Simultaneous Factorization of Multiple Khatri-Rao and Kronecker Matrix Products in MIMO systems

Walter da C. Freitas Jr., Lucas Abdalah and André L. F. de Almeida

***Abstract*— In this paper, we present a singular value decomposition (SVD)-based rank-one approximation algorithm to simultaneously estimate each symbol matrix of the multiple Khatri-Rao product-based space-time (MKRST) coding scheme. We formulate this estimation problem within a multiple input multiple output (MIMO) space-time coding scenario. Our simulation results demonstrate that the proposed algorithm is capable of avoiding the error propagation inherent to the iterative closed-form solution in the context of MKRST coding. Furthermore, the algorithm maintains the same complexity as the high-order SVD (HOSVD) approach.**

***Keywords*— Tensor, PARAFAC, Khatri-Rao and Kronecker product**

I. Introduction

During the last decade the use of tensor decompositions in wireless telecommunications have been widely studied. Tensors are used to represent and analyse multidimensional data. Regarding systems where the signaling transmission could be sent at the spatial, temporal, code and/or frequency dimensions, the received signal can be modeled as a higher-order tensor (usually, third- and fourth-order tensors), where each dimension, or mode, is associated with a type of diversity present in the signal.

Among the various tensor models, the PARAFAC model stands out prominently. The utilization of the PARAFAC model is prevalent in telecommunications, particularly when the data matrix can be organized in a three-dimensional array [1]. In many instances, two of these three dimensions correspond to *space* and *time*, while the third dimension of the third-order tensor depends on the specific wireless communication system. The practical motivation for employing tensor modeling in telecommunications originates from the ability to simultaneously leverage multiple (more than two) forms of diversity for tasks such as multiuser signal separation, equalization, and channel estimation. These tasks benefit from model uniqueness conditions that are more relaxed compared to conventional matrix-based approaches.

Tensor-based receivers have been successfully used for joint symbol and channel estimation in cooperative MIMO communications. In this context, the usefulness of tensor decompositions to derive semiblind receivers has been demonstrated in several works (see, e.g., [2], [3] and references therein). Tensor-based receivers also have been proposed for one-way two-hop MIMO relaying, [4]–[6] and for multi-hop relaying [7].

Traditionally, methods proposed for fitting the PARAFAC model are iterative and rely on the alternate least square (ALS) algorithm. However, the estimated parameters are often obtained after a large number of iterations, and convergence to the global optimum is not guaranteed. To address the drawbacks of the ALS approach, such as slow convergence, susceptibility to local minima, and limited capacity to account for specific structures, non-iterative or closed-form solutions can be employed for estimating the PARAFAC factors from noisy data.

In a non-iterative approach presented in [8], the authors proposed a method for estimating the parameters of a PARAFAC decomposition. This method is applicable when at least one factor exhibits a Toeplitz structure resulting from the Khatri-Rao product of the two other matrix factors, which are assumed to have full column rank. For scenarios involving more than two matrices, this problem can be tackled iteratively by determining the factor matrices through a two-by-two search. Nevertheless, an inherent limitation of this iterative algorithm emerges when the number of factor matrices increases, mainly due to the propagation of errors, leading to a decline in its performance.

Compared with conventional least square (LS) receivers, closed-form tensor-based receivers present two main advantages: i) they avoid accumulation of channel estimation errors, and ii) they can operate under less restrictive (and more flexible) conditions on the required number of antennas, as shown in [9]. In such closed-form tensor-based receivers we may encounter situations where our observations are an estimate of a Khatri-Rao product or a Kronecker product which we would like to factorize.

To overcome this problem, we propose a least-square simultaneous SVD rank-one factorization which operates in a parallel way for estimating each symbol matrix of the factorization of Khatri-Rao (and Kronecker) products. In this work, we present a SVD-based rank-one approximation algorithm to estimate in parallel each symbol matrix of the Khatri-Rao (and Kronecker) products. We formulated this estimation problem within a MIMO space-time coding scenario. Our simulation results demonstrate that the proposed algorithm is capable of avoiding the error propagation inherent in the iterative closed-form solution when dealing with MKRST coding. Remarkably, the proposed algorithm achieves this while maintaining the same complexity as the HOSVD approach.

