The Log-μ Process

Godfred Kumi Tenkorang and Michel Daoud Yacoub

***Abstract*— This paper builds upon the idea of a medium nonlinearity to introduce a general application three-parameter statistical model - the Log-μ model, that presents simple analytical formulations. Its probability density function (PDF) and cumulative distribution function (CDF) appear in closed forms with support in $\mathbb{R}$ but limited in the left. Interestingly, for some particular combination of the parameters, a bimodality shows. Knowing that a physical model is behind the proposal and recognizing its potential for wireless communications applications, we specialize it so that its support now is $\mathbb{R}^+$. In addition, the paper demonstrates an application of the Log-μ fading model by analyzing the performance of the pure-selection combining diversity technique.**

***Keywords*— Channel's Nonlinearity; Fading Model; Pure-Selection Combining Diversity.**

I. Introduction

Statistical modeling plays a pivotal role in numerous scientific and engineering disciplines, enabling researchers and designers to capture, analyze, and predict the behavior of complex systems under uncertainty. Statistical models are widely employed to represent real-world phenomena, support decision making, and optimize system performance. In the field of wireless communications, statistical models are particularly essential for characterizing the behavior of the radio propagation channel, a component that exhibits highly variable and often unpredictable dynamics due to environmental and mobility-related factors.

Wireless communication is one of the fastest growing fields in the communications industry, driven by the continuous demand for higher data rates, lower latency, and greater connectivity. Key advancements shaping this evolution include fifth (5G) and sixth (6G) generation cellular communication systems [1], and the IEEE 802.11ax standard (Wi-Fi 6) [2], all of which significantly enhance network capacity, efficiency, and overall performance.

The received signal in a wireless system is significantly affected by channel impairments, namely shadowing and multipath. These phenomena can cause severe signal fading, which, in extreme cases, may render communication impractical. Consequently, accurate models that describe these radio channel effects are essential for developing reliable systems.

Numerous channel models have been studied in the literature to describe long-term and short-term fading, among worth mentioning Lognormal [3], [4], Rayleigh [5], Hoyt [6], Rice [7], Nakagami-m [8], Weibull [9], α-μ [10], κ-μ [11], η-μ [11], and α-η-κ-μ [12]. Certainly, the greater the number of fading parameters, the more flexible and adaptable the fading model can be to real-world propagation scenarios. However, this comes at the cost of greater mathematical complexity, which may limit its practical applicability.

As wireless communication shifts toward higher frequencies, including the mmWave and THz bands, the nonlinear effects of the propagation medium may become more pronounced [13]. The concept of nonlinearity was first hinted at in the α-μ model, where this is manifested through a power parameter affecting the modulus of fading components. This formulation has proven highly effective, offering a balance between flexibility and mathematical simplicity. As a result, the α-μ model has gained widespread popularity, with numerous studies demonstrating its ability to fit field measurements across various frequency bands, including mmWave and THz [14], [15], [16], [17], [18], [19], [20].

Recently, the authors of [21] explored the suitability of the α-μ model in composite fading scenarios. Their study compared its statistical fitting performance against well-known composite models, Nakagami-Lognormal, Generalized-K, and Fisher-Snedecor, using field measurements at 1.8 GHz in a composite fading environment. Their findings revealed that the α-μ model either outperformed these established models or provided comparable accuracy.

Motivated by this and inspired by the importance of the logarithmic presence in several statistical scenarios, we introduce the Log-μ process. This is a general application three-parameter statistical model that presents simple analytical formulations. Its PDF and CDF appear in closed forms with support in $\mathbb{R}$ but limited in the left. Interestingly, for some particular combination of the parameters, a bimodality shows. Knowing that a physical model is behind the proposal and recognizing its potential for wireless communications applications, we specialize it so that its support now is $\mathbb{R}^+$.

The remainder of this article is structured as follows. Section II introduces the Log-μ process depicting its main statistical formulations and specializing it to accommodate a fading vision. Section III presents some plots to illustrate its behavior under different parameter configurations. Section IV, explores practical applications of the Log-μ fading model, focusing on evaluating the performance of the pure-selection combining diversity technique. Finally, Section V concludes the paper and outlines potential future research.

