Partial Commutativity and Quantum Zero-Error Capacity

Andresso da Silva and Francisco M. de Assis

***Abstract*— Recently, it was discovered that there is a connection between the growth factor of the partially commutative monoid, denoted as $\beta(G)$, and the independence number of the graph. It was shown that $\lfloor\beta(G)\rfloor \geq \alpha(G)$ and that $\log\lfloor\beta(G)\rfloor$ is as an upper bound for the classical zero-error capacity [1]. In this paper, we demonstrate that $\beta(G)$ is not only an upper bound for the Lovász number of G, but is also for the chromatic number of the complement G. Furthermore, we show that $\log\lfloor\beta(G)\rfloor$ is an upper bound for the quantum zero-error capacity.**

***Keywords*— Zero-Error Capacity, Partially Commutative Monoids, Lovász Number.**

I. Introduction

The classical zero-error capacity was defined by Shannon [2] using concepts from graph theory. The definition of the zero-error capacity of a quantum channel was presented in the works by Medeiros and collaborators [3], [4], [5]. Even today, it is not known whether this classical zero-error capacity is, in general, computable in the sense that there is a program that returns the value in a finite time [6], [7]. Whereas quantum zero-error capacity is a problem of the class QMA-Complete [8]. For these reasons, some upper bounds have been proposed [9], [10], [11], [12], [1], [13].

The zero-error capacity, whether classical or quantum, is often estimated using the Lovász number $\vartheta(G)$, where G is the adjacency graph of the channel [9], [13], [14]. Some methods for deriving upper bounds for the quantum zero-error capacity involve establishing the partial commutation between operators that define noncommutative graphs [11], [15], [14]. Duan et al. [11] introduced a quantum version of the Lovász number which can be used to calculate an upper bound for the quantum zero-error capacity. Furthermore, Boreland et al. [14] presented an improved version of the quantum Lovász number proposed in [11].

Recently, it has been discovered that there is a connection between partially commutative monoids and the zero-error capacity C_0 of a classical channel [1]. Partially commutative monoids are defined by a finite alphabet and a partial order relation between the symbols in that alphabet. In this context, partial commutativity relations define equivalent words, which then form equivalence classes. These commutativity relations can be represented using a graph G, known as a commutativity graph. The number of classes can be estimated by the growth factor of the monoid $\beta(G)$, which is calculated from the cliques of G. By interpreting the adjacency graph of the channel as a commutativity graph, the connection between zero-error capacity and partially commutative monoids was established.

In the approach presented in [1], the authors showed that the classical zero-error capacity is upper-bounded by $C_0(G) \leq \log\lfloor\beta(G)\rfloor$, based on the demonstration that $\lfloor\beta(G)\rfloor \geq \alpha(G)$ holds. However, in [1], they did not find the relationship between $\beta(G)$ and $\vartheta(G)$ and did not address the quantum zero-error capacity.

In this paper, we show that $\beta(G)$ is an upper bound for the chromatic number of the complement of G, $\chi(\overline{G})$, and the Lovász number $\vartheta(G)$ of G. These results are used to show that $\log\lfloor\beta(G)\rfloor$ is an upper bound for the zero-error capacity of a quantum channel with adjacency graph G. We also discuss and answer some questions about the bound tightness.

The rest of this article is organized as follows: Section II presents the fundamental concepts needed to understand the results involving graphs, classical and quantum zero-error theory, and partially commutative monoids. In Section III, we demonstrate the relationship between $\beta(G)$ and the chromatic number of G and, from this, we show that $\log\lfloor\beta(G)\rfloor$ is also an upper bound for quantum zero-error capacity. Also, in this section, we discuss some properties of $\beta(G)$ and answer some questions about the bound tightness. Section IV presents the conclusions and future work.

