THE η-μ DISTRIBUTION

Michel Daoud Yacoub

DECOM/FEEC/UNICAMP
C.P. 6101
13083-970 Campinas, SP, Brazil
michel@decom.fee.unicamp.br

ABSTRACT

This paper presents a general fading distribution – the η-μ Distribution – that includes the One-Sided Gaussian, the Rayleigh, and, more generally, the Nakagami distributions as special cases. Rice and Lognormal distributions may also be well-approximated by the η-μ Distribution. Preliminary results show that the η-μ Distribution provides a very good fitting to experimental data.

1. INTRODUCTION

The propagation of energy in a mobile radio environment is characterized by incident waves interacting with surface irregularities via diffraction, scattering, reflection, and absorption. The interaction of the wave with the physical structures generates a continuous distribution of partial waves [1], with these waves showing amplitudes and phases varying according to the physical properties of the surface. The propagated signal then reaches the receiver through multiple paths. If the waves are not resolvable within the available bandwidth or if an appropriate signal treatment is not carried out, the result is a combined signal that fades rapidly, characterizing the short term fading. For surfaces assumed to be of the Gaussian random rough type, universal statistical laws can be derived in a parameterized form [1].

A great number of distributions exists that well describe the statistics of the mobile radio signal. Extensive field trials have been used to validate these distributions and the results show a very good agreement between measurements and theoretical formulas. The long term signal variation is well characterized by the lognormal distribution whereas the short term signal variation is described by several other distributions such as Rayleigh, Rice, Nakagami, and Weibull, though to the latter, originally derived for reliability study purposes, little attention has been paid. It is generally accepted that the path strength at any delay is characterized by the short term distributions over a spatial dimension of a few hundred wavelengths, and by the lognormal distribution over areas whose dimension is much larger [2]. Three other distributions attempt to describe the transition from the local distribution to the global distribution of the path strength, thus combining both fast and slow fading. These composite (or mixed) distributions assume the local mean, which is the mean of the fast fading distribution, to be lognormally distributed. The best known composite distributions are Rayleigh-lognormal, also known as Suzuki, Rice-lognormal, and Nakagami-lognormal.

In fact, the Rayleigh distribution constitutes a special case of the Rice, Nakagami, Weibull, and the composite distributions and can be obtained in an exact manner by appropriately setting the parameters of these distributions. Nakagami-m and Rice are found to approximate each other by some simple equations relating the physical parameters associated to each distribution..

Among these, the Nakagami-m distribution has been given a special attention for its ease of manipulation and wide range of applicability [3]. Although, *in general*, it has been found that the fading statistics of the mobile radio channel may well be characterized by the Nakagami-m, situations are easily found for which other distributions such as Rice and even Weibull yield better results [4, 5]. More importantly, situations are encountered for which no distributions seem to adequately fit experimental data, though one or another may yield a moderate fitting. Some researches [5] even question the use of the Nakagami distribution because its tail does not seem to yield a good fitting to experimental data, better fitting being found around the mean or median.

This paper presents a general fading distribution - the η-μ Distribution - that includes the One-Sided Gaussian, the Rayleigh, and, more generally, the Nakagami-m distributions as special cases. Rice and Lognormal distributions may also be well-approximated by the η-μ Distribution.

2. THE η-μ DISTRIBUTION

The η-μ distribution is a general fading distribution that can be used to better represent the small scale variation of the fading signal. For a fading signal whose envelope is r and whose envelope ρ normalized with respect to the *rms* value is given by $\rho = \frac{r}{\sqrt{E(r^2)}}$, the η-μ probability density function $p(\rho)$ is written as

$$p(\rho) = \frac{4\sqrt{\pi}\,\mu^{\mu+\frac{1}{2}} h^{\mu}}{\Gamma(\mu) H^{\mu-\frac{1}{2}}} \rho^{2\mu} \exp\left(-2\mu h \rho^2\right) I_{\mu-\frac{1}{2}}\left(2\mu H \rho^2\right) \qquad (1)$$