II. The Log-μ Process

The usefulness of the concept of medium nonlinearity has been recognized in several practical applications, as mentioned previously. The importance of the logarithmic presence in several statistical scenarios is well known. Hence, motivated by this, we propose the Log-μ Process, which leads to simple mathematical tractability.

G. K. Tenkorang and M. D. Yacoub are with the Wireless Technology Laboratory (WissTek), Department of Communications, School of Electrical and Computer Engineering, State University of Campinas, Campinas, SP 13083-852, Brazil (e-mail: g218490@dac.unicamp.br; mdyacoub@unicamp.br). This work was partially supported by the São Paulo Research Foundation, FAPESP, (2024/20097-3).

A. Preliminaries

Let $g(P)$ be a nonlinear function of Z, which is a unity-power Gamma random variable, with shape parameter μ, and PDF given as

$$f_Z(z) = \frac{\mu^\mu z^{\mu-1}}{\Gamma(\mu)\exp(\mu z)} \tag{1}$$

where $\Gamma(u) = \int_0^\infty x^{u-1} exp(-x)dx$ is the Gamma function.

For any monotonic $g(P)$, the PDF, $f_P(\rho)$, of the normalized envelope, $P = R/\hat{r}$, is found as $f_P(\rho) = |g'(\rho)| f_Z(g(\rho))$, in which $1/g'(\rho) = d\rho/dz$.

B. Log-μ Model

Defining

$$g(P) = \log(P^\alpha + s) \tag{2}$$

where $\alpha > 0$, and $s \in \mathbb{R}$ are arbitrary parameters, the PDF, $f_P(\rho)$, is found as

$$f_P(\rho) = \frac{\alpha\rho^{\alpha-1}\mu^\mu \log(\rho^\alpha + s)^{\mu-1}}{\Gamma(\mu)(\rho^\alpha + s)^{\mu+1}}, \tag{3}$$

where $\rho^\alpha + s > 1$ and $\mu > 0$. The parameter s plays a dual role, thus acting as both a shaping and a shifting parameter. It is important to note that α is unrestricted for $s \leq 1$; however, for $s > 1$, α must be restricted to odd integers to ensure real-valued expressions. The shape parameter, μ, is given by

$$\mu = V^{-1}(\log(\rho^\alpha + s)), \tag{4}$$

where V(.) denotes the variance operator.

The CDF of the normalized envelope, $F_P(\rho)$, is found in a closed-form as

$$F_P(\rho) = 1 - \frac{\Gamma(\mu, \mu \log(\rho^\alpha + s))}{\Gamma(\mu)}, \tag{5}$$

where $\Gamma(u, v) = \int_0^v x^{u-1}\exp(-x)dx$ is the incomplete Gamma function.

C. Log-μ Fading Model

In linear units, the envelope or power of a fading signal is a positive entity. To enforce a non-negative domain suitable for modeling fading, we define a random variable, Y, as

$$y = \rho - (1-s)^{1/\alpha} > 0. \tag{6}$$

Using the $f_P(\rho)$ obtained in (3), and following the normal procedure for transformation of random variable, the PDF, $f_Y(y)$, is derived as

$$f_Y(y) = \frac{\alpha(y + (1-s)^{1/\alpha})^{\alpha-1}\mu^\mu}{\Gamma(\mu)((y + (1-s)^{1/\alpha})^\alpha + s)^{\mu+1}} \times \log((y + (1-s)^{1/\alpha})^\alpha + s)^{\mu-1}. \tag{7}$$

Consequently, the PDF of the normalized envelope, $f_P(\rho)$, for the Log-μ fading model is expressed exactly as

$$f_P(\rho) = \frac{\alpha(\rho + (1-s)^{1/\alpha})^{\alpha-1}\mu^\mu}{\Gamma(\mu)((\rho + (1-s)^{1/\alpha})^\alpha + s)^{\mu+1}} \times \log((\rho + (1-s)^{1/\alpha})^\alpha + s)^{\mu-1}, \tag{8}$$

where $\rho > 0$.

