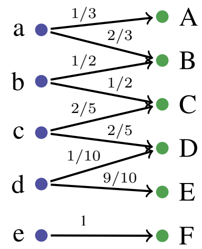
New Bound for Classical Zero-Error Capacity using Partially Commutative Monoids as Counting Tools

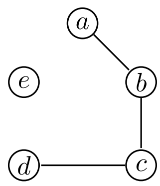
Andresso da Silva, Francisco Marcos de Assis

***Abstract*— In this paper we propose a new bound for the classical zero-error capacity of a communication channel using partially commutative monoids as enumeration tools. Specifically, we analyze the relationship between classical zero-error capacity and the growth factor of the monoid, $\beta(G)$, of a graph G. Although the value of $\beta(G)$ allows us to calculate an upper bound for the classical zero-error capacity, determining it requires counting the number of cliques in a graph, which is an NP-Complete problem. Our main result is that the classical zero-error capacity is always lower than the integer part of $\beta(G)$.**

***Keywords*— Partially Commutative Monoid, Zero-Error Capacity, Partial Order, Cliques.**



(a) DMC representation.



(b) DMC adjacency graph of Fig.2a.

Fig. 2: DMC and Adjacency Graph.

I. Introduction

Shannon [1] defined the zero-error capacity of a discrete memoryless channel (DMC) as the highest information transmission rate at which the error probabilitiy in decoding equals zero. A DMC (see Fig. 1) is defined by using a finite transition matrix $[p(j|i)]$ where each element $p(j|i)$ is the probability of receiving the symbol j given that symbol i was sent. In addition, the probabilities of sending the symbols i and j are independent.

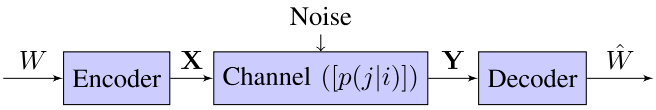


Fig. 1: Communication system.

Symbols i and j are said to be *confusable* in decoding if there exists an output symbol t such that $p(t|i) > 0$ and $p(t|j) > 0$. The error probability is the probability of noise causing the output symbol to be decoded incorrectly. One can define the channel adjacency graph G by considering the error probability. In this graph, the input symbols compose the vertices set, and two symbols u and v are connected if they can be confusable. In Fig. 2 are depicted a DMC and its adjacency graph, where $a, \ldots, e$ and $A, \ldots, F$ are the input and output symbols, respectively.

Even for small adjacency graphs, the zero-error capacity can be difficult to find. In general, zero-error capacity is not computable [7]. For example, the zero-error capacity of the pentagon remained unknown until Lovász [2] defined $\vartheta(G)$, known as Lovász's ϑ function. This function allows one to calculate the zero-error capacity of the pentagon indirectly. The zero-error capacity and the Lovász ϑ function continue to be studied in various contexts [3], [4], [5], [6].

Cartier and Foata [8] investigated combinatorial properties of sequences defined in finite alphabets where elements a and b can have the property that ab and ba are equivalent. These sequences are called partially commutative monoids (PCM). When two elements have this property, they are said to commute, and this commutativity allows some sequences of symbols to be equivalent to others, forming equivalence classes. The commutativity graph represents the commutativity relations, where commuting symbols are connected by an edge.

Fisher [14] showed that determining the number of equivalence classes is an NP-Complete problem because it depends on the number of cliques of the commutativity graph. The number of cliques defines the dependence polynomial where the inverse of its smallest real root is the growth factor of the monoid, β.

By employing the theory of partially commutative monoids (PCM), in this paper, we establish the relationship between the growth factor of the monoid β and the independence number of the graph α, for both computing is an NP-complete problem. From this, we establish an upper bound for the zero-error capacity of DMC channels.

In the Section II are presented the essential concepts involving graphs, zero-error capacity, and partially commutative monoids. In the Section III the main results are introduced. Section IV concludes the paper.