II. Fundamentals

A. Graph Theory

A graph G is a pair (V, E) where V is the set of vertices and E is the set of edges. The edges are represented by tuples of the form (a, b) where $a, b \in V$. Two vertices connected by an edge are called adjacent. Simple, undirected graphs are those in which there are no multiple edges connecting two vertices and no edges of the type (a, a) with $a \in V$. The graphs discussed in this article are simple and undirected. The complement of the graph $G = (V, E)$ is composed of the same set of vertices V, and the edges connect the vertices that are not connected in E. The complement of G is represented by $\overline{G} = (V, \overline{E})$.

In this article, we will frequently discuss some types of graphs, so it is important to define each of them. A complete graph is one in which all vertices are connected to each other, and it's represented by K_k, where k is the number of vertices. An empty graph is one that has no edges, denoted by $|E| = 0$. A cyclic graph, represented by C_k, is a graph that contains only one cycle with all k vertices, where k is greater than or equal to 3. An example of a cyclic graph is the pentagon, which is represented as C_5.

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A subgraph of $G = (V, E)$ is formed by a subset vertex of V and a subset edge of E. Given a subset $S \subseteq V$, a graph induced by S is formed by the vertices S and all the edges of E that connect two elements of S.

The strong product of n copies of $G = (V, E)$ is denoted by G^n and consists of the graph formed by the vertices resulting from the Cartesian product between n copies of V. Two vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ of G^n are adjacent if there is at least one i where $(u_i, v_i) \in E$.

A clique of a graph is a subset of vertices in which all these vertices are two by two adjacent. The clique number $\omega(G)$ is the cardinality of the largest clique of G.

Proper graph coloring is the color assignment to vertices so that no two adjacent vertices have the same color. The chromatic number $\chi(G)$ is the minimum number of colors used in a proper coloring of G. A graph is perfect if $\omega(G) = \chi(G)$. Obtaining the chromatic number of a graph belongs to the class of NP-hard problems.

A stable set of a graph is a subset of vertices that are two by two non-adjacent. The independence number $\alpha(G)$ is the cardinality of the largest stable set of G. Therefore, it should be noted that $\alpha(G) = \omega(\overline{G})$. Furthermore, obtaining graph independence or clique numbers are NP-complete problems.

Lovász [9] introduced an upper bound for $\alpha(G)$ that can be calculated in polynomial time since it is a semidefinite programming relaxation.

Definition 1 (Lovász number [9]): Let $G = (V, E)$ be a adjacency graph and let $i, j \in V$ be two vertices. Define the matrix T whose elements (i, j) are $T_{i,j} = 0$ if $(i, j) \in E$ or $i = j$. The Lovász number of G is defined as

$$\vartheta(G) = \max\{\|I + T\| : I + T \geq 0\}, \tag{1}$$

where I is the identity matrix and T is a $|V| \times |V|$ Hermitian matrix.

The Lovász number $\vartheta(G)$ is related to other graph properties as stated in the Lovász Sandwich Theorem.

Theorem 1 (Lovász Sandwich Theorem [16], [17]): If G is a graph and its complement $\overline{G}$, then the following holds

$$\alpha(G) \leq \vartheta(G) \leq \chi(\overline{G}). \tag{2}$$

The number $\vartheta(G)$ lies between two quantities used to solve problems in the NP complexity class. For perfect graphs G, it is true that $\alpha(G) = \vartheta(G) = \chi(\overline{G})$. In 2006 Chudnovsky et al. [18] proved the so-called strong perfect graph theorem.

Theorem 2 (The strong perfect graph [18]): A graph is perfect if and only if it contains no induced cycles of the form C_k and no complement of a cyclic graph $\overline{C}_k$, where $k \geq 5$ and k is odd.

Although $\vartheta(G)$ is widely used as the upper bound for $\alpha(G)$, Feigi [19] has shown that there is a gap between $\vartheta(G)$, $\alpha(G)$ and $\chi(\overline{G})$ proportional to the number of k vertices for some graphs.

