Zero-Error Capacity and Invariant Subspace of a Quantum Channel

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***Abstract*— In this paper, we explore the connection between the zero-error capacity and the invariant subspace of a quantum channel. We begin by demonstrating that a quantum channel with two invariant subspaces, which only have trivial intersections, has a positive zero-error capacity. Additionally, we establish that an unital quantum channel with a non-trivial invariant subspace of at least two dimensions also has a positive zero-error capacity.**

***Keywords*— Quantum channel, Zero-error capacity, Invariant Subspace.**

I. INTRODUÇÃO

The field of quantum information theory is rooted in the principles of quantum mechanics, and its primary focus is on issues related to the processing and transmission of information through a quantum channel [1], [2]. This research area is diverse, encompassing potential applications in computing [3], as well as the investigation of the zero-error capacity of a quantum channel [4], [5], and exploring new approaches [6]. This article will specifically delve into the latter area.

The zero-error capacity of a discrete memoryless quantum channel was first defined in 2006, by Medeiros and Assis [5], as the highest rate at which a quantum channel can transmit classical information with a probability of error equal to zero. This idea extends the classical zero-error capacity of a discrete memoryless channel, which was first introduced by Shannon [7] in 1956.

The study of zero-error capacity, particularly focused on determining whether a quantum channel has a positive zero-error capacity [5], [8], [9], [10], allows us to explore the relationship between zero-error capacity and the invariant subspace of a quantum channel. The notion of an common invariant subspace of a quantum channel [11] is directly linked to the concept of a common subspace [12] of the Kraus operators that represent the channel [1].

The concept of invariant subspaces of a matrix (or linear operator) is extensively studied in mathematical research. There is significant knowledge about matrices that have non-trivial common invariant subspaces [12]. One important topic of discussion is the conditions for the existence of non-trivial common invariant subspaces for two or more matrices [13], [14]. This concept is also relevant in the study of quantum channels represented by Kraus matrices, particularly in the discussion of quantum channel irreducibility [15], [16]. In this paper we aim to explore the relationship between invariant subspaces and the zero-error capacity of a quantum channel.

To achieve the main objective of this article, the document is organized as follows: Section II presents the definition of a quantum channel and some basic properties. Section III discusses the zero-error capacity of a quantum channel, which is fundamental to understanding the main objective of this article. Section IV covers the concept of the common invariant subspace of a quantum channel and some of its properties. Section V demonstrates two results related to the link between zero-error capacity and the concept of common invariant subspace of a quantum channel. Finally, Section VI presents some conclusions and proposals for future developments on the subject.

II. DEFINITIONS AND QUANTUM CHANNEL

Let $\mathcal{H}$ be a Hilbert space of dimension d. We denote by $\mathcal{B}(\mathcal{H})$, the Hilbert space of linear operators of $\mathcal{H}$, of dimension d^2.

A quantum channel, defined on the Hilbert space $\mathcal{H}$, is a linear map that is completely positive and trace-preserving, and it acts on the density matrices. This quantum channel is denoted as $\mathcal{E} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$. By using a set of Kraus operators $\{A_i\}_{i=1}^{\kappa}$, we can represent the quantum channel $\mathcal{E}$, and we can express it as

$$\mathcal{E}(\rho) = \sum_{i=1}^{\kappa} A_i \rho A_i^\dagger \text{ and } \sum_{i=1}^{\kappa} A_i^\dagger A_i = I, \tag{1}$$

for all $\rho \in \mathcal{B}(\mathcal{H})$.

Furthermore, we will denote by $\mathcal{C}(\mathcal{H})$ the set formed by all quantum channels in the Hilbert space $\mathcal{H}$, which is closed concerning convex combination and multiplication (recursive application), i.e., for $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{C}(\mathcal{H})$ and $p \in [0,1]$ the transformations defined by the maps

$$p\mathcal{E}_1(\rho) + (1-p)\mathcal{E}_2(\rho), \quad \rho \in \mathcal{B}(\mathcal{H}) \tag{2}$$

$$(\mathcal{E}_1 \circ \mathcal{E}_2)(\rho) := \mathcal{E}_1(\mathcal{E}_2(\rho)), \quad \rho \in \mathcal{B}(\mathcal{H}) \tag{3}$$

are also elements of $\mathcal{C}(\mathcal{H})$.