II. Fundamentals

A. Graph Theory

An undirected simple graph is a pair $G = (V, E)$ where V is called the vertex set and $E \subset V \times V$ is called the edge set. In G multiple edges connecting two vertices are not allowed, nor are edges leaving and arriving at the same vertex. Two

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vertices $a, b \in V$ are adjacent if $ab \in E$ and are called non-adjacent otherwise. The complement of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$ with the same set of vertices V, where any two distinct vertices in V are adjacent in $\overline{G}$ if and only if they are non-adjacent in G.

A stable set of a graph G consists of vertices $S \subseteq V$ such that no pair of elements belonging to S are adjacent. A stable set is maximal if it is not a subset of another stable set. Among the maximal stable sets of a graph, the cardinality of the largest one is called *independence number*, denoted by $\alpha(G)$.

A *clique* of a graph G is a set of mutually adjacent vertices $C \subseteq V$. A clique is maximal if it is not a subset of another clique. The cardinality of the largest clique is called *clique number*, $\omega(G)$. Thus, the vertex set C is a clique of G if and only if C is a stable set of the complement of G, i.e., $\omega(G) = \alpha(\overline{G})$. Finding the clique and the independence number are NP-Complete problems [10].

A k-coloring of a graph is the assignment of k colors to its vertices. A *coloring is proper* when no pair of adjacent vertices has the same color. The k-coloring can be understood as the partitioning of V into $\{V_1, V_2, \ldots, V_k\}$, where V_i denotes the sets (possibly empty) of vertices that have color i, also called color classes. The minimum k value of colors needed to make a proper coloring of G is called *chromatic number*, denoted by $\chi(G)$. In a graph with the largest clique having $\omega(G)$ mutually connected vertices, at least $\omega(G)$ colors are required, that is, $\chi(G) \geq \omega(G)$. Perfect graphs are those in which $\chi(G) = \omega(G)$.

The *strong product* of two graphs G and H is denoted by $G \boxtimes H$ and the strong product of k copies of $G = (V, E)$ is denoted by G^k. Each vertex of G^k represents a word of length k defined in V. Two vertices (words) $\mathbf{u} = (u_1, u_2, \ldots, u_k)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_k)$ of G^k are connected by an edge if there exists at least one pair u_i and v_i such that $u_i v_i \in E$ and this two words are called *confusable*.

It is possible to select the largest number of words of length k such that they are not mutually adjacent so that there is no error in decoding. These words correspond to the largest stable set of G^k, and the number of words corresponds to the independence number of G^k, $\alpha(G^k)$. Furthermore, it is known that $\alpha(G)^k \leq \alpha(G^k)$ [2].

B. Classical Zero-Error Capacity

The zero-error transmission capacity associated with the graph G is defined by some authors, e.g., Lovász [2] as

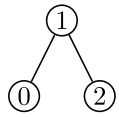
$$\Theta(G) \triangleq \sup_{n} \sqrt[n]{\alpha(G^n)} = \lim_{n \to \infty} \sqrt[n]{\alpha(G^n)}, \tag{1}$$

and is also called Shannon Capacity. However, it is interesting to recall that Shannon himself has defined the zero-error capacity as $C_0 = \log_2 \Theta(G)$. In this paper, by convenience, we utilize (1) to refer to the zero-error capacity.

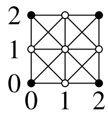
Shannon [1] demonstrated (see Theorem 1) that if the graph G is perfect, then $\alpha(G^k) = \alpha^k(G)$ and therefore, in that case, $\Theta(G) = \sup_k \sqrt[k]{\alpha(G^k)} = \alpha(G)$. However, this does not hold for most graphs, even the simplest ones, as in the pentagon.

Theorem 1 ([1]): If G is a perfect graph, then $\Theta(G) = \alpha(G)$.