The CDF of the normalized envelope, $F_P(\rho)$, is found in a closed-form as

$$F_P(\rho) = 1 - \frac{\Gamma\left(\mu, \mu \log\left(\left(\rho + (1-s)^{1/\alpha}\right)^\alpha + s\right)\right)}{\Gamma(\mu)}. \tag{9}$$

III. Some Sample Plots

This section presents illustrative plots of the Log-μ process under different parameter configurations, showcasing its flexibility in capturing a variety of behaviors through its PDFs and CDFs. One of the most distinctive features of the Log-μ process is its ability to exhibit bimodal characteristics for certain parameter configurations, particularly when the parameter $s > 1$. The bimodality phenomenon has been observed in the mmWave and Terahertz (THz) communication channels, where the fading signal PDF can exhibit such traits, as noted in the literature [22], [23], [24], [25]. Therefore, developing distributions with such characteristics is crucial for modeling and designing communication systems, particularly in emerging high-frequency bands where fading behaviors often deviate from conventional models.

A. Log-μ Model

Figs. 1 and 2, respectively, show the PDF and CDF of the Log-μ model for $\alpha = 1$, and three values of s: 3, 1, and -2, under varying values of μ. These plots reflect the generality of the Log-μ model beyond standard fading distributions. Similarly, Figs. 3 and 4, respectively, show the PDF and CDF for $\alpha = 3$, with s set to 5 and 2, and varying μ. In these cases, the emergence of bimodal behavior becomes evident. Figs. 5 and 6, respectively, depict the effect of varying α while keeping $\mu = 4$ and $s = 3$ constant.

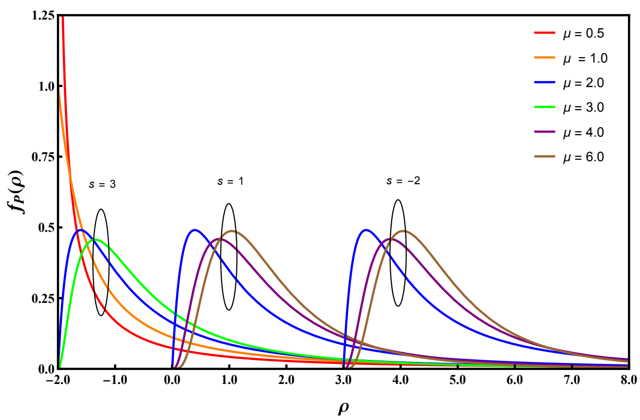

Fig. 1. Various shapes of the PDF of the Log-μ model for $\alpha = 1$ and different s values.

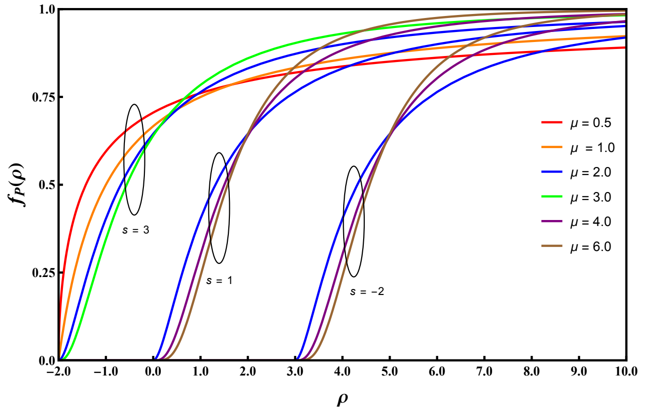

Fig. 2. Various shapes of the CDF of the Log-μ model for $\alpha = 1$ and s = 3.

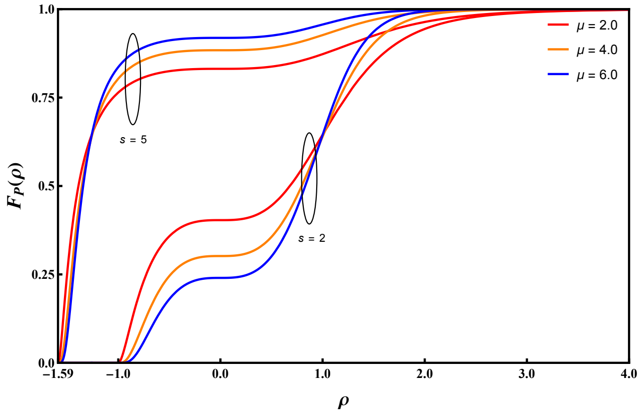

Fig. 4. Various shapes of the CDF of the Log-μ model for $\alpha = 3$ and different s values.