Theorem 3 ([19]): For a constant $\varepsilon > 0$, there exists an infinite family of graphs with k vertices in which holds that

$$\chi(\overline{G}) > \vartheta(G) k^{1-\varepsilon}. \tag{3}$$

B. Zero-Error Capacity

The zero-error capacity, whether classical or quantum, is associated with the number of messages that can be transmitted by a channel with zero probability of decoding errors [2]. In the classical case, the channel will be a discrete memoryless (DMC) defined by a finite input alphabet $\mathcal{X}$, a finite output alphabet $\mathcal{Y}$, and a transition probability matrix $P = [p(y_i|x_j)]$. The elements $p(y_i|x_j)$ correspond to the probability of receiving y_i at the channel output, given that x_i was the input. In this way, you can define when two states are confusable or indistinguishable.

Definition 2 (Adjacency of Classical States): Given a DMC ($\mathcal{X}$, $\mathcal{Y}$, P), the states $x_i \in \mathcal{X}$ and $x_j \in \mathcal{X}$ are adjacent (indistinguishable) if there exists a $y \in \mathcal{Y}$ such that $p(y|x_i) > 0$ and $p(y|x_j) > 0$.

Based on the adjacency relations between the states, it is possible to define a graph representing the indistinguishability relations between the channel's input states.

Definition 3 (Classical Adjacency Graph): Given a DMC($\mathcal{X}$, $\mathcal{Y}$, P), the adjacency graph $G = (V, E)$ has its vertices associated with the elements of $\mathcal{X} = \{x_1, \ldots, x_l\}$ and the edges connect states that are confusable at the output of the channel, i.e.,

1) $V = \{1, \ldots, l\}$ and
2) $E = \{(i, j) : \exists y, p(y|x_i) > 0 \ \ and \ \ p(y|x_j) > 0\}$.

Note that the adjacency between two words of length n defined in $\mathcal{X}$ corresponds to finding the adjacency between vertices in the strong product of n copies of G. It is, therefore, possible to define the classical zero-error capacity using the adjacency graph.

Definition 4 (Classical Zero-Error Capacity): Given a DMC channel and its adjacency graph G, the classical zero-error capacity is defined as

$$C_0(G) = \sup_n \frac{1}{n} \log \alpha(G^n), \tag{4}$$

where $\alpha(G^n)$ is the independence number of G^n, G^n is the strong product of n copies of G and the logarithm is base 2.

In general, if $\alpha^k(G) \leq \alpha(G^k)$, then $C_0(G) \geq \log \alpha(G)$. Shannon [2] demonstrated that if the graph G is perfect, then $\alpha(G^k) = \alpha^k(G)$ and $C_0(G) = \log \alpha(G)$. Thus, estimating the classical zero-error capacity is an NP-complete problem [8], but it is not known whether it is computable in general [6], [7].

In the quantum case, a channel is a completely positive trace-preserving map (CPTP) represented by Kraus operators. While there are other possible representations, this paper will adopt this one.

Definition 5 (Quantum Channel): Let $\mathcal{E}$ be a quantum channel represented by the Kraus operators $\{E_i\}_{i=1}^m$ and $\sum_{i=1}^m E_i^\dagger E_i = I$. The effect of the $\mathcal{E}$ channel on the ρ state is such that

$$\mathcal{E}(\rho) = \sum_{i=1}^m E_i \rho E_i^\dagger. \tag{5}$$

The support of a state is the space spanned by its eigenvectors with nonzero eigenvalues. Two quantum states ρ_i and ρ_j can be perfectly discriminated if, and only if, they have supports in orthogonal subspaces [20]. If ρ_i and ρ_j have supports in orthogonal states, then $\mathrm{Tr}(\rho_i \rho_j) = 0$, allowing us to define the adjacency between quantum states.

Definition 6 (Adjacency of Quantum States): Let $\rho_i, \rho_j \in \mathcal{S}$ be two states, such that $i \neq j$. The states ρ_i and ρ_j are adjacent (indistinguishable) if ρ_i and ρ_j do not have supports in orthogonal subspaces, i.e., $\mathrm{Tr}(\rho_i \rho_j) \neq 0$, and are said to be non-adjacent otherwise.

