THE κ-μ DISTRIBUTION

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ABSTRACT

This paper presents a general fading distribution – the κ-μ Distribution - that includes the Rice and the Nakagami-m distributions as special cases. Therefore the One-Sided Gaussian and the Rayleigh distributions also constitute special cases and the Lognormal distribution may be well-approximated by the κ-μ Distribution. Preliminary results show that the κ-μ Distribution provides a very good fitting to experimental data.

1. INTRODUCTION

The propagation of energy in a mobile radio environment is characterized by incident waves interacting with surface irregularities via diffraction, scattering, reflection, and absorption. The interaction of the wave with the physical structures generates a continuous distribution of partial waves [1], with these waves showing amplitudes and phases varying according to the physical properties of the surface. The propagated signal then reaches the receiver through multiple paths. If the waves are not resolvable within the available bandwidth or if an appropriate signal treatment is not carried out, the result is a combined signal that fades rapidly, characterizing the short term fading. For surfaces assumed to be of the Gaussian random rough type, universal statistical laws can be derived in a parameterized form [1].

A great number of distributions exists that well describe the statistics of the mobile radio signal. Extensive field trials have been used to validate these distributions and the results show a very good agreement between measurements and theoretical formulas. The long term signal variation is well characterized by the Lognormal distribution whereas the short term signal variation is described by several other distributions such as Rayleigh, Rice, Nakagami-m, and Weibull, though to the latter, originally derived for reliability study purposes, little attention has been paid. It is generally accepted that the path strength at any delay is characterized by the short term distributions over a spatial dimension of a few hundred wavelengths, and by the Lognormal distribution over areas whose dimension is much larger [2]. Three other distributions attempt to describe the transition from the local distribution to the global distribution of the path strength, thus combining both fast and slow fading. These composite (or mixed) distributions assume the local mean, which is the mean of the fast fading distribution, to be lognormally distributed. The best known composite distributions are Rayleigh-lognormal, also known as Suzuki, Rice-lognormal, and Nakagami-m-lognormal.

In fact, the Rayleigh distribution constitutes a special case of the Rice, Nakagami-m, Weibull, and of the composite distributions and can be obtained in an exact manner by appropriately setting the parameters of these distributions. Nakagami-m and Rice are found to approximate each other by some simple equations relating the physical parameters associated to each distribution.

Among these, the Nakagami-m distribution has been given a special attention for its ease of manipulation and wide range of applicability [3]. Although, *in general*, it has been found that the fading statistics of the mobile radio channel may well be characterized by the Nakagami-m, situations are easily found for which other distributions such as Rice and even Weibull yield better results [4, 5]. More importantly, situations are encountered for which no distributions seem to adequately fit experimental data, though one or another may yield a moderate fitting. Some researches [5] even question the use of the Nakagami-m distribution because its tail does not seem to yield a good fitting to experimental data, better fitting being found around the mean or median.

This paper presents a general fading distribution - the κ-μ Distribution - that includes the Nakagami-m, Rice, One-Sided Gaussian, and Rayleigh distributions as special cases. The Lognormal distribution may also be well-approximated by the κ- Distribution.

2. THE κ-μ DISTRIBUTION

The κ-μ distribution is a general fading distribution that can be used to represent the small scale variation of the fading signal. For a fading signal with envelope r and normalized envelope $\rho = \frac{r}{\hat{r}}$, $\hat{r} = \sqrt{E\left(r^2\right)}$ being the *rms* value of r, theκ- probability density function $p(\rho)$ is written as

$$p(\rho) = \frac{2\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)}\rho^{\mu}\exp\left(-\mu(1+\kappa)\rho^2\right)I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)}\rho\right) \quad (1)$$

where $\kappa \geq 0$ is the ratio between the total power of the dominant components and the total power of the scattered waves, $\mu \geq 0$ is given by $\mu = \frac{E^2\left(r^2\right)}{Var\left(r^2\right)} \times \frac{1+2\kappa}{(1+\kappa)^2}$ (or equivalently,

$\mu = \frac{1}{Var(\rho^2)} \times \frac{1+2\kappa}{(1+\kappa)^2}$), $\frac{\mu(1+\kappa)^2}{1+2\kappa} \geq \frac{1}{2}$, and $I_\nu(.)$ is the modified Bessel function of the first kind and arbitrary order ν (ν real).