where $h = \frac{2+\eta^{-1}+\eta}{4}$, $H = \frac{\eta^{-1}-\eta}{4}$, $\mu = \frac{E^2(r^2)}{Var(r^2)} \times \frac{1+\eta^2}{(1+\eta)^2}$ (or equivalently, $\mu = \frac{1}{Var(\rho^2)} \times \frac{1+\eta^2}{(1+\eta)^2}$), $\Gamma(.)$ is the Gamma function, $I_\nu(.)$ is the modified Bessel function of the first kind and arbitrary order ν (ν real), with $\frac{\mu(1+\eta)^2}{1+\eta^2} \geq \frac{1}{2}$, and $0 \leq \eta \leq 1$. (In fact, the distribution is symmetrical for $1 \leq \eta < \infty$, or equivalently $0 \leq \eta^{-1} \leq 1$, in which case $H = \frac{\eta-\eta^{-1}}{4}$. Therefore, more generally, we may write $0 \leq \eta < \infty$ and $H = \left|\eta - n^{-1}\right|/4$. But, due to the symmetry around 1, it suffices to consider η within one of the ranges only.) From the restrictions given we observe that that $\mu \geq \frac{1}{4}$, obtainable for $\eta = 1$.

For a fading signal with power $w = r^2/2$ and normalized power $\omega = \frac{w}{E(w)}$ the η-μ probability density function $p(\omega)$ is given by

$$p(\omega) = \frac{2\sqrt{\pi}\mu^{\mu+\frac{1}{2}}h^{\mu}}{\Gamma(\mu)H^{\mu-\frac{1}{2}}}\omega^{\mu-\frac{1}{2}}\exp(-2\mu h\omega)I_{\mu-\frac{1}{2}}(2\mu H\omega) \tag{2}$$

In particular, we may also write $\mu = \frac{E^2(w)}{Var(w)} \times \frac{1+\eta^2}{(1+\eta)^2}$ (or equivalently $\mu = \frac{1}{Var(\omega)} \times \frac{1+\eta^2}{(1+\eta)^2}$).

3. PHYSICAL MODEL

The well-known fading distributions have been derived assuming a homogeneous diffuse scattering field, resulting from randomly distributed point scatterers. With such an assumption, the central limit theorem leads to complex Gaussian processes with in-phase and quadrature Gaussian distributed variables *x* and *y* having zero means and equal standard deviations. In case one cluster of multipath wave is considered then the Rayleigh distribution can be obtained. If a specular component predominates over the scattered waves, then the Rice distribution is accomplished. The Nakagami signal can be understood as composed of clusters of multipath waves so that within any one cluster the phases of scattered waves are random and have similar delay times with delay-time spreads of different clusters being relatively large. The assumption of a homogeneous diffuse scattering field is certainly an approximation because the surfaces are spatially correlated characterizing a non-homogeneous environment [1].

The fading model for the η-μ Distribution considers a signal composed of clusters of multipath waves propagating in an non-homogeneous environment. Within any one cluster, the phases of the scattered waves are random and have similar delay times with delay-time spreads of different clusters being relatively large.

4. DERIVATION OF THE η-μ DISTRIBUTION

Given the physical model for the η-μ Distribution the envelope r can be written in terms of the in-phase and quadrature components of the fading signal as

$$r^2 = \sum_{i=1}^{n}\left(x_i^2 + y_i^2\right) \tag{3}$$

where x_i and y_i are mutually independent Gaussian processes with $E(x_i) = E(y_i) = 0$, $E(x_i^2) = \sigma_x^2$ and $E(y_i^2) = \sigma_y^2$. Now we form the process $r_i^2 = x_i^2 + y_i^2$, so that $r^2 = \sum_{i=1}^{n} r_i^2$. In the same way we may write $w = \sum_{i=1}^{n} w_i$, where $w = r^2/2$ and $w_i = r_i^2/2$. We proceed to find the density of r_i. This can be carried by following the standard, but long and tedious, procedure so that

$$p(r_i) = \frac{\sqrt{\eta}r_i}{\sigma_x^2}\exp\left[-\frac{(1+\eta)r_i^2}{4\sigma_x^2}\right]I_0\left[\frac{(\eta-1)r_i^2}{4\sigma_x^2}\right]$$