The paper is organized as follows. In Section II we describe the PARAFAC tensor decomposition and its application as a

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model to the MIMO space-time system. Section III describes the simultaneous SVD-based rank-one factorization and the case when one of the factors is known. Section IV presents simulation results and in section V we discuss the conclusions and perspectives.

Notation: Scalars are denoted by lower-case letters $(a, b, \ldots)$, vectors are written as boldface lower-case letters $(\mathbf{a}, \mathbf{b}, \ldots)$, matrices as boldface capitals $(\mathbf{A}, \mathbf{B}, \ldots)$, and tensors as calligraphic letters $(\mathcal{A}, \mathcal{B}, \ldots)$. $\mathbf{A}_{i.} \in \mathbb{C}^{1\times R}$ denotes the i-th row of $\mathbf{A} \in \mathbb{C}^{I\times R}$. $\mathbf{A}_{.r} \in \mathbb{C}^{I\times 1}$ is the r-th column of $\mathbf{A}$. The operator diag$(\mathbf{a})$ forms a diagonal matrix from its vector argument. The *Kruskal*-rank (k-rank) of $\mathbf{A}$, denoted by $k_{\mathbf{A}}$, is the greatest integer k such that *every* set of k columns of $\mathbf{A}$ is linearly independent. The Kronecker and Khatri-Rao products are denoted by $\otimes$ and $\diamond$, respectively:

$$\mathbf{A} \diamond \mathbf{B} = \begin{bmatrix} \mathbf{B}\text{diag}(\mathbf{A}_{1.}) \\ \vdots \\ \mathbf{B}\text{diag}(\mathbf{A}_{I.}) \end{bmatrix}, \tag{1}$$

with $\mathbf{A} = [\mathbf{A}_{.1}, \ldots, \mathbf{A}_{.R}] \in \mathbb{C}^{I\times R}$, $\mathbf{B} = [\mathbf{B}_{.1}, \ldots, \mathbf{B}_{.R}] \in \mathbb{C}^{J\times R}$. We use the superscripts $(\cdot)^T$, $(\cdot)^H$, $(\cdot)^{-1}$, $(\cdot)^\dagger$ and $(\cdot)^*$ for matrix transposition, Hermitian transposition, inversion, the Moore-Penrose pseudo inverse of matrices, and complex conjugation, respectively. A third-order tensor $\mathcal{A} \in \mathbb{C}^{I\times J\times K}$, with entries denoted as $a_{i,j,k}$, can be restructured into mode-n matrices. In this reorganization, the mode-n elements form the columns of the resulting matrix. The matrices $\mathbf{A}_{JK\times I}$, $\mathbf{A}_{KI\times J}$, and $\mathbf{A}_{IJ\times K}$ represent the tall mode-1, mode-2, and mode-3 unfoldings, respectively. Here, $a_{i,j,k}$ corresponds to $[\mathbf{A}_{JK\times I}]_{(k-1)J+j,i}$, $[\mathbf{A}_{KI\times J}]_{(i-1)K+k,j}$, and $[\mathbf{A}_{JI\times K}]_{(j-1)I+i,k}$. The operator vec$(\cdot)$ transforms a matrix into a column vector by stacking the columns of its matrix argument while the operator $\text{unvec}(\cdot)$ corresponds to the inverse transformation.

II. PARAFAC Tensor Decomposition

Let $\mathcal{X} \in \mathbb{C}^{I\times J\times K}$ be a third-order tensor with entries $x_{i,j,k}$; $i = 1, 2, \cdots, I$, $j = 1, 2, \cdots, J$ and $k = 1, 2, \cdots, K$. The third order tensor could be decomposed as

$$\mathcal{X} = \mathcal{I}_M \times_1 \mathbf{A}^{(1)} \times_2 \mathbf{A}^{(2)} \times_3 \mathbf{A}^{(3)}, \tag{2}$$