Definition 7 (Adjacency Graph of a Quantum Channel): If $\mathcal{E}$ is a quantum channel and $\mathcal{S} = \{\rho_1, \ldots, \rho_l\}$ is a set of input states, the characteristic graph $G = (V, E)$ has its vertices associated with the elements of $\mathcal{S}$ and the edges connect states that are indistinguishable at the output of $\mathcal{E}$, i.e.,

1) $V = \{1, \ldots, l\}$ and
2) $E = \{(i, j) : \rho_i, \rho_j \in \mathcal{S}, \mathrm{Tr}(\mathcal{E}(\rho_i)\mathcal{E}(\rho_j)) \neq 0\}$.

Given the quantum adjacency graph, it is possible to define the quantum zero-error capacity of a channel [3], [4]. In essence, the definition is similar to the classical case, but in the quantum case, it is necessary to look for the set of input states that maximize the capacity value.

Definition 8 (Quantum Zero-Error capacity): The zero-error capacity of a quantum channel $\mathcal{E}$ is given by

$$C^{(0)}(\mathcal{E}) = \sup_{\mathcal{S}} \sup_{n} \frac{1}{n} \log \alpha(G^n), \tag{6}$$

where the supremum is taken over all input sets $\mathcal{S}$ and all codes of length n.

The Lovász number is an upper bound for the classical zero-error capacity when G is a classical adjacency graph [9], $C_0(G) \leq \log \vartheta(G)$. When G is an adjacency graph associated to a quantum channel $\mathcal{E}$ [10], it also holds that $C^{(0)}(\mathcal{E}) \leq \log \vartheta(G)$, even if entanglement is used. Duan *et al.* [11] proposed a generalization of the Lovász number using quantum operators that is also an upper bound for the quantum zero-error capacity.

Beigi and Shor [8] proved that calculating the zero-error capacity of a channel is a problem of the QMA-Complete class, as it is associated with the problem of finding the quantum stable set of a graph. The QMA class is the quantum analog of the NP class, but it also contains the NP complexity class [21].

C. Partially Commutative Monoids and Adjacency Graphs

Given a finite alphabet Σ and a commutativity relation $ab \equiv ba$ between two elements $a, b \in \Sigma$. Commutativity relations can be represented using a graph, where the vertices are associated with the elements of Σ and the edges connect two vertices that commute. This graph is called a commutativity graph.

Swapping the order of commutative symbols in words defined in Σ makes it possible to form equivalent words. For example, the word $\mathbf{u} = abcba$ with the relation $ab \equiv ba$ makes it possible to obtain the equivalent words $\mathbf{v} = bacba$, $\mathbf{w} = abcab$ and $\mathbf{x} = bacab$. The equivalence relation between two words $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \equiv \mathbf{v}$. The set of words equivalent to $\mathbf{u}$ is called an equivalence class, and the set of all equivalence classes is called a partially commutative monoid.

The number $\tau_G(n)$ of equivalence classes formed by words of length n can be obtained using the number of cliques of the commutativity graph G.

Definition 9 (Dependence Polynomial [22]): Let G be a commutativity graph, then the dependence polynomial is defined as

$$D(G, z) = \sum_{i=0}^{\omega(G)} (-1)^i c_i z^i, \tag{7}$$

where c_i is the number of complete subgraphs (cliques) with i vertices of G.

The dependence polynomial can be used to calculate the generating function of the monoid.

Definition 10 (Monoid Generating Function [22]):

$$\frac{1}{D(G, z)} = \sum_{n=0}^{\infty} \tau_G(n) z^n. \tag{8}$$

The relationship between the partially commutative monoid and the strong product of graphs was described in [1]. The adjacency graph of the channel was interpreted as a commutative graph, giving rise to Lemma 1.

Lemma 1 ([1]): If two words of length n belong to the same equivalence class according to G, then they are connected in G^n and are confusable.

By combining the Lemma 1 and the concept of monoid growth factor, the authors in [1] arrived at an upper bound for the classical zero-error capacity of a channel represented by an adjacency graph G. This result is expressed in Theorem 4.