To illustrate how the strong product is applied to adjacency graphs, consider the case with the adjacency channel shown in Fig. 3a, where $A = \{0, 1, 2\}$, then the possible words of length $k = 2$ are $V(G^2) = \{00, 01, 02, 10, 11, 12, 20, 21, 22\}$ (Fig. 3b). One can verify that 00 is adjacent to 01, 10, and 11 but not adjacent to 12, 02, 21, 20, or 22. Similarly, 01 is adjacent to 00, 02, 11, 10, and 12 but not to 20, 21, or 22. By proceeding with the analysis, we arrive at the graph G^2 shown in Fig. 3b. The largest stable set of G^2 is represented by the black vertices in Fig. 3b and contains 00, 02, 20, and 44, so $\alpha(G^2) = 4$. Thus there exist at most 4 words of length $k = 2$ that can be transmitted without error in decoding.



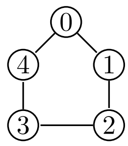
(a) Adjacency graph G of a channel with 3 input symbols.



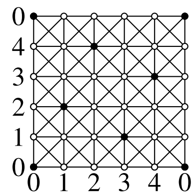
(b) Resulting graph G^2 of the strong product of $k = 2$ copies of G.

Fig. 3: Example of a strong single-channel product with 3 input symbols.

As another example, consider a channel with the pentagon adjacency graph $G = C_5$ shown in Fig. 4a, the alphabet is $A = \{0, 1, 2, 3, 4\}$, and each symbol i can be confused with $i-1$ or $i+1$(mod 5). Vertex 00 is adjacent to vertices 01, 04, 10, 11, 14, 40, 41 and 44. Vertex 01 is adjacent to vertices 00, 02, 10, 11, 12, 40, 41 and 42. Following this procedure, we obtain the graph G^2 shown in Fig. 4b. The largest stable set of G^2 is composed of 00, 12, 24, 31, and 43, thus $\alpha(G^2) \geq 5$. Thus, only these 5 words of length 2 can be transmitted without error.



(a) Adjacency graph C_5.



(b) Graph C_5^2.

Fig. 4: Example pentagon graph C_5.

The zero-error capacity of the pentagon remained unknown until Lovász [2] presented the function that is named after him. This function corresponds to an upper bound for the zero-error capacity and uses the so-called orthonormal representation of the adjacency graph.

Definition 1 (Orthonormal Representation): Let $G = (V, E)$ be a adjacency graph on k vertices and let $i, j \in V$ be two vertices. The orthonormal representation of G is the mapping of vertices onto unit vectors $\mathbf{u_i} \in \mathbb{R}^n, i \in V$ such that

$$\mathbf{u}_i^T \mathbf{u}_j = \begin{cases} 0, & \text{if } ij \notin E \\ \epsilon, & \text{if } ij \in E, \epsilon > 0 \end{cases} \tag{2}$$

Definition 2 (Lovász Theta Function [2]):

$$\vartheta(G) = \min_{\mathbf{c}} \max_{1\leq i\leq k} \frac{1}{(\mathbf{c}^T\mathbf{u}_i)^2} \tag{3}$$

where $\mathbf{c}$ is a unit vector in $\mathbb{R}^k$ and $\mathbf{u_i}$ is an orthonormal representation of G in $\mathbb{R}^k$.

By employing Definition 2, Lovász proved that the capacity of the pentagon is exactly $\Theta(C_5) = \sqrt{5}$. Knuth [11] demonstrated that $\alpha(G) \leq \vartheta(G) \leq \chi(\overline{G})$, where $\alpha(G)$ and $\chi(\overline{G})$ both have NP-complete complexity. For perfect graphs, $\omega(G) = \chi(G)$, then $\alpha(G) = \vartheta(G)$.

C. Partially Commutative Monoids

Let Σ be a finite alphabet and Σ^* be the set of all finite words formed by the elements of Σ, including the empty word ε. The set Σ^* is called the free monoid generated by Σ. Furthermore, let Σ^n be the set of all words of length n defined in the alphabet Σ. Throughout this paper, word and sequence are used as synonyms.