where $\mathcal{I}_M$ is a identity tensor, $\mathbf{A}^{(1)} \in \mathbb{C}^{I\times M}$, $\mathbf{A}^{(2)} \in \mathbb{C}^{J\times M}$ and $\mathbf{A}^{(3)} \in \mathbb{C}^{K\times M}$. By stacking these matrix slices, we get three unfolded matrices $\mathbf{X}_{JK\times I}$, $\mathbf{X}_{KI\times J}$ and $\mathbf{X}_{JI\times K}$ that contain all the data of the tensor $\mathcal{X}$. It can be shown that these unfolded matrices are given by:

$$\begin{aligned} \mathbf{X}_{JK\times I} &= \left(\mathbf{A}^{(3)} \diamond \mathbf{A}^{(2)}\right) \mathbf{A}^{(1)T}, \\ \mathbf{X}_{KI\times J} &= \left(\mathbf{A}^{(1)} \diamond \mathbf{A}^{(3)}\right) \mathbf{A}^{(2)T}, \\ \mathbf{X}_{JI\times K} &= \left(\mathbf{A}^{(2)} \diamond \mathbf{A}^{(1)}\right) \mathbf{A}^{(3)T}. \end{aligned} \tag{3}$$

The problem of estimating the factors of a Khatri-Rao (and Kronecker) product, as seen in Equation (3), is commonly referred to as the nearest Kronecker product (NKP) problem. Its origins can be traced back to its initial exposition in [10], and then in [3]. The fundamental concept underlying this factorization approach is that any product can be viewed as a collection of all pair-wise products of its elements. This is equivalent to a rank-one matrix constructed from the outer product of two vectors if we arrange the corresponding elements into the matrix in the correct manner. Therefore, in the presence of noise, the matrix approximates a rank-one structure. Yet, the truncated SVD provides the best rank-one approximation in the LS sense. A similar problem was also tackled in [11], and we review this problem next.

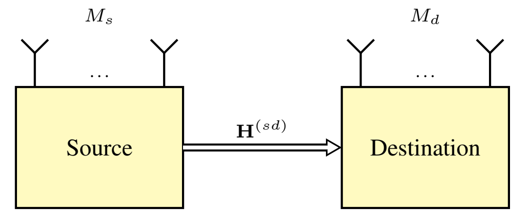

Fig. 1. MIMO system model.

Given the Khatri-Rao product $\mathbf{D} = \mathbf{A} \diamond \mathbf{B}$, the factor matrices $\mathbf{A} \in \mathbb{C}^{I\times M}$ and $\mathbf{B} \in \mathbb{C}^{J\times M}$ can be estimated by calculating the rank-one approximation of the matrix defined for each column $(m = 1, \cdots, M)$ as

$$\mathbf{F}_{.m} = \text{unvec}_{I\times J}[\mathbf{D}_{.m}] = (\mathbf{A})_{.m}(\mathbf{B})^T_{.m}. \tag{4}$$

Defining the SVD of $\mathbf{F}_{.m} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$, the m-th column of $\mathbf{A}$ and $\mathbf{B}$ are given by $(\hat{\mathbf{A}})_{.m} = \sqrt{\sigma_1}\mathbf{V}^*_{.1}$ and $(\hat{\mathbf{B}})_{.m} = \sqrt{\sigma_1}\mathbf{U}_{.1}$, where $\mathbf{U}_{.1}$ and $\mathbf{V}_{.1}$ represent the first column of $\mathbf{U}$ and $\mathbf{V}$ associated with the largest singular value σ_1 of $\mathbf{F}_{.m}$, respectively.

Note that the estimates of the factor matrices $(\mathbf{A}, \mathbf{B})$ of the Khatri-Rao product $\mathbf{A} \diamond \mathbf{B}$ are obtained up to a scalar scaling factor for each column $m_{\mathbf{S}} = 1, \cdots, M$. Therefore, to eliminate these scaling ambiguities, one needs to know one element for each column, i.e. one row of $\mathbf{A}$ or $\mathbf{B}$, the *a priori* knowledge of the first row of each $\mathbf{S}_q$ for $q = 1 \cdots, Q$, is needed to carry out the MKRST decoding without ambiguity.

In cases where the product involves more than two matrices, such as when $Q > 2$, let $\mathbf{A} = \mathbf{A}_1 \diamond \mathbf{A}_2 \diamond \cdots \diamond \mathbf{A}_Q$. By associating $\mathbf{A} = \mathbf{A}_1$ and $\mathbf{B} = \mathbf{A}_{2:Q}$, this process can be iteratively repeated to estimate all the factors $\mathbf{A}_q$ $(q = 1, 2, \ldots, Q)$ of the product.