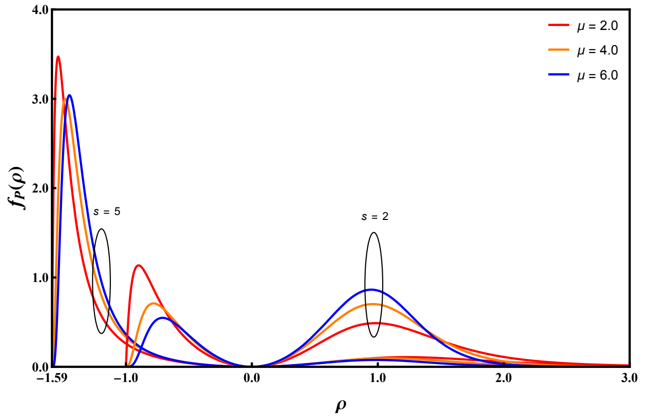

Fig. 3. Various shapes of the PDF of the Log-μ model for $\alpha = 3$ and different s values.

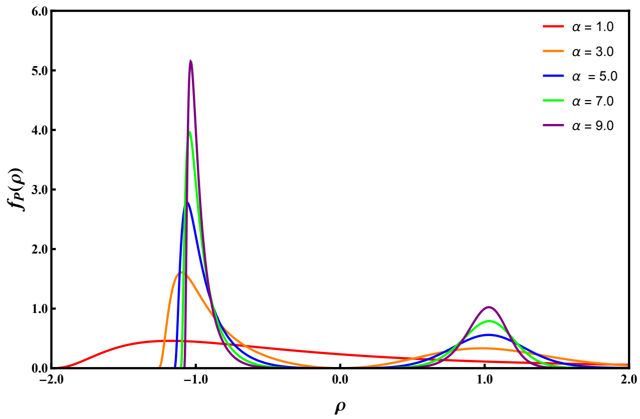

Fig. 5. Various shapes of the PDF of the Log-μ model for $\mu = 4$ and s = 3.

B. Log-μ Fading Model

Figs. 7 and 8, respectively, show the PDF and CDF of the Log-μ fading model for $\alpha = 2$, $s = 0.5$, with varying μ. As observed, the fading envelope remains strictly positive regardless of s, which in this case acts solely as a shaping parameter. Finally, Figs. 9 and 10 illustrate, respectively, the PDF and CDF of the Log-μ fading model for $\alpha = 3$ and $s = 2$, considering different values of μ. These results further confirm the model's ability to capture bimodal behavior.

IV. Some Application of the Log-μ Fading Model

Several techniques have been proposed in the literature to mitigate the effects of fading on transmitted signals. Among these are diversity, adaptive equalization, and coding. In reception systems, diversity combining methods can significantly enhance signal quality. This section presents a performance analysis of the pure-selection combining technique.

A. Pure-Selection Combining

In the pure-selection combining technique, the received signals are continuously monitored to select the branch with the highest signal-to-noise ratio (SNR). Consequently, at the receiver, the combiner output envelope R is given by

$$R = \max_{i=1,\ldots,M} \{R_i\}. \tag{10}$$

Considering the SNR in each branch i defined as $\gamma_i = {r_i}^2$, the probability that γ_i, is less than or equal to a given SNR, Γ, is derived as

$$F(\Gamma) = 1 - \frac{\Gamma\left(\mu, \mu \log\left(\left(\left(\frac{\Gamma}{\gamma_0}E(P^2)\right)^{1/2} + (1-s)^{1/\alpha}\right)^{\alpha} + s\right)\right)}{\Gamma(\mu)}, \tag{11}$$

where $\gamma_0 = E(R^2) = \hat{r}^2 E(P^2)$ is the mean power.