Since $\mathcal{E}(\rho) = \sum_{i=1}^{\kappa} A_i \rho A_i^\dagger$ is a quantum channel with Kraus representation $\{A_i\}_{i=1}^{\kappa}$, let's define the adjoint of $\mathcal{E}$, denoted by $\mathcal{E}^\dagger$, as the quantum channel represented by the adjoints of the Kraus operators, i.e.,

$$\mathcal{E}^\dagger(\rho) = \sum_{i=1}^{\kappa} A_i^\dagger \rho A_i, \tag{4}$$

for all $\rho \in \mathcal{B}(\mathcal{H})$.

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It is also important to note that $\mathcal{E}$ and $\mathcal{E}^\dagger$ have identical spectra, i.e., they have the same eigenvalues.

On the other hand, suppose that $\mathcal{E}$ and $\mathcal{E}^\dagger$ are represented as in (1) and (4), respectively. So we denote by $\mathcal{E}^\dagger \circ \mathcal{E}$, the quantum channel obtained from the recursive application of $\mathcal{E}^\dagger$ and $\mathcal{E}$ with representation by Kraus operators and we can write

$$(\mathcal{E}^\dagger \circ \mathcal{E})(\rho) = \sum_{i=1}^{\kappa}\sum_{j=1}^{\kappa} A_i^\dagger A_j \rho A_j^\dagger A_i. \tag{5}$$

Note that for all $\rho \in \mathcal{B}(\mathcal{H})$, we have

$$(\mathcal{E}^\dagger \circ \mathcal{E})(\rho) = \sum_{i=1}^{\kappa}\sum_{j=1}^{\kappa} A_i^\dagger A_j \rho A_j^\dagger A_i \tag{6}$$

$$= \sum_{i=1}^{\kappa} A_i^\dagger \left(\sum_{j=1}^{\kappa} A_j \rho A_j^\dagger\right) A_i \tag{7}$$

$$= \sum_{i=1}^{\kappa} A_i^\dagger \mathcal{E}(\rho) A_i, \tag{8}$$

in other words, the recursive application $\mathcal{E}^\dagger \circ \mathcal{E}$ in the state ρ means that the received state $\mathcal{E}(\rho)$ is sent by the adjunct $\mathcal{E}^\dagger$.

III. Quantum Zero-Error Capacity

Let $\mathcal{X}$ be the set of possible input states for the quantum channel $\mathcal{E}$. If $\rho \in \mathcal{X}$, we denote by $\sigma = \mathcal{E}(\rho)$, the state received when ρ is transmitted by the quantum channel $\mathcal{E}$. Define $\mathcal{S} \subset \mathcal{X}$ to be a finite subset and let $\rho_i \in \mathcal{S}$. If Bob performs measurements using a Positive Operator-Valued Measurement (POVM) $\{M_j\}$, where $\sum_j M_j = I$. Then, define $p(j|i)$ as the probability that Bob measures j given that Alice sent the state ρ_i. Thus,

$$p(j|i) = \mathrm{tr}[\sigma_i M_j] = \mathrm{tr}[\mathcal{E}(\rho_i) M_j]. \tag{9}$$

Quantum zero-error capacity is defined for product states at the channel input. The tensor product for any n input states is called the quantum input word

$$\overline{\rho}_i = \rho_{i_1} \otimes ... \otimes \rho_{i_n}, \tag{10}$$

which belongs to a Hilbert space of dimension d^n, which we denote by $\mathcal{H}^n$.

A mapping of K classical messages $X_1, ..., X_K$ into a subset of input quantum words will be called a block quantum code of length n. The output states are called quantum code words. Thus, $\frac{1}{n}\log K$ is the rate of this code. A sequence of n output indices obtained from POVM measurements with $\{M_1, ..., M_b\}$ elements will be called a output word, $w \in \{1, ..., b\}^n$.

For a quantum code consisting of blocks of length n, a decoding scheme is a function that uniquely associates each output word with integers from 1 to K, representing the classical messages. The probability of error for this code is larger than zero if the system identifies a message other than the one sent.

The discussions in the preceding paragraphs enable us to determine the zero-error capacity of quantum channels as follows.