A. *Example of a PARAFAC Tensor Decomposition – MIMO MKRST Systems*

We consider a one-way MIMO system, assuming a simplified Khatri-Rao space-time (KRST) coding consisting of a time spreading of the symbol matrix by means of a code matrix $\mathbf{C} \in \mathbb{C}^{P\times M_s}$, where P is the spreading length. The system is illustrated by means of Fig. 1, where M_s and M_d denote the numbers of antennas at the source and destination nodes, with $(M_s, M_d) \geq 2$. The source-destination channel, $\mathbf{H}^{(sd)} \in \mathbb{C}^{M_d\times M_s}$ is assumed to be Rayleigh flat-fading and quasi-static during the transmission protocol. Let $\tilde{\mathcal{X}} = \mathcal{X} + \mathcal{N}$ be the noisy received signal tensor, the entries of the noise tensor $\mathcal{N}$ being zero-mean circularly-symmetric complex-valued Gaussian random variables.

The source transmits the MKRST encoded symbols proposed in [6] and defined as multiple Khatri-Rao products of $Q \geq 2$ symbol matrices $\mathbf{S}_q \in \mathbb{C}^{N_q \times M_s}$ with $q = 1, \cdots, Q$, i.e., $\mathbf{S} = \mathbf{S}_1 \diamond \cdots \diamond \mathbf{S}_q \diamond \cdots \diamond \mathbf{S}_Q = \overset{Q}{\underset{q=1}{\diamond}} \mathbf{S}_q \in \mathbb{C}^{N \times M_s}$, with $N = \prod_{q=1}^{Q} N_q$. The signals received at the destination are given by

$$\tilde{\mathbf{X}}^{(sd)}_{M_d \times PN} = \mathbf{H}^{(sd)} (\mathbf{C} \diamond \mathbf{S})^T + \mathbf{N}_{M_d \times PN}, \tag{5}$$

where $\mathbf{N}_{M_d \times PN}$ is the noise term added at the relay.

These signals define a third-order tensor $\tilde{\mathcal{X}}^{(sd)} \in \mathbb{C}^{M_d \times P \times N}$ which satisfies a PARAFAC model $\|\mathbf{H}^{(sd)}, \mathbf{C}, \mathbf{S}; M_s\|$, with $\mathbf{S} = \overset{Q}{\underset{q=1}{\diamond}} \mathbf{S}_q$ on the coding used at the source.

Assume the code matrix $\mathbf{C}$ have a truncated discrete Fourier transform (DFT) structure and is known at the destination node. We derive a closed-form receiver for jointly estimating the individual channel ($\mathbf{H}^{(sd)}$) and the transmitted symbols ($\mathbf{S}_q$, $q = 1, \cdots, Q$).

The transmitted symbol matrix $\mathbf{S}$ can be estimated at the destination using the following tall 3-mode unfolding of $\tilde{\mathcal{X}}^{(sd)}$ by permuting the matrix factors

$$\tilde{\mathbf{X}}^{(sd)}_{M_d N \times P} = \left(\mathbf{H}^{(sd)}_{M_d \times M_s} \diamond \mathbf{S} \right) \mathbf{C}^T + \mathbf{N}_{M_d \times PN}. \tag{6}$$

The source code matrix $\mathbf{C}$ being assumed column-orthonormal ($\mathbf{C}^T \mathbf{C}^* = \mathbf{I}_{M_s}$), which implies $P \geq M_s$, a LS estimate of the Khatri-Rao product $\mathbf{R} = \mathbf{H}^{(sd)}_{M_d \times M_s} \diamond \mathbf{S}$ is given by

$$\hat{\mathbf{R}} = \tilde{\mathbf{X}}^{(sd)}_{M_d N \times P} \mathbf{C}^* \in \mathbb{C}^{M_d N \times M_s}. \tag{7}$$

Once $\mathbf{R}$ is estimated, the factor matrices $(\mathbf{H}^{(sd)}_{M_d \times M_s}, \mathbf{S})$ of the Khatri-Rao product can be obtained by applying the SVD rank-one factorization. This approach avoids the error propagation that occurs when Equation (4) is applied in a two-by-two search.