Let $G = (V, E)$ be an undirected simple graph called *commutativity graph*. The elements of V are associated with a finite alphabet Σ by means of a bijective function $\lambda : V \mapsto \Sigma$ and we use Σ or V indistinctly if there is no risk of confusion. If the vertices $x, y \in V$ are adjacent they commute. Two symbols $x, y \in V$ commute when xy is equivalent in some way to yx, represented by $xy \equiv_G yx$. If they do not commute, the relation is represented by $xy \not\equiv_G yx$. The complement of the commutativity graph, $\overline{G} = (V, \overline{E})$, is called the non-commutativity graph and it is a simple graph in which vertices connected by an edge are associated with symbols that *do not* commute.

Definition 3 (Partially Commutative Monoid): The partially commutative monoid $\mathcal{M}(\Sigma, G) = \Sigma^* \backslash \equiv_G$ is the quotient of the free monoid Σ^* by the congruence relation $\equiv_G$.

Two words $\mathbf{u}, \mathbf{v} \in \Sigma^*$ are equivalent if one can be obtained from the other by swapping the positions of the consecutive symbols that commute according to $\equiv_G$. If two words are equivalent, they belong to the same equivalence class and are called congruent. A word $\mathbf{u} \in \Sigma^*$ belongs to a class in $\mathcal{M}(\Sigma, G)$, so, with some abuse of notation, one can say that $\mathbf{u} \in \mathcal{M}(\Sigma, G)$.

Definition 4 (Equivalence Class): Let $\mathcal{E}_G(\mathbf{u})$ be the set of words equivalent to a word $\mathbf{u} \in \mathcal{M}(\Sigma, G)$ according to the relation $\equiv_G$. The set $\mathcal{E}_G(\mathbf{u})$ is called the equivalence class of $\mathbf{u}$.

Definition 5: (Word Projection [12]) For any subset A of the alphabet Σ and any word $\mathbf{w}$ defined in Σ^*, the projection $\pi_A(\mathbf{w})$ of the word $\mathbf{w}$ into A is obtained by deleting from $\mathbf{w}$ all symbols that are not present in A.

Theorem 2: (Perrin [12, p.330]) The necessary and sufficient conditions for two words $\mathbf{w}, \mathbf{u}$ to be congruent (i.e., to be in the same equivalence class) is that they have the same frequency of occurrence of symbols (the same type) and that $\pi_{\{x,y\}}(\mathbf{w}) = \pi_{\{x,y\}}(\mathbf{u})$ for all symbols $xy \in \overline{E}$.

The number $\tau_G(n)$ of equivalence classes of length n considering the relation $\equiv_G$ corresponds to the number of words $\mathbf{u}_i, i = 1, 2, \dots, \tau_G(n)$ defined in the monoid $\mathcal{M}(\Sigma, G)$ and of length n that are pairwise non-congruent. Thus, representing the subset of classes of the monoid $\mathcal{M}(\Sigma, G)$ that have length n by $\mathcal{M}^n(\Sigma, G) = \Sigma^n \backslash \equiv_G$, then

$$\mathcal{M}^n(\Sigma, G) = \mathcal{E}_G(\mathbf{u}_1) \cup \mathcal{E}_G(\mathbf{u}_2) \cup \cdots \cup \mathcal{E}_G(\mathbf{u}_{\tau_G(n)}), \tag{4}$$

in which $\mathcal{E}_G(\mathbf{u}_i) \cap \mathcal{E}_G(\mathbf{u}_j) = \emptyset$ for $i \neq j$, $|\mathbf{u}_i| = n, i = 1, 2, \dots, \tau_G(n)$ and $\tau_G(n) = |\mathcal{M}^n(\Sigma, G)|$.

Fisher [13], [14] developed methods for determining the number $\tau_G(n)$ of equivalence classes of length n based on the theory of partially commutative monoids presented by Cartier and Foata [8]. The main tools for determining $\tau_G(n)$ are the dependence polynomial of a commutativity graph G and the monoid generating function [14].