where $\eta = \sigma_x^2/\sigma_y^2$ and $I_0(.)$ is the modified Bessel function of the first kind order zero. Note that $0 \leq \eta \leq 1$ defines the region within which $\sigma_x^2 \leq \sigma_y^2$, whereas $0 \leq \eta^{-1} \leq 1$ defines the region within which $\sigma_y^2 \leq \sigma_x^2$. It is possible to show that $\hat{r}_0^2 \overset{\Delta}{=} E(r_i^2) = (1+\eta^{-1})\sigma_x^2$. Therefore

$$p(r_i) = \frac{2\sqrt{h}r_i}{\hat{r}_0^2}\exp\left[-h\left(\frac{r_i}{\hat{r}_0}\right)^2\right]I_0\left[H\left(\frac{r_i}{\hat{r}_0}\right)^2\right]$$

where h and H are as previously defined. The density $p(w_i)$ of the power w_i is easily found by a simple transformation of variables and it is given by

$$p(w_i) = \frac{\sqrt{h}w_i}{\overline{w}_0}\exp\left(-\frac{hw_i}{\overline{w}_0}\right)I_0\left(\frac{Hw_i}{\overline{w}_0}\right)$$

where $\overline{w}_0 = E(w_i)$. The Laplace transform $L[p(w_i)]$ of $p(w_i)$ is found in an exact manner as [6, pag. 1025, Eq. 29.3.60]

$$L[p(w_i)] = \frac{\sqrt{h/\overline{w}_0}}{\sqrt{(s + h/\overline{w}_0)^2 - (H/\overline{w}_0)^2}}$$

Knowing that w_i, i = 1, 2, ..., n, are independent, the Laplace transform $L[p(w)]$ of $p(w)$ is found as

$$L[p(w)] = \left[\frac{\sqrt{h/\overline{w}_0}}{\sqrt{(s + h/\overline{w}_0)^2 - (H/\overline{w}_0)^2}}\right]^n$$

whose inverse is given by [6, pag. 1025, Eq. 29.3.60]

$$p(w) = \frac{\sqrt{\pi}\, h^{\frac{n}{2}}}{(2H)^{\frac{n-1}{2}}\Gamma\left(\frac{n}{2}\right)}\left(\frac{w}{\overline{w}_0}\right)^{\frac{n-1}{2}} \exp\left(-\frac{hw}{\overline{w}_0}\right) I_{\frac{n-1}{2}}\left(\frac{Hw}{\overline{w}_0}\right)$$

We note, however, that $\overline{w} = E(w) = n\overline{w}_0$. Therefore

$$\overline{w}p(w) = \frac{\sqrt{\pi}\, n^{\frac{n+1}{2}} h^{\frac{n}{2}}}{(2H)^{\frac{n-1}{2}}\Gamma\left(\frac{n}{2}\right)}\left(\frac{w}{\overline{w}}\right)^{\frac{n-1}{2}} \exp\left(-\frac{nhw}{\overline{w}}\right) I_{\frac{n-1}{2}}\left(\frac{nHw}{\overline{w}}\right) \quad (4)$$

The corresponding density of the envelope is found to be

$$\hat{r}p(r) = \frac{2\sqrt{\pi}\, n^{\frac{n+1}{2}} h^{\frac{n}{2}}}{(2H)^{\frac{n-1}{2}}\Gamma\left(\frac{n}{2}\right)}\left(\frac{r}{\hat{r}}\right)^{n} \exp\left[-nh\left(\frac{r}{\hat{r}}\right)^2\right] I_{\frac{n-1}{2}}\left[nH\left(\frac{r}{\hat{r}}\right)^2\right] \quad (5)$$

From Equation 3 we find that $E(r^2) = n(1+\eta)\sigma_y^2$ and $Var(r^2) = 2n(1+\eta^2)\sigma_y^4$. Thus

$$\frac{E^2(r^2)}{Var(r^2)} = \frac{n}{2} \times \frac{(1+\eta)^2}{(1+\eta^2)} \quad (6)$$