Definition 3.1 (Quantum Zero-Error Capacity [5]): The quantum zero-error capacity of a quantum channel $\mathcal{E}(\cdot)$, denoted by $C^{(0)}(\mathcal{E})$, is the supremum of the rates achievable with decoding error probability equal to zero,

$$C^{(0)}(\mathcal{E}) = \sup_{\mathcal{S}} \sup_{n} \frac{1}{n}\log m \tag{11}$$

where m is the maximum number of classical messages that the system can transmit without error when using a zero-error quantum block code (m, n) and the input alphabet is $\mathcal{S}$.

Two quantum states are distinguishable if, and only if, the Hilbert subspaces generated by the supports of these quantum states are orthogonal. Thus, given two quantum states $\rho_i, \rho_j \in \mathcal{S}$ with $i \neq j$, then we say that ρ_i and ρ_j are *non-adjacent* (or distinguishable) at the output of the quantum channel $\mathcal{E}$ if $\mathcal{E}(\rho_i)$ and $\mathcal{E}(\rho_j)$ belong to orthogonal Hilbert subspaces. Otherwise, we say that ρ_i and ρ_j are *adjacent* (or indistinguishable) in the output of $\mathcal{E}$.

The zero-error capacity of a quantum channel $\mathcal{E}$ is, by definition, related to the concept of distinguishability of quantum states at the channel output. Thus, a quantum channel $\mathcal{E}$ has positive zero-error capacity if, and only if, there are at least two non-adjacent quantum states [5, Proposition 1], [8, Proposition 3]. In other words, if there are two states ρ_1 and ρ_2 in which $\mathrm{tr}[\mathcal{E}(\rho_1)\mathcal{E}(\rho_2)] = 0$, then the $\mathcal{E}$ channel has positive zero-error capacity.

IV. Invariant subspace of a quantum channel

Let $W \subseteq \mathcal{H}$ be a vector subspace. We say that W is an invariant subspace of an operator $A \in \mathcal{B}(\mathcal{H})$, or A-*invariant*, if $A|\nu\rangle \in W$ for all $|\nu\rangle \in W$ [12]. We also say that W is a common invariant subspace for a set $A_1, ..., A_s \in \mathcal{B}(\mathcal{H})$, if W is A_i-invariant for all $i = 1, ..., s$, i.e., $A_i|\nu\rangle \in W$ for all $|\nu\rangle \in W$.

In studies involving invariant subspaces of linear operators, the null subspaces and the Hilbert space $\mathcal{H}$ are always invariant and are called trivial invariant subspaces. If there are others, they are called non-trivial invariant subspaces. In the field of quantum information theory, we can transpose the concept of subspace common to linear operators to quantum channels.

Definition 4.1 (Inv. Subspace of a Quantum Channel [11]): Let $\mathcal{E} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a quantum channel represented by Kraus operators $A_1, ..., A_\kappa$. A subspace $W \subseteq \mathcal{H}$ is invariant for a quantum channel $\mathcal{E}$, when W is a common invariant subspace for operators A_i with $i = 1, ..., \kappa$, i.e., $A_i|\nu\rangle \in W$ for all $|\nu\rangle \in W$.

In the definition of the invariant subspace of a quantum channel with representation in terms of the Kraus operators A_i, it is important to note that the invariance of the subspace $W \subseteq \mathcal{H}$ to the quantum channel $\mathcal{E}$ depends on the subspace W being common invariant to the operators A_i with $i = 1, ..., \kappa$. The condition that W is an invariant subspace for a quantum channel $\mathcal{E}$, can be expressed equivalently by the following properties [17], [11]:

1) $A_i P = P A_i P$ for $i = 1, ..., s$ and P is a projector onto W;
2) $\mathcal{E}(P) = P\mathcal{E}(P)P$;

3) $\text{Supp}[\mathcal{E}(\rho)] \subseteq W$ for all states ρ with $\text{Supp}(\rho) \subseteq W$ ($\text{Supp}(\rho)$ represents the support of ρ).

Lemma 4.2 ([11]): Let $\rho, \sigma \in \mathcal{B}(\mathcal{H})$, thus

$$\rho = q\sigma + (1-q)\tau \tag{12}$$

where $q \in (0,1]$ and $\tau \in \mathcal{B}(\mathcal{H})$ if, and only if, $\text{Supp}(\sigma) \subseteq \text{Supp}(\rho)$.