Definition 6: (Dependence Polynomial [14]) The dependence polynomial of the commutativity graph G is defined by

$$D(G, z) = \sum_{k=0}^{\omega} (-1)^k c_k z^k, \tag{5}$$

where c_k denotes the number cliques of size k in the graph G and ω is the clique number of G.

From the definition of dependence polynomial, it is also useful to define the class of graphs that have the same dependence polynomial.

Definition 7 (Dependence Class of a Graph): Let G be a graph. The dependence class $\mathcal{D}(G)$ is formed by all graphs $G_1, \dots, G_m$ that have the same dependence polynomial $D(G, z)$ as G.

The generating function $\zeta_{\mathcal{M}}$ of the monoid $\mathcal{M}(\Sigma, G)$ can be used to analyze the asymptotic behavior of the sequences $\{\tau_G(n)\}_{n\geq 0}$.

Definition 8: (Monoid Generating Function) The generating function of a monoid $\mathcal{M}(\Sigma, G)$ is defined as

$$\zeta_{\mathcal{M}}(z) = \sum_{n=0}^{\infty} \tau_G(n) z^n. \tag{6}$$

It is possible to obtain an expression for $\tau_G(n)$ via the dependence polynomial as presented in Corollary 1.

Corollary 1: [14, p.251] The generating function $\zeta_{\mathcal{M}}(z)$ can be obtained by

$$\zeta_{\mathcal{M}}(z) D(G, z) = 1. \tag{7}$$

III. Upper Bound for Classical Zero-Error Capacity

Partially commutative monoids (PCM) were initially developed to analyze concurrent systems. The congruence relation $ab \equiv_G ba$ between two symbols a and b can be interpreted as the equivalence relation arising from the permutation of these symbols. In this context, the equivalence relation could come from the representation of two events as a and b that may occur concurrently in a given system. Furthermore, it can also be understood as an invariance property of the system outcome given that the operations ab or ba have been executed. However, in the present paper, these interpretations of the PCM do not seem to be entirely suitable for application to the zero-error theory.

Starting from the definition of the congruence relation between two symbols as the possibility of confusion in the channel, one can interpret the adjacency graph of the channel as a commutativity graph. By developing this interpretation, the definition of channel confusable words in an adjacency graph G^n will be rewritten in terms of PCM.

Definition 9 (Confusable words): Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be two words defined in G^n. The word $\mathbf{u}$ and $\mathbf{v}$ are confusable if there exist a pair u_i and v_i such that $u_i v_i \equiv_G v_i u_i$.

By using the definition of confusable words, one can obtain at the Lemma 1.

Lemma 1: If two words of length n are in the same equivalence class of $\mathcal{M}(\Sigma, G)$, then they are confusable in G^n.

Proof: Let be the word $\mathbf{u} = (u_1, \ldots, u_i, u_{i+1}, \ldots, u_n)$. Suppose that there exists at least a pair of congruent contiguous symbols (i.e, $u_i u_{i+1} \equiv_G u_{i+1} u_i$), then swapping positions of these symbols form a word $\mathbf{u}' = (u_1, \ldots, u_{i+1}, u_i, \ldots, u_n)$, which is congruent to $\mathbf{u}$ considering G. By Definition 9, $\mathbf{u}$ and $\mathbf{u}'$ are also confusable. ∎

From Lemma 1, it is possible to infer that an equivalence class is formed by words that are confusable and have the same type. If two words are confusable, they cannot be selected to compose the non-confusable code-words. Note that words in an equivalence class correspond to a subset of confusable words, thus the maximum number of non-confusable words of length n equals the number $\tau_G(n)$ of equivalence classes according to the channel adjacency graph G. In general, equivalence classes correspond to cliques (not necessarily maximal) of G^n

Consider as a example the pentagon C_5 and the graph C_5^2 depicted in Fig. 5. The equivalence classes of length $n = 2$ defined in $\mathcal{M}(\Sigma, C_5)$ are 00, 01 $\equiv$ 10, 02, 03, 04 $\equiv$ 40, 11, 12 $\equiv$ 21, 13, 14, 20, 22, 23 $\equiv$ 32, 24, 30, 31, 33, 34 $\equiv$ 43, 41, 42, 44, forming 20 classes. One can see that the classes represent clique on C_5^2. Since these clique are not necessarily maximal, many confusable words do not belong to the same class. For example, the words $22, 23, 32, 33$ are confusable with each other (they form a clique in C_5^2). However these words correspond to 3 distinct classes in $\mathcal{M}(\Sigma, C_5)$, $22, 23 \equiv 32, 33$, because only 23 and 32 have the same type.