Note from Equation 6 that $n/2$ may be totally expressed in terms of physical parameters such as mean squared value, variance of the power, and power of the in-phase and quadrature components of the fading signal. Note also that whereas these physical parameters are of a continuous nature, $n/2$ is of a discrete nature (integer multiple of $1/2$). It is plausible to presume that if these parameters are to be obtained by field measurements, their ratios, as defined in Equation 6, will certainly lead to figures that may depart from the exact $n/2$. Several reasons exist for this. One of them, probably the most significant one, is that, although the model proposed here is general, it is in fact an approximate solution to the so-called random phase problem, as are approximate solution to the random phase problem all the other well-known fading models. The limitation of the model can be made less stringent by defining μ as

$$\mu = \frac{E^2(r^2)}{Var(r^2)} \times \frac{1+\eta^2}{(1+\eta)^2} \quad (7)$$

μ being the *real* extension of $n/2$. (We note that in derivation of the Nakagami model [7], the parameter n, which describes the number of “component signals”, therefore discrete, is also written in terms of the Nakagami continuous parameter m as $m = n/2$.) It has been observed experimentally by Nakagami [7] that $\frac{E^2(r^2)}{Var(r^2)} \geq \frac{1}{2}$. Therefore, for the η-μ Distribution

$$\frac{\mu(1+\eta)^2}{1+\eta^2} \geq \frac{1}{2} \quad (8)$$

with

$$0 \leq \eta \leq 1 \quad (9)$$

(or equivalently $0 \leq \eta^{-1} \leq 1$) and μ assuming any *real* value according to Equation 8. In particular, the maximum of $\frac{(1+\eta)^2}{1+\eta^2}$ is obtained at $\eta = 1$, in which case $\mu \geq \frac{1}{4}$, i.e. the minimum value that may be assumed by μ is 0.25.

The densities can now be written as

$$\hat{r}p(r) = \frac{4\sqrt{\pi}\,\mu^{\mu+\frac{1}{2}} h^{\mu}}{\Gamma(\mu) H^{\mu-\frac{1}{2}}}\left(\frac{r}{\hat{r}}\right)^{2\mu} \exp\left[-2\mu h\left(\frac{r}{\hat{r}}\right)^2\right] I_{\mu-\frac{1}{2}}\left[2\mu H\left(\frac{r}{\hat{r}}\right)^2\right] \quad (10)$$

and

$$\overline{w}p(w) = \frac{2\sqrt{\pi}\mu^{\mu+\frac{1}{2}}h^{\mu}}{\Gamma(\mu)H^{\mu-\frac{1}{2}}}\left(\frac{w}{\overline{w}}\right)^{\mu-\frac{1}{2}}\exp\left(-\frac{2\mu hw}{\overline{w}}\right)I_{\mu-\frac{1}{2}}\left(\frac{2\mu Hw}{\overline{w}}\right) \quad (11)$$

which, in the normalized form, are given as in Equations 1 and 2, respectively.

5. THE η-μ DISTRIBUTION AND THE OTHER FADING DISTRIBUTIONS

The η-μ Distribution is a general fading distribution that includes the One-Sided Gaussian, the Rayleigh, and, more generally, the Nakagami distributions as special cases. Rice and Lognormal distributions may also be well-approximated by the η-μ Distribution. We note that the One-Side Gaussian and the Rayleigh distributions can be obtained from the Nakagami distribution by setting the Nakagami parameter $m = 0.5$ and $m = 1$, respectively. Therefore, in order to relate the η-μ Distribution with these two distributions it suffices to relate it with the Nakagami one.