III. Simultaneous SVD Rank-one Factorization

In the general case we have, $\mathbf{S}_q \in \mathbb{C}^{N_q \times M_s}$ (or $\mathbf{S}_q \in \mathbb{C}^{N_q \times M_{s_q}}$), with $q = 1, \cdots, Q$, such that $\hat{\mathbf{S}} = \overset{Q}{\underset{q=1}{\diamond}} \mathbf{S}_q$ (or $\hat{\mathbf{S}} = \overset{Q}{\underset{q=1}{\otimes}} \mathbf{S}_q$). The decoding problem consists in estimating the sub-matrices $\mathbf{S}_q \in \mathbb{C}^{N_q \times M_s}$ (or $\mathbf{S}_q \in \mathbb{C}^{N_q \times M_{s_q}}$), with $q = 1, \cdots, Q$, such that $\hat{\mathbf{S}} = \overset{Q}{\underset{q=1}{\diamond}} \mathbf{S}_q$. This problem can be solved iteratively by determining the symbol matrices $\mathbf{S}_q$ with a two-by-two search. This basic algorithm was proposed in [8] for estimating two matrix factors of a Khatri-Rao product associated with a third-order PARAFAC model. This solution is named here as successive approach. A drawback of this iterative algorithm is that its performance degrades when Q increases, due to error propagation.

To overcome this problem, we propose a decoding procedure which operates in a parallel way for estimating each symbol matrix. Before presenting this procedure, let us recall the following formula for permuting the matrix factors ($\mathbf{A} \in \mathbb{C}^{I \times R}, \mathbf{B} \in \mathbb{C}^{J \times S}$) of a Kronecker product

$$\mathbf{A} \otimes \mathbf{B} = \mathbf{\Pi}_{I,J} \left(\mathbf{B} \otimes \mathbf{A} \right) \mathbf{\Pi}_{S,R} \tag{8}$$

where $\mathbf{\Pi}_{I,J}$ and $\mathbf{\Pi}_{S,R}$ are two permutation matrices of dimensions $(IJ \times JI)$ and $(SR \times RS)$, respectively, defined as

$$\begin{aligned} \mathbf{\Pi}_{I,J} &= \sum_i \sum_j \left(\mathbf{e}_i^{(I)} \mathbf{e}_j^{(J)^T} \right) \otimes \left(\mathbf{e}_j^{(J)} \mathbf{e}_i^{(I)^T} \right), & (9) \\ \mathbf{\Pi}_{S,R} &= \sum_s \sum_r \left(\mathbf{e}_s^{(S)} \mathbf{e}_r^{(R)^T} \right) \otimes \left(\mathbf{e}_r^{(R)} \mathbf{e}_s^{(S)^S} \right), & (10) \end{aligned}$$

$\mathbf{e}_j^{(J)}$ being the j-th canonical basis vector of the Euclidean space $\mathbb{R}^J$.

To illustrate the proposed decoding procedure, consider the case $Q = 4$, with $\mathbf{S} = \mathbf{S}_1 \otimes \mathbf{S}_2 \otimes \mathbf{S}_3 \otimes \mathbf{S}_4$, denoted as $\mathbf{S}_{(1,2,3,4)}$ for simplicity. The matrices $\mathbf{S}_1$ and $\mathbf{S}_4$ can be estimated by applying the following two decompositions of $\mathbf{S}$

$$\begin{aligned} \mathbf{S} &= \mathbf{S}_1 \otimes \mathbf{S}_{(2,3,4)} \;\text{ with }\; \mathbf{S}_{(2,3,4)} = \mathbf{S}_2 \otimes \mathbf{S}_3 \otimes \mathbf{S}_4, & (11) \\ \mathbf{S} &= \mathbf{S}_{(1,2,3)} \otimes \mathbf{S}_4 \;\text{ with }\; \mathbf{S}_{(1,2,3)} = \mathbf{S}_1 \otimes \mathbf{S}_2 \otimes \mathbf{S}_3. & (12) \end{aligned}$$