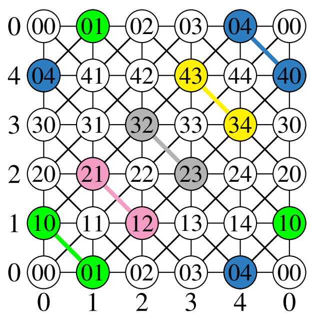


Fig. 5: Adjacency graph C_5^2.

In order to obtain a bound for the zero-error capacity, it is necessary to formalize the relations between the number of equivalence classes and the independence number. Henceforth, we define $\beta(G) = \lim_{n\to\infty} \tau_G(n)^{\frac{1}{n}}$. The $\beta(G)$ can be interpreted as the growth factor of the monoid, since $\tau_G(n) \sim \beta(G)^n$, where $\beta(G)$ corresponds to the smallest real root of multiplicity 1 of the dependence polynomial $D(G, z)$ and $f(n) \sim g(n)$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. For more details, one can consult [15].

Theorem 3: Let G be a adjacency graph, then

$$\beta(G) \geq \alpha(G), \tag{8}$$

in which $\beta(G) = \lim_{n\to\infty} \tau_G(n)^{\frac{1}{n}}$.

Proof: Let G be a graph and the number $\tau_G(n)$ of equivalence classes of length n. It is known that each equivalence class $\mathcal{E}_G(\mathbf{u}_i), |\mathbf{u}_i| = n$, corresponds to a clique in G^n. The stable set is formed by vertices not connected pairwise and cannot be formed using two vertices belonging to the same clique. Thus, the independence number $\alpha(G^n)$ has as its upper bound $\tau_G(n) \geq \alpha(G^n)$. Finally, using the fact that $\alpha(G)^n \leq \alpha(G^n)$, the desired result is obtained. ∎

Theorem 4: Let $\Theta(G)$ be the zero-error capacity of a channel represented by an adjacency graph G, then

$$\Theta(G) \leq \lfloor \beta(G) \rfloor. \tag{9}$$

Proof: By the definition of zero-error capacity, it is known $\Theta(G) = \lim_{n\to\infty} \sqrt[n]{\alpha(G^n)}$ and, by Theorem 3, $\tau_G(n) \geq \alpha(G^n)$, then

$$\Theta(G) = \lim_{n\to\infty} \sqrt[n]{\alpha(G^n)} \leq \beta(G). \tag{10}$$

The independence number $\alpha(G^n)$ is always an integer, and by using the floor function proprieties [16, cap.3], we obtain

$$\alpha(G^n) \leq \left\lfloor \sqrt[n]{\tau_G(n)} \right\rfloor \leq \sqrt[n]{\tau_G(n)}, \tag{11}$$

which concludes the proof. ∎

A. Examples

As a first example, we consider the case when G is a complete graph, where $\tau_G(n)$ is given by

$$\tau_G(n) = \binom{k+n-1}{n}. \tag{12}$$

It is known that for a complete graph (all symbols are confusable) $\Theta(G) = 1$ (see Theorem 1). Thus, by using (12) in Theorem 4, we obtain that $\Theta(G) \leq \lim_{n\to\infty} \sqrt[n]{\tau_G(n)} \leq 1$.