The Nakagami distribution can be obtained in an exact manner from the η-μ Distribution for $\mu = m$ and $\eta \to 0$ (or equivalently $\eta \to \infty$) or, in the same way, for $\mu = m/2$ and $\eta \to 1$. This result is not straightforwardly seen from the densities here derived. We observe, nonetheless, that for these conditions all the Gaussian variates present identical variances and the fading model proposed here deteriorates into that of [8], where the Nakagami distribution is obtained in an exact manner. For intermediate values of η the η-μ distribution and the Nakagami distribution relate to each other for $\frac{\mu(1+\eta)^2}{1+\eta^2} = m$. This is a very interesting result which shows that an infinite number of curves of the η-μ distribution can be found that presents the same Nakagami parameter m, conditioned on the fact that the constraints $m/2 \le \mu \le m$ and $\eta = \frac{\frac{\mu}{m} - \sqrt{2\frac{\mu}{m} - 1}}{1 - \frac{\mu}{m}}$ are satisfied. The Lognormal distribution, given as a function of m in Equation 13 of [7], can also be approximated by the η-μ Distribution for $e^{-1} \le \rho \le e$, and for η, μ, and m satisfying the relations given above for the Nakagami case. In the same way, an infinite number of curves of the η-μ Distribution can be found that presents the same Nakagami parameter for the Lognormal distribution. The Rice distribution can be approximated by the η-μ distribution for $\frac{\mu(1+\eta)^2}{1+\eta^2} = \frac{(1+k)^2}{1+2k}$, where $k \ge 0$ is the Rice parameter. In the same way, this result shows that an infinite number of curves of the η-μ Distribution can be found that presents the same Rice parameter *k*, conditioned on the fact that the constraints $\frac{(1+k)^2/2}{2k+1} \le \mu \le \frac{(1+k)^2}{2k+1}$ and $\eta = \frac{\frac{\mu(1+2k)}{(1+k)^2} - \sqrt{\frac{2\mu(1+2k)}{(1+k)^2} - 1}}{1 - \frac{\mu(1+2k)}{(1+k)^2}}$ are satisfied.

6. SAMPLE EXAMPLES OF THE η-μ DISTRIBUTION

This section shows some plots of the η-μ Distribution. Figure 1 depicts a sample of the various shapes of the η-μ probability density function $p(\rho)$ as a function of the normalized envelope ρ for the same Nakagami parameter *m* = 0.5 (One-Sided Gaussian). Figure 2, 3, and 4 do the same but respectively for *m* = 0.75, 1.0 (Rayleigh), and 1.25. Figure 5 illustrates a sample of the various shapes of the η-μ probability distribution function $P(\rho)$ as a function of the normalized envelope ρ for the same Nakagami parameter *m* = 1.0 (Rayleigh). Figure 6 does the same but for *m* = 1.25. The plots are carried out for $\eta \to 1$, η = 0.62, 0.4, 0.25, 0.16, 0.10, 0.05, 0.015, and $\eta \to 0$ (which in decibels correspond to approximately 0, -4, -8, -12, -16, -20, -26, -36, and -∞ dB). The curves for which $\eta \to 1$ and $\eta \to 0$ coincide with each other and also with the Nakagami one, and this is indicated in all Figures by the arrow sign. In such cases, $\mu = m/2$ and $\mu = m$, respectively.

It can be seen that, although the normalized variance (parameter *m*) is kept constant for each Figure, the curves are substantially different from each other. And this is particularly noticeable for the distribution function, in which case the lower tail of the distribution may yield differences in the probability of some orders. Moreover, the curves present a very interesting feature, as described next. For the same value of *m* and departing from the condition for which $\eta \to 1$, as η diminishes the curves depart from that for which $\eta \to 1$, initially keeping a similar shape. As η diminishes the shapes of the curves change substantially. As η diminishes even further and as $\eta \to 0$, the curves merge with that of the initial shape but such curves present shapes very different from those obtained as $\eta \to 1$. This feature renders the η-μ Distribution very flexible and this flexibility can be used in order to adjust the curves to practical data.

7. CONCLUSIONS

This paper presented a general fading distribution – the η-μ Distribution - that can be used to better represent the small scale variation of the fading signal. The distribution includes the One-Sided Gaussian, the Rayleigh, and, more generally, the Nakagami-m distributions as a special cases and offers a higher degree of freedom. It also approximates the Rice and Lognromal distributions. Preliminary results show that the η-μ Distribution provides a very good fitting to experimental data.

8. REFERENCES

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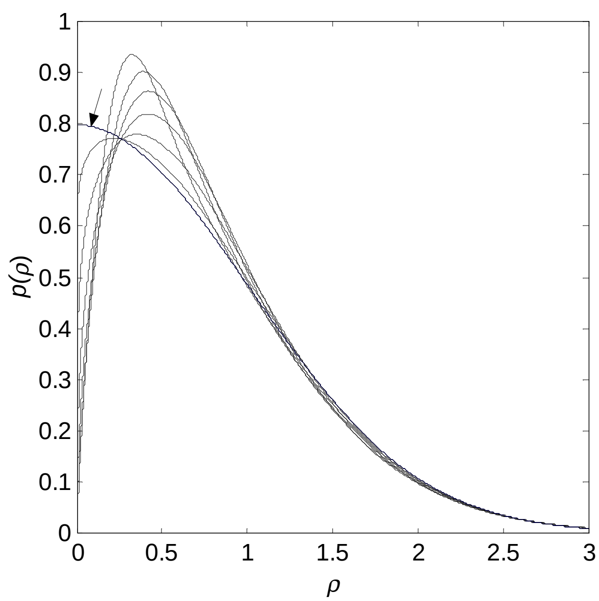