Definition 11 (Monoid Growth Factor):

$$\beta(G) = \lim_{n \to \infty} \tau_G(n)^{\frac{1}{n}}. \tag{9}$$

The value $\beta(G)$ also corresponds to the inverse of the smallest real root of the dependence polynomial $D(G, z)$. When $1/\beta(G)$ is a root with multiplicity 1, it is possible to perform the approximation $\tau_G(n) \sim \beta(G)^n$. For this reason, $\beta(G)$ ć is known as the monoid growth factor. The notation $f(n) \sim g(n)$ denotes $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$ (see [23]).

Theorem 4 (Upper Bound for Classical Zero-Error Capacity):

$$C_0(G) \leq \log \lfloor \beta(G) \rfloor \tag{10}$$

It has been shown[1] that for cyclic graphs C_k it is true that

$$\alpha(C_k) \leq \vartheta(C_k) \leq \lfloor \beta(C_k) \rfloor. \tag{11}$$

However, the general relationship between these properties was not found by the authors in [1].

III. Upper Bound for Quantum Zero-error Capacity

In this section, we extend the results presented in [1] by presenting the relationship between $\lfloor \beta(G) \rfloor$, the Lovász number $\vartheta(G)$ and the chromatic number $\chi(\overline{G})$. The relationship is presented in Theorem 5.

Theorem 5: Let G be a graph, then

$$\lfloor \beta(G) \rfloor \geq \chi(\overline{G}). \tag{12}$$

Proof: Using the Lemma 1, it is known that words of length n congruent in $\mathcal{M}(\Sigma, G)$ are connected in G^n. Thus, the equivalence classes $\mathcal{E}(\mathbf{u}_i)$, with $|\mathbf{u}_i| = n$, correspond to cliques in G^n and, consequently, to stable sets in $\overline{G^n}$. Therefore, there are at most $\tau_G(n)$ unconnected vertices in $\overline{G^n}$. Since the elements of a stable set can be colored using the same color, it would take at most $\tau_G(n)$ colors to properly color $\overline{G^n}$, i.e. $\tau_G(n) \geq \chi(\overline{G^n})$. By using the fact that $\chi(\overline{G^n}) \leq \chi(\overline{G})^n$ (see [24]), we get $\tau_G(n) \geq \chi(\overline{G})^n$. Consequently, $\beta(G) \geq \chi(\overline{G})$ and, knowing that $\chi(\overline{G})$ is a natural number, we get $\lfloor \beta(G) \rfloor \geq \chi(\overline{G})$. ■

From Theorem 5, the Lovász sandwich theorem (see Theorem 1) can be supplemented with one more layer

$$\alpha(G) \leq \vartheta(G) \leq \chi(\overline{G}) \leq \lfloor \beta(G) \rfloor, \tag{13}$$

where obtaining $\beta(G)$ is also, in general, a problem of the NP-complete complexity class because it involves the dependence polynomial.

In this way, the relationship between $\beta(G)$ and the quantum zero-error capacity can be established, as stated in Theorem 6.

Theorem 6:

$$C^{(0)}(\mathcal{E}) \leq \log \lfloor \beta(G) \rfloor \tag{14}$$

Proof: It follows from Theorem 5 and the fact that $C^{(0)}(\mathcal{E}) \leq \log \vartheta(G)$ [10]. ■

Note that Theorem 4 could be proved similarly. In both cases, $\log \vartheta(G)$ is an upper bound for the zero-error capacity. The quantum case is more delicate because it requires searching among the states of a set of input states, and these states define the adjacency graph.

Whereas quantum zero-error capacity is a QMA-complete problem, calculating $\beta(G)$ involves an NP-complete problem and can be calculated for several families of graphs. Additionally, calculating $\beta(G)$ only requires the adjacency graph, unlike the zero-error capacity which involves the strong product of G. In this context, we will provide some examples of how to calculate $\lfloor \beta(G) \rfloor$.

Example 1 (Complete and Empty Graphs): These graphs are perfect and we know that $\beta(S_k) = \alpha(S_k)$ [1].