Assuming that the selector is ideal and that the best signal is always present at the output, the probability that the SNRs in all of the M branches are simultaneously less than or equal to a given SNR, Γ_s, is given by the CDF of the output SNR, denoted $F_{SEL}(\Gamma_s)$. Under the assumption of independent and identically distributed (i.i.d.) Log-μ fading across all branches, the CDF is given by

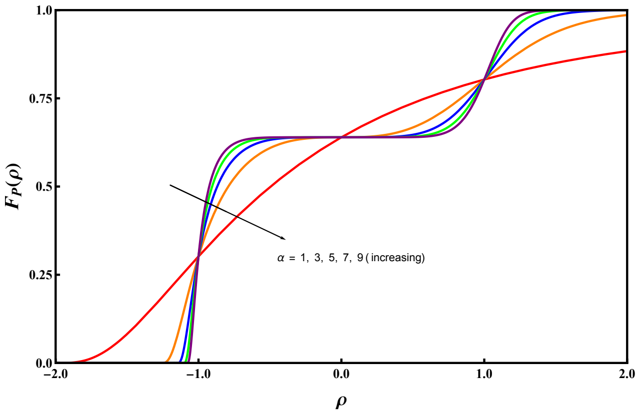

Fig. 6. Various shapes of the CDF of the Log-μ model for $\mu = 4$ and s = 3.

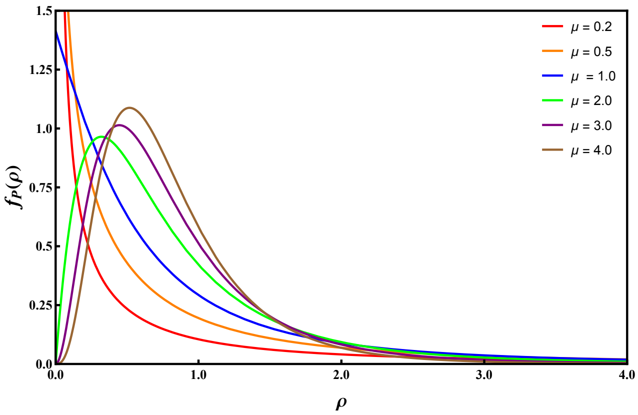

Fig. 7. Various shapes of the PDF of the Log-μ fading model for $\alpha = 2$ and s = 0.5.

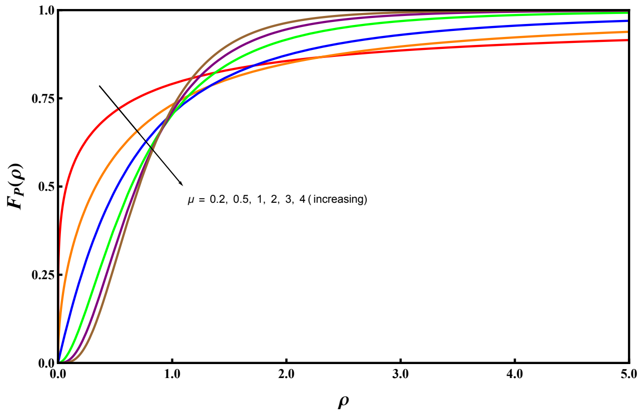

Fig. 8. Various shapes of the CDF of the Log-μ fading model for $\alpha = 2$ and s = 0.5.

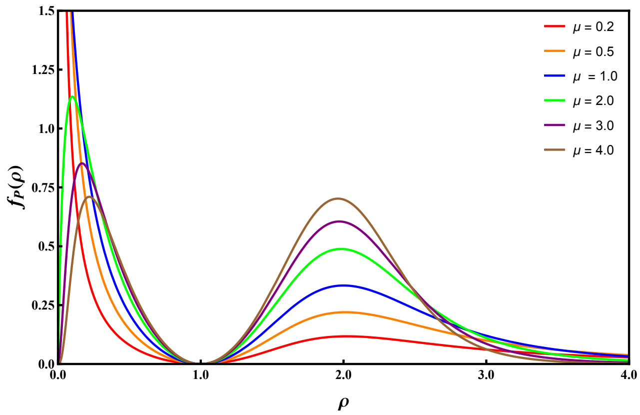

Fig. 9. Various shapes of the PDF of the Log-μ fading model for $\alpha = 3$ and s = 2.

$$F_{SEL}(\Gamma_s) = \left(1 - \frac{\Gamma\left(\mu, \mu \log\left(\left(\left(\frac{\Gamma_s}{\gamma_0} E(P^2)\right)^{1/2} + (1-s)^{1/\alpha}\right)^{\alpha} + s\right)\right)}{\Gamma(\mu)}\right)^{M}. \quad (12)$$

B. Sample Shapes of Distribution of the SNR at the Output of the Pure-Selection Combiner

In this section, we present some outage probability plots, $F_{SEL}(\Gamma_s)$, for fixed parameters α and μ, while varying M. As shown in Fig. 11, the diversity technique enhances system performance, specifically, increasing the number of branches improves the SNR. However, while adding more diversity branches always improves SNR, the incremental benefit diminishes as the number of branches increases.