Figure 1: A sample of the various shapes of the η-μ probability density function for the same Nakagami parameter m = 0.5. The arrow indicates the conditions $\eta \to 1$, $\mu = 0.25$ and $\eta \to 0$, $\mu = 0.5$, for which the curves coincide with that of Nakagami.

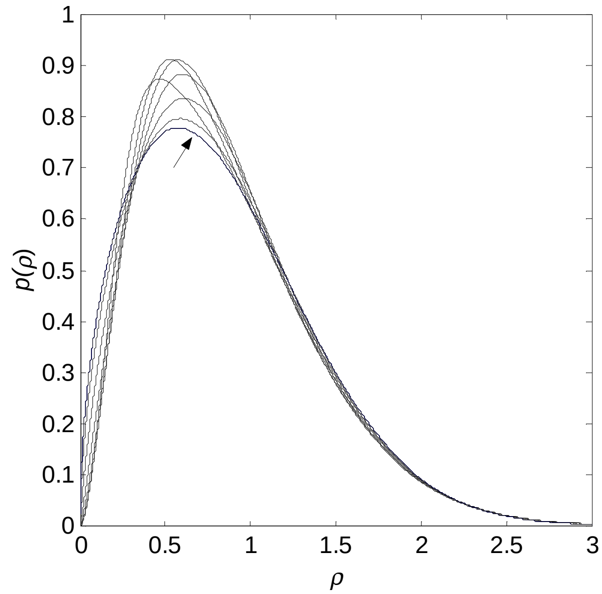

Figure 2: A sample of the various shapes of the η-μ probability density function for the same Nakagami parameter m = 0.75. The arrow indicates the conditions $\eta \to 1$, $\mu = 0.375$ and $\eta \to 0$, $\mu = 0.75$, for which the curves coincide with that of Nakagami.

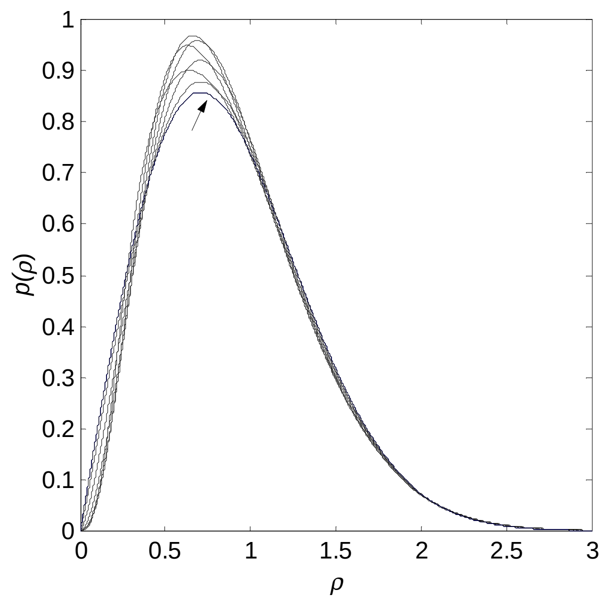

Figure 3: A sample of the various shapes of the η-μ probability density function for the same Nakagami parameter m = 1.0. The arrow indicates the conditions $\eta \to 1$, $\mu = 0.5$ and $\eta \to 0$, $\mu = 1.0$, for which the curves coincide with that of Nakagami.