A proof of this result can be found in [11, Lemma 8].

Proposition 4.3 ([11]): If ρ is a fixed point of $\mathcal{E}$, then the subspace generated by $\text{Supp}(\rho)$ is an invariant subspace of $\mathcal{E}$. Furthermore, if $W \subset \mathcal{H}$ is an invariant subspace of $\mathcal{E}$, then there exists a fixed point $\rho_W \in W$ such that $\text{Supp}(\rho_W) \subset W$.

Proof: Given $|\psi\rangle \in \text{Supp}(\rho)$, then by Lemma 4.2 then there exists a probability $p > 0$ and a state σ, such that

$$\rho = p\,|\psi\rangle\langle\psi| + (1-p)\sigma. \tag{13}$$

Thus, by applying the $\mathcal{E}$ channel to Eq. (13) and using linearity, we obtain

$$\mathcal{E}(\rho) = p\mathcal{E}(|\psi\rangle\langle\psi|) + (1-p)\mathcal{E}(\sigma). \tag{14}$$

By lemma 4.2, we have that

$$\text{Supp}[\mathcal{E}(|\psi\rangle\langle\psi|)] \subseteq \text{Supp}[\mathcal{E}(\rho)]. \tag{15}$$

Now let ρ be the fixed point of $\mathcal{E}$, so $\mathcal{E}(\rho) = \rho$ and from the Eq. (15), one can obtain

$$\text{Supp}[\mathcal{E}(|\psi\rangle\langle\psi|)] \subseteq \text{Supp}(\rho). \tag{16}$$

The conclusion drawn is that $\text{Supp}(\rho)$ is an invariant subspace of $\mathcal{E}$. On the other hand, if W is an invariant subspace of $\mathcal{E}$, then considering only states of W as input states, denoted by $\mathcal{E}|_W$, we find that $\mathcal{E}|_W$ is also a channel in $\mathcal{C}(W)$. Therefore, $\mathcal{E}|_W$ has a fixed point $\rho_W \in W$, such that $\mathcal{E}|_W(\rho_W) = \rho_W$. Using the first part of this proposition and the equivalent property (3) of Definition 4.1, we can conclude that $\text{Supp}(\rho_W) \subset W$. ∎

Proposition 4.4 ([11]): Let $\mathcal{E} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a quantum channel. A subspace $W \subset \mathcal{H}$ is an invariant subspace for $\mathcal{E}$ if, and only if, $W^{\perp} = \{|\varphi\rangle \in W : \langle\varphi|\psi\rangle = 0,\ \forall\,|\psi\rangle \in W\}$ is an invariant subspace for $\mathcal{E}^{\dagger}$.

Proof Outline: Suppose the projectors P and $P^{\perp}$ of the subspaces W and $W^{\perp}$, respectively. Since W is an invariant subspace of $\mathcal{E}$, it follows that $\mathcal{E}(P) = P\mathcal{E}(P)P$, so it is possible to show that $\langle\mathcal{E}^{\dagger}(P^{\perp}), P\rangle = 0$, implying that $P\mathcal{E}^{\dagger}(P^{\perp})P = 0$. Thus, we have $\mathcal{E}^{\dagger}(P^{\perp}) = P^{\perp}\mathcal{E}^{\dagger}(P^{\perp})P^{\perp}$, implying that $W^{\perp}$ is an invariant subspace of $\mathcal{E}^{\dagger}$. The idea behind the proof of the reciprocal is analogous.

V. Zero-Error Capacity and Invariant Subspace of a Quantum Channel

In this section, we will present two results linking the quantum zero-error capacity and the invariant subspace of a quantum channel. The first result shows that if the invariant subspaces of a quantum channel intersect only trivially, this implies the positive zero-error capacity of the quantum channel.

Theorem 5.1: Let $\mathcal{E} \in \mathcal{C}(\mathcal{H})$ be a quantum channel represented by Kraus operators $A_1, ..., A_{\kappa}$. Let $W_1 \neq \{0\}$ and $W_2 \neq \{0\}$ be vector subspaces of $\mathcal{H}$ invariants of $\mathcal{E}$, such that $W_1 \cap W_2 = \{0\}$, then $C^{(0)}(\mathcal{E})$ is positive.