For estimating $\mathbf{S}_2$, we use the following equation obtained by permuting the factors $\mathbf{S}_1$ and $\mathbf{S}_2$

$$\begin{aligned} \mathbf{S}_{(2,1,3,4)} &= \mathbf{\Pi}_{N_2,N_1} \left(\mathbf{S}_1 \otimes \mathbf{S}_2 \right) \mathbf{\Pi}_{M_{s_1},M_{s_2}} \otimes \mathbf{S}_3 \otimes \mathbf{S}_4 \\ &= \underbrace{\left(\mathbf{\Pi}_{N_2,N_1} \otimes \mathbf{I}_{N_3 N_4} \right)}_{\mathbf{\Pi}_2^{\text{row}} - \text{row permutation}} \mathbf{S} \underbrace{\left(\mathbf{\Pi}_{M_{s_1},M_{s_2}} \otimes \mathbf{I}_{M_{s_3} M_{s_4}} \right)}_{\mathbf{\Pi}_2^{\text{column}} - \text{column permutation}} \\ &= \mathbf{S}_2 \otimes \mathbf{S}_{(1,3,4)}. \end{aligned} \tag{13}$$

Applying such idea allows to estimate $\mathbf{S}_2$ and $\mathbf{S}_{(1,3,4)}$. Similarly, by permuting $\mathbf{S}_3$ with $\mathbf{S}_1 \otimes \mathbf{S}_2$, we obtain

$$\begin{aligned} \mathbf{S}_{(3,1,2,4)} &= \underbrace{\left(\mathbf{\Pi}_{N_3,N_1 N_2} \otimes \mathbf{I}_{N_4} \right)}_{\mathbf{\Pi}_3^{\text{row}}} \mathbf{S} \underbrace{\left(\mathbf{\Pi}_{M_{s_1} M_{s_2},M_{s_3}} \otimes \mathbf{I}_{M_{S_4}} \right)}_{\mathbf{\Pi}_3^{\text{column}}} \\ &= \mathbf{S}_3 \otimes \mathbf{S}_{(1,2;4)}. \end{aligned} \tag{14}$$

In summary, for $Q = 4$, we can use Eqs. (11), (13), (14) and (12) to estimate $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$ and $\mathbf{S}_4$ simultaneously. Such an approach can be generalized to any Q. Each symbol matrix $\mathbf{S}_q$ is estimated by applying the following equation

$$\mathbf{S}_q \otimes \mathbf{S}_1 \cdots \otimes \mathbf{S}_{q-1} \otimes \mathbf{S}_{q+1} \otimes \cdots \otimes \mathbf{S}_Q = \mathbf{\Pi}_q^{\text{row}} \mathbf{S} \mathbf{\Pi}_q^{\text{column}}, \tag{15}$$

where

$$\begin{aligned} \mathbf{\Pi}_q^{\text{row}} &= \mathbf{\Pi}_{N_q, N_1 \cdots N_{q-1}} \otimes \mathbf{I}_{N_{q+1} \cdots N_Q}, & (16) \\ \mathbf{\Pi}_q^{\text{column}} &= \mathbf{\Pi}_{M_{s_1} \cdots M_{s_{q-1}}, M_{s_q}} \otimes \mathbf{I}_{M_{s_{q+1}} \cdots M_{s_Q}}. & (17) \end{aligned}$$

Besides avoiding error propagation, another advantage of this approach is that the estimation of the symbol matrices $\mathbf{S}_q$, $q = 1, \cdots, Q$, can be parallelized. In the case of the MKRST coding where $\mathbf{S} = \mathbf{S}_1 \diamond \cdots \diamond \mathbf{S}_q \diamond \cdots \diamond \mathbf{S}_Q$, the same procedure can be applied without column permutation.

IV. Simulation Results

Simulation results are provided to evaluate the performance of the proposed simultaneous SVD rank-one factorization in terms of bit error rate (BER) and complexity which are plotted as a function of the symbol energy to noise spectral density ratio (E_s/N_0). Each BER curve represents an average over at least 4×10^4 Monte Carlo runs. Each run corresponds to a different realization of the channels, transmitted symbols and noise. The symbols are randomly drawn from a unit energy quadrature phase-shift keying (QPSK) alphabet. The number of data symbols and antennas are $N_q = M_s = M_d = 2$. Recall that the code matrices are DFT matrices.