Consider an empty graph (no symbol is confusable) of k edges and $\tau_G(n) = k^n$. In this case, $\Theta(G) = k$. By applying now the Theorem 4, one has

$$\Theta(G) \leq \lim_{n\to\infty} \sqrt[n]{\tau_G(n)} = \sqrt[n]{k^n} = k. \tag{13}$$

The pentagon C_5 is considered as another example, where $\tau_{C_5}(n) \sim \frac{1}{2}\left(5 + \sqrt{5}\right)$. For this graph, it is known that $\Theta(C_5) = \sqrt{5} \approx 2.23$. Now applying Theorem 4,

$$\Theta(C_5) \leq \left\lfloor \lim_{n\to\infty} \sqrt[n]{\tau_{C_5}(n)} \right\rfloor = \left\lfloor \frac{1}{2}\left(5+\sqrt{5}\right) \right\rfloor = 3. \quad (14)$$

The value obtained in (14) is larger than the upper bound presented by Shannon [1], $\sqrt{5} \leq \Theta(G) \leq \frac{5}{2}$. The reason why the bound obtained by using Theorem 4 is not tight in this case is detailed in the next section.

In general, for cyclic graphs of k vertices and k edges the number of equivalence classes of length n is given by

$$\tau_{C_k}(n) \sim \left(\frac{1}{2}\left(k+\sqrt{k-4}\sqrt{k}\right)\right)^n. \quad (15)$$

By applying Theorem 4, it is obtained that

$$\Theta(C_k) \leq \left\lfloor \frac{1}{2}\left(k+\sqrt{k-4}\sqrt{k}\right) \right\rfloor. \quad (16)$$

The Lovász ϑ function for cyclic graphs is known [16] and is given by

$$\vartheta(C_k) = \begin{cases} \frac{k\cos(\pi/k)}{1+\cos(\pi/k)} & \text{if } k \text{ id odd} \\ \frac{k}{2} & \text{if } k \text{ is even.} \end{cases} \quad (17)$$

The comparison between the values of $\lfloor \beta(C_k) \rfloor$ and $\vartheta(C_k)$ for cyclic graphs is illustrated in Fig. 6. As the k increases $\lfloor \beta(C_k) \rfloor$ becomes about twice the value of $\vartheta(C_k)$. The values $\lfloor \beta(C_k) \rfloor$ and $\vartheta(C_k)$ are equal only for $k = 2$.

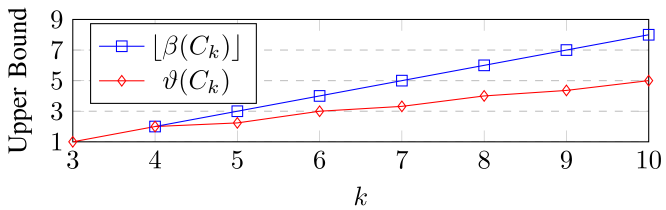


Fig. 6: Values of $\lfloor \beta \rfloor$ and ϑ considering cyclic graphs of k vertices.

B. Bounds Accuracy

The $\tau_G(n)$ is defined by employing the number of cliques $c_0, c_1, \ldots, c_\omega$ of the adjacency graph. In this way, $\tau_G(n)$ provides an upper bound for the entire dependence class $\mathcal{D}(G)$. Without loss of generality, one can write the zero-error capacities of graphs of the same dependence class as $\Theta(G_1) \leq \Theta(G_2) \leq \cdots \leq \Theta(G_m)$. As an example, consider the graph G from Fig. 7. This graph has $c_0 = 1, c_1 = 5$ and $c_2 = 5$, the same values as the pentagon C_5. Thus, the bound obtained in Eq. 14 is also valid for G. However, one can verify that $\Theta(G) = 3$, i.e., $\Theta(C_5) \leq \Theta(G)$.

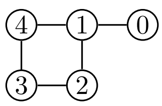


Fig. 7: Adjacency graph G of a channel with 5 input symbols.