Proof: By hypothesis, we have that $W_1 \neq \{0\}$ and $W_2 \neq \{0\}$ are vector subspaces of $\mathcal{H}$ invariant under $\mathcal{E}$ and with the property that $W_1 \cap W_2 = \{0\}$. So by Proposition 4.3, we can say that there are distinct fixed points $\rho_{W_1} \in W_1$ and $\rho_{W_2} \in W_2$ of the quantum channel $\mathcal{E}$, such that $\text{Supp}(\rho_{W_1}) \subset W_1$ and $\text{Supp}(\rho_{W_2}) \subset W_2$. Thus, the subspaces generated by $\text{Supp}(\rho_{W_1})$ and $\text{Supp}(\rho_{W_2})$ are orthogonal subspaces [11, Corollary 1]. Therefore, for the states $\rho_{W_1} \in W_1$ and $\rho_{W_2} \in W_2$, we have that $\text{tr}[\mathcal{E}(\rho_{W_1})\mathcal{E}(\rho_{W_2})] = 0$, and this means that the zero-error capacity of the quantum channel $\mathcal{E}$ is positive. ∎

To establish the second result on the relationship between the zero-error capacity and invariant subspace of a quantum channel, we must first examine the noise commutator of a quantum channel.

If $\mathcal{E}$ is a quantum channel represented by Kraus operators $A_1, ..., A_{\kappa}$, we denote the noise commutator [17] of $\mathcal{E}$ as $\mathcal{A}'$ and define as the set of all $\mathcal{B}(\mathcal{H})$ operators that commute with A_i and $A_i^{\dagger}$, i.e.,

$$\mathcal{A}' = \{\rho \in \mathcal{B}(\mathcal{H}); [\rho, A] = 0 = [\rho, A^{\dagger}], \forall A \in \{A_i\}_{i=1}^{\kappa}\}. \tag{17}$$

For unital channels, i.e., quantum channels that fix the identity operator ($\mathcal{E}(I) = I$), all $\rho \in \mathcal{A}'$ satisfy $\mathcal{E}(\rho) = \rho$. Similarly, the adjoint channel $\mathcal{E}^{\dagger}$ also fixes the identity ($\mathcal{E}^{\dagger}(I) = I$) [11] and so for all $\rho \in \mathcal{A}'$, we have that $\mathcal{E}^{\dagger}(\rho) = \rho$. Since $\rho \in A'$, then

$$\mathcal{E}(\rho) = \sum_{i=1}^{\kappa} A_i \rho A_i^{\dagger} = \sum_{i=1}^{\kappa} \rho A_i A_i^{\dagger} = \sum_{i=1}^{\kappa} \rho\mathcal{E}(I) = \rho \tag{18}$$

and

$$\mathcal{E}^{\dagger}(\rho) = \sum_{i=1}^{\kappa} A_i^{\dagger} \rho A_i = \sum_{i=1}^{\kappa} \rho A_i^{\dagger} A_i = \sum_{i=1}^{\kappa} \rho\mathcal{E}^{\dagger}(I) = \rho. \tag{19}$$

On the other hand, the points fixed by the unit channels $\mathcal{E}$ and $\mathcal{E}^{\dagger}$ belong to $\mathcal{A}'$ [18]. In other words, the fixed points of $\mathcal{E}$ and $\mathcal{E}^{\dagger}$ are the same.

In the following, we will show that if an unital quantum channel has an invariant subspace with a dimension of at least two, then the channel has a positive zero-error capacity.

Theorem 5.2: Let $\mathcal{E} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be an unital quantum channel with Kraus operators $A_1, ..., A_{\kappa}$. If $W \subset \mathcal{H}$ is an invariant subspace of $\mathcal{E}$ with $dim\ W = s$ and $2 \leq s < d$, then $C^{(0)}(\mathcal{E})$ is positive.