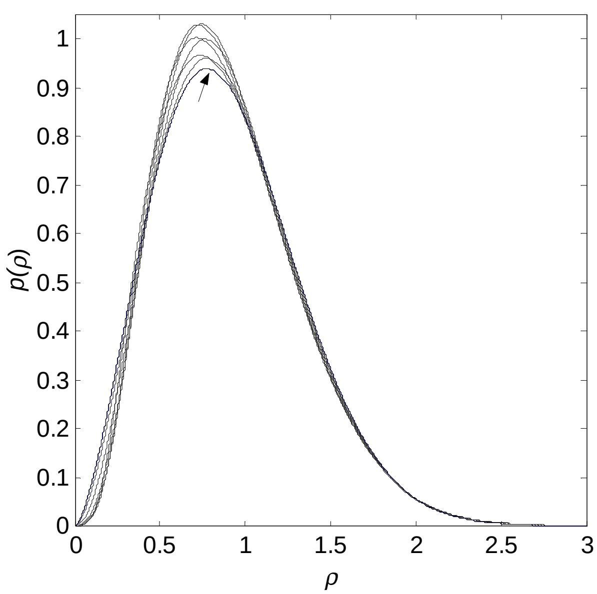

Figure 4: A sample of the various shapes of the η-μ probability density function for the same Nakagami parameter m = 1.25. The arrow indicates the conditions $\eta \to 1$, $\mu = 0.625$ and $\eta \to 0$, $\mu = 1.25$, for which the curves coincide with that of Nakagami.

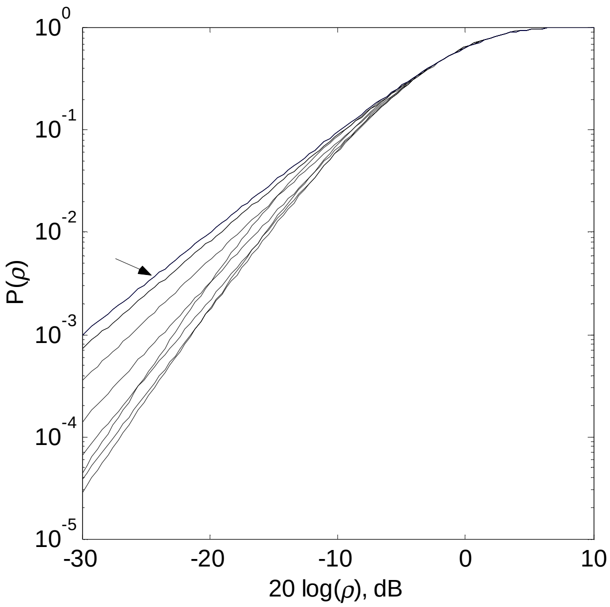

Figure 5: A sample of the various shapes of the η-μ probability distribution function for the same Nakagami parameter m = 1.0. The arrow indicates the conditions $\eta \to 1$, $\mu = 0.5$ and $\eta \to 0$, $\mu = 1.0$, for which the curves coincide with that of Nakagami.

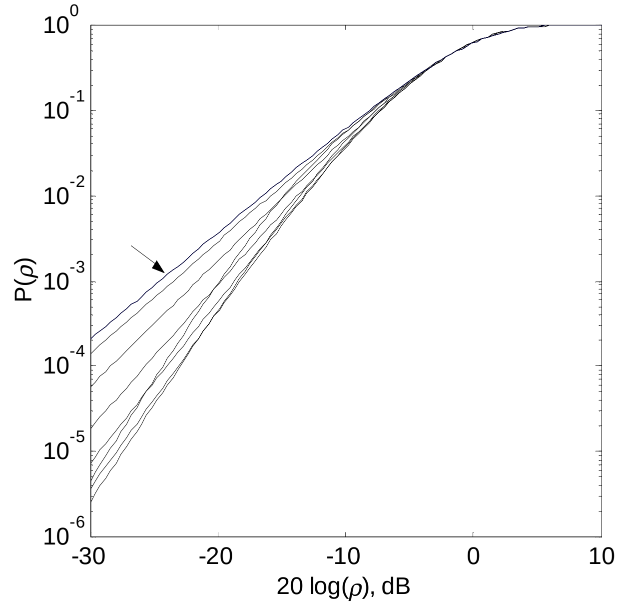

Figure 6: A sample of the various shapes of the η-μ probability distribution function for the same Nakagami parameter m = 1.25. . The arrow indicates the conditions $\eta \to 1$, $\mu = 0.625$ and $\eta \to 0$, $\mu = 1.25$, for which the curves coincide with that of Nakagami.