Example 2 (Cycle Graphs): For cyclic graphs C_k, $k \geq 4$, $\beta(C_k)$ is given by

$$\beta(C_k) = \frac{1}{2}\left(k + \sqrt{k-4}\sqrt{k}\right). \tag{15}$$

In addition, for odd $k \geq 3$, we have that

$$\chi(\overline{C_k}) = \frac{k+1}{2} \quad (16) \qquad \vartheta(G) = \frac{k\cos(\pi/k)}{1+\cos(\pi/k)}. \quad (17)$$

Lovász [9] showed that $C_0(C_5) = \vartheta(C_5) = \frac{1}{2}\log 5$. However, the zero-error capacity of channels with C_k as an adjacency where k is an odd number greater than 5, remains unknown. In general, for odd k and $k \geq 5$, the graphs C_k are non-perfect, according to Theorem 2.

The comparison between some values of $\lfloor \beta(C_k) \rfloor$, $\vartheta(G)$ and $\chi(\overline{C_k})$ is shown in Figure 1.

By using Eq. (15), Eq. (16) and Eq. (17), we can show that the asymptotic behavior of $\lfloor \beta(C_k) \rfloor$, $\vartheta(C_k)$ and $\chi(\overline{C_k})$ have the following property

$$\lim_{k\to\infty} \frac{\beta(C_k)}{\vartheta(C_k)} = \lim_{k\to\infty} \frac{\beta(C_k)}{\chi(\overline{C_k})} = 2. \tag{18}$$

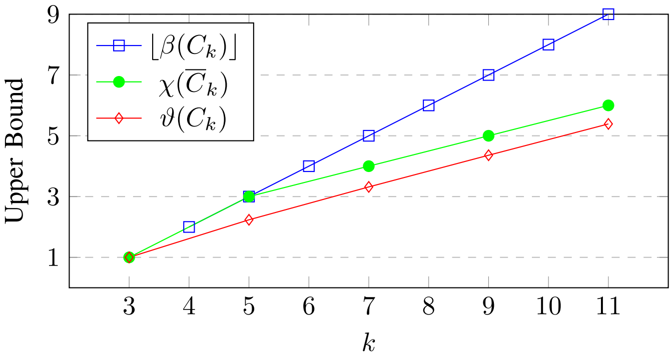

Fig. 1: Values of $\lfloor \beta(C_k) \rfloor$ considering cyclic graphs of k vertices, $\vartheta(G)$ and $\chi(\overline{C_k})$ for k odd, $3 \leq k \leq 11$.

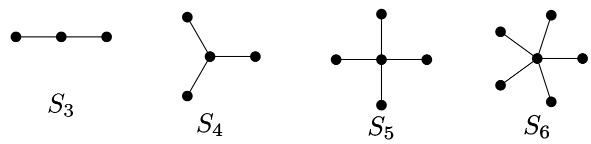

Fig. 2: Star Graphs S_k, $k = 3, 4, 5, 6$.

Example 3 (Star Graphs): Star graphs, represented by S_k with $k > 1$, have k vertices and $k-1$ edges. Figure 2 shows some examples of this graph.

The chromatic number of S_k is $\chi(S_k) = 2$ and $\omega(S_k) = 2$, so these graphs are perfect. To find the chromatic number of $\overline{S}_k$, it should be noted that the complement of S_k will consist of a complete graph K_{k-1} and a vertex that is disconnected from all the others. This means that at least $k-1$ colors are needed to properly color $\overline{S}_k$, so $\chi(\overline{S}_k) = k-1$.

In addition, the dependence polynomial of S_k will be

$$D(S_k, z) = 1 - kz + (k-1)z^2, \tag{19}$$

in such a way that

$$\tau_{S_k}(n) \sim (k-1)^n \tag{20}$$

and therefore,

$$\beta(S_k) = (k-1). \tag{21}$$

In this case, $\beta(S_k) = \chi(\overline{S}_k)$. Since the graphs S_k are perfect, then $\beta(S_k) = \chi(\overline{S}_k) = \vartheta(S_k) = \alpha(S_k)$.