Proof: We have that the Hilbert space $\mathcal{H}$ can be written as the direct sum of the invariant subspace W and the orthogonal subspace $W^{\perp}$, that is, $\mathcal{H} = W \oplus W^{\perp}$, and since $dim\ W \geq 2$, then we can say that $dim\ W^{\perp} \geq 1$. Furthermore, by hypothesis that the subspace W is invariant of $\mathcal{E}$, then by Proposition 4.4, the subspace $W^{\perp}$ is invariant of $\mathcal{E}^{\dagger}$. Also, since W is an invariant subspace of $\mathcal{E}$, we have, by Proposition 4.3, that there exists a $|\psi\rangle \in W$ which is a fixed point of $\mathcal{E}$, i.e. $\mathcal{E}(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi|$. By what we proved in equation (19), we have that $|\psi\rangle$ is also a fixed point for $\mathcal{E}^{\dagger}$, i.e., $\mathcal{E}^{\dagger}(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi|$. On the other hand, we assume that $|\phi\rangle \in W^{\perp}$, then we have that $|\psi\rangle$ and $|\phi\rangle$ are orthogonal states. Let us denote $a = |\psi\rangle\langle\psi|$ and $b = |\phi\rangle\langle\phi|$, then to prove that $C^{(0)}(\mathcal{E})$ is positive, in line with the discussion developed in Section III of this paper, it suffices to prove that

$$\mathrm{tr}(\mathcal{E}(a)\mathcal{E}(b)) = 0. \tag{20}$$

By expanding Eq. (20), we have

$$\begin{aligned}
\mathrm{tr}(\mathcal{E}(a)\mathcal{E}(b)) &= \mathrm{tr}\left(\sum_{i=1}^{\kappa} A_i\,|\psi\rangle\langle\psi|\,A_i^\dagger \sum_{j=1}^{\kappa} A_j\,|\phi\rangle\langle\phi|\,A_j^\dagger\right) && (21)\\
&= \mathrm{tr}\left[\left(\sum_{i,j=1}^{\kappa} A_j^\dagger A_i\,|\psi\rangle\langle\psi|\,A_i^\dagger A_j\right)|\phi\rangle\langle\phi|\right] && (22)\\
&= \mathrm{tr}\big[(\mathcal{E}^\dagger \circ \mathcal{E})(|\psi\rangle\langle\psi|)\,|\phi\rangle\langle\phi|\big] && (23)\\
&= \mathrm{tr}\big[(\mathcal{E}^\dagger(\mathcal{E}(|\psi\rangle\langle\psi|))\,|\phi\rangle\langle\phi|\big] && (24)\\
&= \mathrm{tr}\big[\mathcal{E}^\dagger(|\psi\rangle\langle\psi|)\,|\phi\rangle\langle\phi|\big] && (25)\\
&= \mathrm{tr}[|\psi\rangle\langle\psi|\,|\phi\rangle\langle\phi|] = 0. && (26)
\end{aligned}$$

Thus, if $W \subset \mathcal{H}$ is an invariant subspace of a unital quantum channel $\mathcal{E}$, with $dim\ W = s$ and $2 \leq s < d$, then $C^{(0)}(\mathcal{E})$ is positive. ■

The following example shows the mathematical model of a unital quantum channel, with an invariant subspace of dimension two.

Example 5.3: Let a quantum channel $\mathcal{E}$ be represented by Kraus operators A_1, A_2 and A_3 given by

$$A_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

The quantum channel $\mathcal{E}$ is unital, since $\mathcal{E}(I) = \sum_{i=1}^{3} A_i I A_i^\dagger = I$. Furthermore, the subspace

$$W = span\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\} \tag{27}$$

where $|\psi_1\rangle = (0,0,1,0,0)$ and $|\psi_2\rangle = (0,0,0,1,0)$, $|\psi_3\rangle = (0,0,0,0,1)$ has dimension three and is an invariant subspace of $\mathcal{E}$, so by the Theorem 5.2, we have that $C^{(0)}(\mathcal{E}) \geq \log 3$ is positive.

VI. Conclusion

In this paper, we explored the connection between invariant subspaces and the zero-error capacity of quantum channels. We have established two theorems to demonstrate this connection. The first theorem (Theorem 5.1) states that quantum channels with invariant subspaces that only intersect trivially have a positive zero-error capacity. The second theorem (Theorem 5.2) shows that an unital quantum channel with an invariant subspace of dimension at least two also has a positive zero-error capacity. The last result was obtained for unital channels, so it is natural to ask if it is possible to generalize it in any way. This task will be pursued in future research.

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