Analyzing Fig. 11 for $\alpha = 1$ and $\mu = 4$, it is observed that a 99% reliability level, 1-$F_{SEL}(\Gamma_s)$, is achieved at an SNR of -23 dB for $M = 1$ (no diversity), -16 dB for $M = 2$, -12 dB for $M = 3$, and -10 dB for $M = 4$. The SNR gains when increasing from $M = 1$ to $M = 2$, $M = 2$ to $M = 3$, $M = 3$ to $M = 4$ are approximately 30%, 25%, and 17%, respectively. These results highlight that, while increasing the number of diversity branches consistently enhances SNR, the relative improvement decreases as M increases.

V. Conclusions

In this paper, we introduce the Log-μ process, a novel framework for capturing the nonlinear effects of the propagation medium in which multipath clustering may be present. The Log-μ fading model stems from the idea of the nonlinearity of the propagation environment and the importance of the logarithmic presence in several statistical scenarios. Notably, the PDF and CDF of the Log-μ model lead to a mathematically simple formulation while maintaining a complexity level comparable to other general distributions. Due to the alternative treatment of nonlinearity, the Log-μ fading model is conjectured to provide improved adaptability for certain wireless communication applications than the existing models in the literature. This conjecture arises from the fact that a popular nonlinear fading model alone, namely α-μ, was able to yield very good fitting to practical data for a composite (short-term and long-term) fading environment.

As future work, further investigation into the nonlinearity of the propagation medium could be conducted potentially leading to analytically tractable formulations. Given the physical

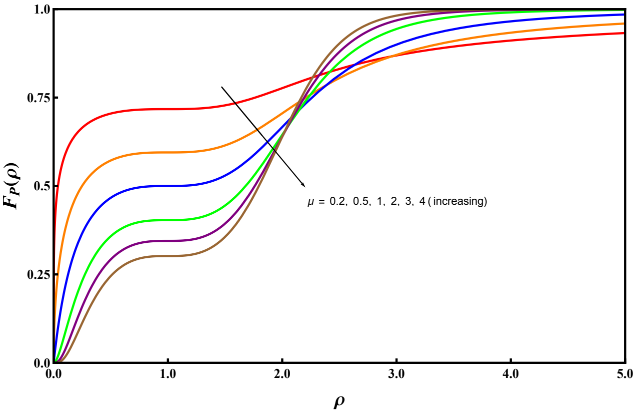

Fig. 10. Various shapes of the CDF of the Log-μ fading model for $\alpha = 3$ and s = 2.

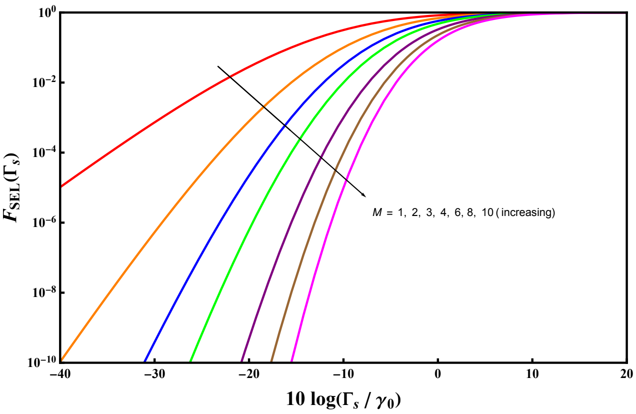

Fig. 11. Distribution of the SNR at the output of the pure-selection combiner for $\alpha = 1$, $\mu = 4$ and s = 1.

basis of the proposed model, an extension to include higher-order statistical measures, such as the level crossing rate (LCR) and average fading duration (AFD), would be a valuable contribution. While this paper focuses on the pure-selection combining diversity technique, exploring additional techniques such as threshold-selection, maximal ratio, and equal gain combining could provide deeper insights into the model's performance. Finally, an experimental validation through field measurements, comparing the Log-μ model against well-established fading models, would further substantiate its practical relevance.

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