It appears that there is a higher likelihood of $\lfloor \beta(G) \rfloor$ being equal to $\chi(\overline{G})$ when G is a perfect graph, based on the examples. However, we found that even for perfect graphs, it is not always the case that $\lfloor \beta(G) \rfloor = \alpha(G)$, as shown in the following example.

Example 4 (King Graph): The graph G_k represents a $k \times k$ chessboard, showing the king's possible moves. Figure 3 displays examples of these graphs.

It is possible to obtain the dependency polynomial of these graphs by inspection as

$$D(G_k, z) = 1 - k^2 z + (2k-2)(2k-1)z^2 - 4(k-1)^2 z^3 + (k-1)^2 z^4. \tag{22}$$

The analytical expression for $\tau_G(n)$ as a function of k becomes increasingly large and complex for larger values of

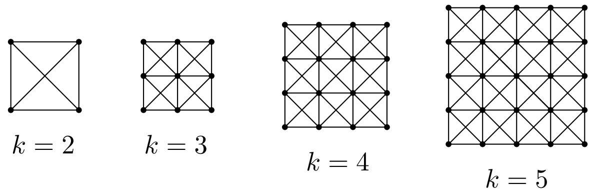

Fig. 3: King Graphs G_k, $k = 2, 3, 4, 5$.

k. Therefore, we use $k = 3$ as an example and

$$D(G_3, z) = 1 - 9z + 20z^2 - 16z^3 + 4z^4. \tag{23}$$

The smallest real root of $D(G, z)$ is approximately $z \approx 0.16243$, so $\tau_{G_3}(n) \sim 6.1565^n$ and $\lfloor \beta(G_3) \rfloor = 6$. On the other hand, it is known that

$$\alpha(G_k) = \left\lfloor \frac{k+1}{2} \right\rfloor^2, \tag{24}$$

and that the graphs G_k are perfect, according to Theorem 2. In particular, $\chi(\overline{G}_3) = 4$ and $\alpha(G_3) = 4$ so that $\lfloor \beta(G_3) \rfloor > \chi(\overline{G}_3)$.

Based on Theorems 5 and 3, it is established that for a constant $\varepsilon > 0$, there exists an infinite family of graphs with k vertices where $\lfloor \beta(G) \rfloor > \vartheta(G) k^{1-\varepsilon}$. It has been noted that non-perfect graphs, like cyclic graphs with $k \geq 5$ and odd k, exhibit a gap proportional to n between $\lfloor \beta(G) \rfloor$ and $\vartheta(G)$. By using the king's graph with $k = 3$, it follows that perfect graphs also have a gap proportional to n between $\lfloor \beta(G) \rfloor$ and $\vartheta(G)$. For cyclic graphs, it is established that $\chi(\overline{C}_k) = k/2$ for even k and $\lfloor \beta(C_k) \rfloor > \chi(\overline{C}_k)$. This leads to the conjecture that there exists an infinite family of graphs and a $\varepsilon > 0$ for which $\lfloor \beta(G) \rfloor > \chi(\overline{G}) k^{1-\varepsilon}$.

IV. Conclusions

In this article, we explore partial commutation relations and zero-error theory. We show that $\beta(G)$ is an upper bound of the chromatic number $\chi(\overline{G})$ of the complement of G and, consequently, of the Lovász number $\vartheta(G)$ of a graph G. From this, we also show that it is possible to use $\beta(G)$ to determine an upper bound for the zero-error capacity of a quantum channel with adjacency graph G employing $C^0(\mathcal{E}) \leq \log \lfloor \beta(G) \rfloor$. We show that this bound may not be tight even if the graph G is perfect.

As future work, we intend to identify which families of graphs have $\beta(G) = \chi(G)$ and answer the conjecture whether there is an infinite family of graphs in which $\lfloor \beta(G) \rfloor > \chi(G)$. In addition, one can investigate applications of $\lfloor \beta(G) \rfloor$ as an upper bound of $\chi(\overline{G})$, which also has applications in quantum information theory.

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