Figure 2 compares the BER performance per layer ($\mathbf{S}_q$) of the proposed simultaneous approach for the cases in which the MKRST has $Q = 3$. We assume that the first row is composed of all ones in the symbol matrix to enable ambiguities elimination. As a reference for comparison, we present the performance of the successive approach, wherein the estimation of $\mathbf{S}_q$ is conducted through a two-by-two search and utilizing the HOSVD algorithm [1]. The experiment demonstrates that, in the sequential decoding approach, errors introduced in earlier stages can have a cascading effect on subsequent estimations as the decoding process progresses. Layers that are decoded earlier are particularly vulnerable to errors because they depend on estimates derived from previous factorization steps. Consequently, these errors have the potential to accumulate and exacerbate as decoding proceeds through subsequent layers, ultimately impacting its BER.

Figure 3 compares the BER performance of the proposed simultaneous approach for the cases of the MKRST has $Q = 4$. As a reference for comparison, we show the performance of the successive approach, where the estimation of $\mathbf{S}_q$ are done in a two-by-two search and the HOSVD algorithm. Differently of the case which the Khatri-Rao factorization is done successively, doing the factorization simultaneously as proposed, we are able to avoid the error propagation improving the whole BER performance achieving a result very close to the HOSVD. Once Q increases, also the gap between the successive and simultaneous approaches due to the error propagation.

Figure 4 compares the computational complexity of the three approaches in terms of the number of antennas at the source (M_s). The dominant complexity cost is associated with the SVD-based rank-one approximations to compute the factors of the Khatri-Rao matrix products. Note that, for a matrix of dimensions $J \times K$, the complexity of its SVD is $\mathbb{O}(\min(J, K)JK)$ [12]. Therefore, the computational complexity of the three approaches when $Q = 3$ are:

$$\begin{aligned} &\mathbb{O}(\min(N_1, N_2N_3)N_1N_2N_3M_s) + \ldots \\ &\mathbb{O}(\min(N_2, N_3)N_2N_3M_s) \end{aligned} \tag{18}$$

for the successive case, equation 18,

$$\begin{aligned} &\mathbb{O}(\min(N_1, N_2N_3)N_1N_2N_3M_s) + \ldots \\ &\mathbb{O}(\min(N_1N_2, N_3)N_1N_2N_3M_s) + \ldots \\ &\mathbb{O}(\min(N_1N_3, N_2)N_1N_3N_2M_s) \end{aligned} \tag{19}$$

for the simultaneous, equation 19, and

$$\begin{aligned} M_s(&\mathbb{O}(\min(N_1, N_2N_3)N_1N_2N_3) + \ldots \\ &\mathbb{O}(\min(N_1N_2, N_3)N_1N_2N_3) + \ldots \\ &\mathbb{O}(\min(N_1N_3, N_2)N_1N_3N_2)) \end{aligned} \tag{20}$$

for the HOSVD, equation 20.

We can observe that as the number of antennas at the source (M_s) increases, the computational complexity impact becomes higher in the successive approach due to the lower-dimensional SVD computations performed in each step. However, the successive approach is prone to error propagation. On the other hand, the computational complexity of the simultaneous approach and HOSVD is nearly identical.

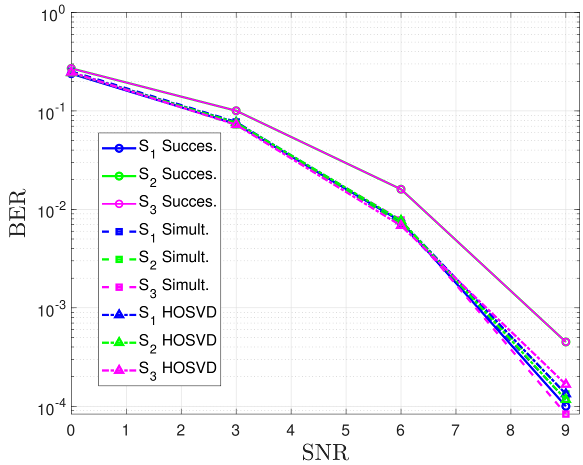

Fig. 2. BER performance per each matrix of the MKRST $Q = 3$ coding scheme.

