of Experiment 1 (illustrated by Fig. 2b), with 20 different realizations of the additive noise generated from an α-stable distribution, for different values of p in the interval $0.1 \leq p \leq 2$ (with a step 0.1). The box-plot of the Pearson correlation coefficient for each tested value of p is shown in Fig. 6. It can be observed that the values of p between 0.3 and 0.7 improve estimation consistency, with higher median correlation and lower dispersion. This behavior supports the hypothesis that ℓ_p quasi-norms with $p < 1$ offer advantages in impulsive noise scenarios.

VI. Conclusion

In this paper, was proposed the usage of the ℓ_p-norm, with $p < 1$, for the deconvolution of sparse signals in a non-Gaussian environment, modeled by the α-stable distribution.



(a)



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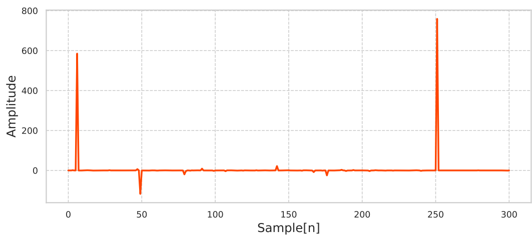

(c)

Fig. 4: (a) Reflectivity, (b) seismic trace, (c) and seismic trace added with α-stable noise with $\alpha = 0.6$, $\beta = 0 = \delta$ and $\gamma = 0.02$.

We have seen that, even when the impulsiveness of the additive noise is very high, the use of the ℓ_p-norm with $p < 1$ provides a better solution than the traditional ℓ_1-norm, which results in a gain in resolution.

In future works, we want to study the usage of the ℓ_p-norm for regularization and also explore it in blind deconvolution problems in which both the reflectivity series and the wavelet must be estimated.

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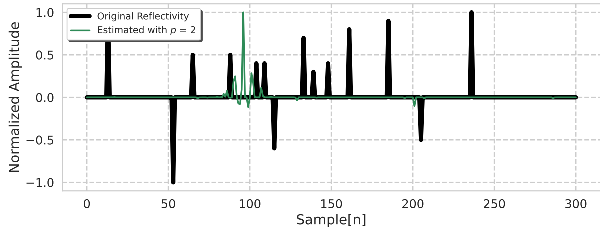
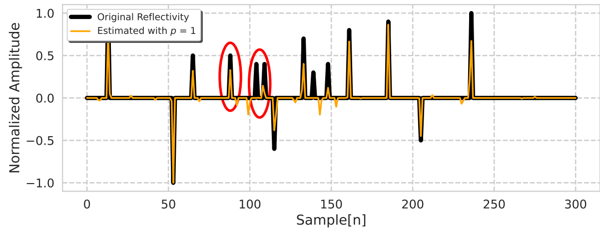
REFERENCES

[1] E. A. Robinson and S. Treitel, *Geophysical Signal Analysis*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1981.
[2] B. Yue and Z. Peng, “A validation study of α-stable distribution characteristic for seismic data,” *Signal Processing*, vol. 106, 2015.
[3] J. V. T. Lima, *Abordagens robustas para problemas inversos baseados em entropias generalizadas de Rényi, Tsallis e de Kaniadakis*. Dissertação de Mestrado, Programa de Pós-Graduação em Física, Universidade Federal do Rio Grande do Norte, 2022.
[4] Min Shao and Chrysostomos L. C.L. Nikias, “Signal Processing with Fractional Lower Order Moments: Stable Processes and Their Applications,” *Proceedings of the IEEE*, vol. 81, no. 7, pp. 986–1010, 1993.



(a)



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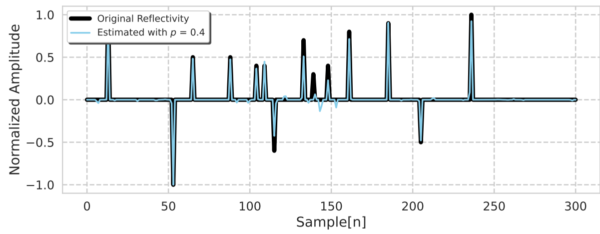

(c)

Fig. 5: (a) Estimated reflectivity for $p = 2$ ($\lambda = 6$, $\mu = 0.01$), (b) estimated reflectivity for $p = 1$ ($\lambda = 2$, $\mu = 0.01$), and estimated reflectivity for $p = 0.4$ ($\lambda = 4$, $\mu = 0.001$). For all simulations were considered $n = 2000$.

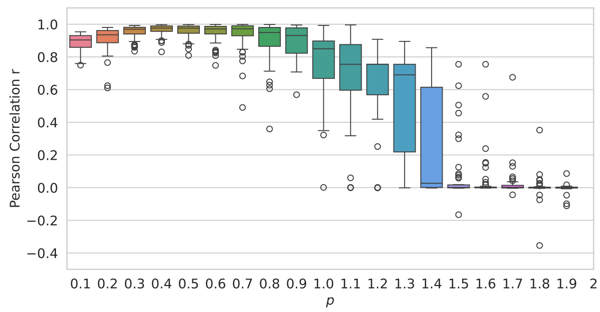

Fig. 6: Box-plot of the Pearson correlation coefficient for each value of p tested.

[5] J. F. Claerbout and F. Muir, “Robust modeling with erratic data,” *Geophysics*, vol. 38, no. 5, pp. 812–985, 1973.
[6] G. Song, X. Guo, and L. Ma, “The Alpha Stable Distribution in Ocean Ambient Noise Modelling,” in *MATEC Web of Conferences*, vol. 283, p. 08001, 2018.
[7] P. G. Georgiou, P. Tsakalides, and C. Kyriakakis, Alpha-stable modeling of noise and robust time-delay estimation in the presence of impulsive noise, *IEEE Trans. Multimedia*, vol. 1, no. 3, pp. 291–301, Sep. 1999.
[8] K. Nose-Filho, R. Lopes, R. D. B. Brotto, T. C. Senna, and J. M. T. Romano, *Algorithms for Sparse Multichannel Blind Deconvolution*, IEEE Transactions on Geoscience and Remote Sensing, vol. 61, 2023.
[9] W. Rudin, *Real and Complex Analysis*, 3rd ed. New York, NY, USA: McGraw-Hill, 1987.
[10] Anthony W. Knapp, *Basic Real Analysis: with an appendix “Elementary Complex Analysis”*. Digital Second Edition, New York, 2016.