The upper bound of Theorem 4 is tight on some graphs and may even be equal to the zero-error capacity of the channel. As an example, the graph of Fig. 7 has zero-error capacity $\Theta(G) = 3$ and $\lfloor \beta(G) \rfloor = 3$, as can be verified in Eq. 14. The equality is also the case for other small graphs. For cyclic graphs, on the other hand, the bound is not tight because we know that, in this case, $\beta(G) \geq \vartheta(G) \geq \Theta(G)$.

In general, the bound accuracy for any graph is not known. The accuracy ascertainment depends on difficult problems such as the graph dependence class determination and the zero-error capacity of each class element. More research needs to be done in this direction.

IV. Conclusions

In this paper we presented a connection between classical zero-error theory and partially commutative monoids and provided an upper bound for the classical zero-error capacity. The central finding of this study is that the zero-error capacity is always less than the integer part of the monoid growth factor, $\beta(G)$. However, the computational complexity of counting cliques of a graph makes it infeasible to determine the exact $\beta(G)$ when there are a large number of vertices and edges.

To further advance the research in this area, it is necessary to investigate the tightness of the obtained bounds and explore the relationships between these bounds and the Lovász function $\vartheta(G)$. Furthermore, it would be relevant to practical uses to investigate cases where the bound equals the zero-error capacity.

References

[1] C. Shannon, "The zero error capacity of a noisy channel," *IEEE Transactions on Information Theory*, vol. 2, no. 3, pp. 8–19, 1956.

[2] L. Lovasz, "On the shannon capacity of a graph," *IEEE Transactions on Information Theory*, vol. 25, no. 1, pp. 1–7, 1979.

[3] T. S. Cubitt, D. W. Leung, W. Matthews, and A. J. Winter, "Improving zero-error classical communication with entanglement," *Physical review letters*, vol. 104 23, p. 230503, 2009.

[4] A. Cabello, S. Severini, and A. Winter, "Graph-theoretic approach to quantum correlations," *Phys. Rev. Lett.*, vol. 112, p. 040401, Jan 2014.

[5] J. Zuiddam, "The asymptotic spectrum of graphs and the shannon capacity," *Combinatorica*, vol. 39, pp. 1173 – 1184, 2018.

[6] A. Schrijver, "On the shannon capacity of sums and products of graphs," *Indagationes Mathematicae*, 2022.

[7] H. Boche and C. Deppe, "Computability of the zero-error capacity of noisy channels," in *2021 IEEE Information Theory Workshop (ITW)*, pp. 1–6, 2021.

[8] C. Pierre and D. Foata, *Problèmes combinatoires de commutation et réarrangements*. Lecture notes in mathematics, Berlin Heidelberg New York: Springer-Verlag, 1969.

[9] A. Mazurkiewicz, "Concurrent program schemes and their interpretations," *DAIMI Report Series*, vol. 6, Jul. 1977.

[10] S. Arora and B. Barak, *Computational Complexity: A Modern Approach*. Cambridge University Press, 2006.

[11] D. E. Knuth, "The sandwich theorem.," *The Electronic Journal of Combinatorics [electronic only]*, vol. 1, 1994.

[12] D. Perrin, "Words over a partially commutative alphabet," in *Combinatorial Algorithms on Words. NATO ASI Series (Series F: Computer and Systems Sciences)*, vol. 12, pp. 329–340, Springer, Berlin, Heidelberg, 1985.

[13] D. Fisher, "The number of words of length n in a graph monoid," *Am. Math. Monthly*, vol. 96, p. 610–614, Aug. 1989.

[14] D. Fisher and A. Solow, "Dependence polynomials," *Discrete Mathematics*, vol. 82, no. 3, pp. 251 – 258, 1990.

[15] A. da Silva and F. M. de Assis, "A new algorithm for compression of partially commutative alphabets," *Information Sciences*, vol. 611, pp. 107–125, 2022.

[16] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*. Reading: Addison-Wesley, 1989.

[17] T. M. Cover and J. A. Thomas, *Elements of Information Theory (Wiley Series in Telecommunications and Signal Processing)*. USA: Wiley-Interscience, 2006.