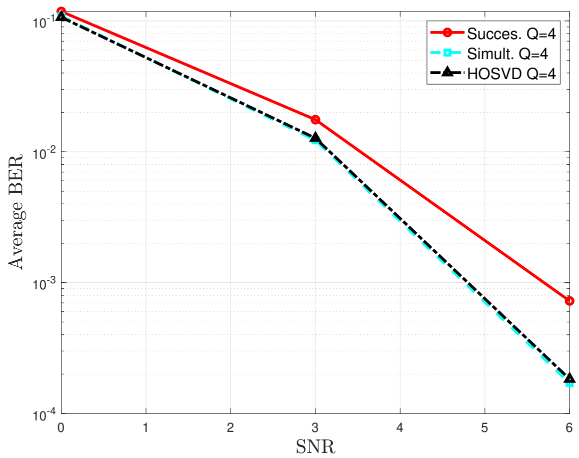

Fig. 3. Average BER performance vs. SNR.

V. Conclusions

The MKRST coding is a valuable tool in signal processing, particularly applicable to multichannel systems such as

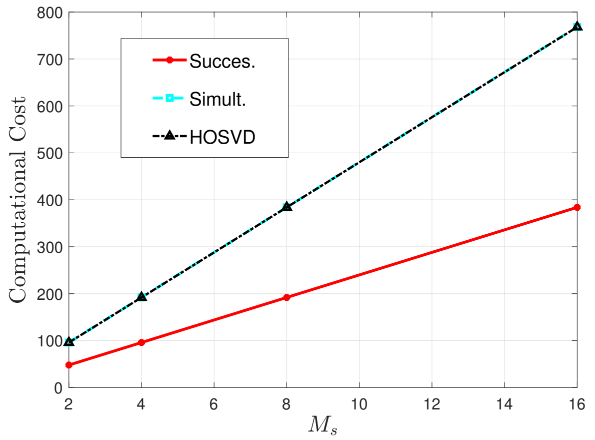

Fig. 4. Complexity comparison vs. number of antennas.

MIMO and array processing. Nevertheless, existing methods for simultaneous estimation of each symbol matrix in MKRST coding suffer from limitations regarding complexity performance and error propagation. To overcome these challenges, this work proposes a SVD-based rank-one factorization technique for MIMO space-time coding scenarios. The proposed technique's performance is extensively evaluated through simulations and compared with a state-of-the-art method, focusing on two crucial aspects: BER and computational cost, specifically in relation to the number of source antennas.

The main findings of the study reveal that the simultaneous approach and HOSVD demonstrate similar BER performance, while the successive approach may accumulate errors in earlier stages, resulting in slightly inferior overall BER performance. However, the successive approach provides a significant advantage in terms of computational complexity compared to both the simultaneous approach and HOSVD, particularly as the number of antennas increases. These outcomes highlight a trade-off between error propagation and computational cost. In the presented experiments, the successive approach yielded slightly higher overall BER, but its computational complexity was reduced by half compared to the other approaches.

Despite the promising results and contributions of this study, it is important to acknowledge its limitations. Firstly, the evaluation of the proposed technique was limited to simulated scenarios, which may not fully capture the complexities and variations present in real-world environments. Furthermore, the comparison of the proposed approach was primarily focused on a specific state-of-the-art method. Considering a broader range of comparative methods would offer a more comprehensive understanding of the proposed technique's strengths and weaknesses, further enhancing its applicability and performance assessment. Addressing these limitations through future research would contribute to a more comprehensive and robust understanding of the proposed technique's effectiveness.

In conclusion, the simulation results provide substantial evidence supporting the effectiveness of the proposed simultaneous techniques. It successfully mitigates the adverse impact of error propagation on BER performance, outperforming the successive approach. Moreover, the simultaneous approach achieves BER performance comparable to that of the HOSVD while maintaining a similar computational complexity. Although the successive approach exhibits lower computational complexity, it comes at the cost of increased BER. These findings emphasize the potential of the proposed approach in improving the performance of communication systems, particularly in scenarios with a higher number of symbols.

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