For a fading signal with power $w = r^2/2$ and normalized power $\omega = \frac{w}{\overline{w}}$, where $\overline{w} = E(w)$, the κ-μ probability density function $p(\omega)$ is given by

$$p(\omega) = \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}} \exp(\mu\kappa)} \omega^{\frac{\mu-1}{2}} \exp(-\mu(1+\kappa)\omega) I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)\omega}\right) \quad (2)$$

In particular, we may also write $\mu = \frac{E^2(w)}{Var(w)} \times \frac{1+2\kappa}{(1+\kappa)^2}$ (or equivalently $\mu = \frac{1}{Var(\omega)} \times \frac{1+2\kappa}{(1+\kappa)^2}$).

3. PHYSICAL MODEL

The well-known fading distributions have been derived assuming a homogeneous diffuse scattering field, resulting from randomly distributed point scatterers. With such an assumption, the central limit theorem leads to complex Gaussian processes with in-phase and quadrature Gaussian distributed variables *x* and *y* having zero means and equal standard deviations. In case one cluster of multipath wave is considered, then the Rayleigh distribution can be obtained. If a specular component predominates over the scattered waves, then the Rice distribution is accomplished. The Nakagami-m signal can be understood as composed of clusters of multipath waves so that within any one cluster the phases of scattered waves are random and have similar delay times with delay-time spreads of different clusters being relatively large. The κ-μ distribution describes the fading behavior of a signal propagating in an non-homogeneous environment.

The fading model for the κ-μ Distribution considers a signal composed of clusters of multipath waves propagating in an homogeneous environment. Within any one cluster, the phases of the scattered waves are random and have similar delay times with delay-time spreads of different clusters being relatively large. The clusters of multipath waves are assumed to have the scattered waves with identical powers but within *each* cluster a dominant component is found that presents an arbitrary power.

4. DERIVATION OF THE κ-μ DISTRIBUTION

Given the physical model for the κ-μ Distribution the envelope, the envelope r can be written in terms of the in-phase and quadrature components of the fading signal as

$$r^2 = \sum_{i=1}^{n} (x_i + p_i)^2 + \sum_{i=1}^{n} (y_i + q_i)^2 \quad (3)$$

where x_i and y_i are mutually independent Gaussian processes with $E(x_i) = E(y_i) = 0$, $E(x_i^2) = E(y_i^2) = \sigma^2$, and p_i and q_i are respectively the mean values of the in-phase and quadrature components of the multipath waves of cluster *i*. Now we form the processes $\xi_i = (x_i + p_i)^2$ and $\psi_i = (y_i + q_i)^2$, so that $\gamma = r^2 = \sum_{i=1}^{n} \xi_i + \sum_{i=1}^{n} \psi_i$. Define $p(\xi_i)$ and $p(\psi_i)$ as the densities of ξ_i and ψ_i respectively. In such a case

$$p(\lambda_i) = \frac{1}{\sqrt{2\pi\lambda_i}\sigma} \exp\left(-\frac{\lambda_i + s_i^2}{2\sigma^2}\right) \cosh\left(\frac{\sqrt{\lambda_i} s_i}{\sigma^2}\right)$$

where $\lambda_i = \xi_i$ and $s_i = p_i$ or $\lambda_i = \psi_i$ and $s_i = q_i$ The Laplace transform $L[p(\lambda_i)]$ of $p(\lambda_i)$ is found in an exact manner as [6, page 1026, Eq. 29.3.77]

$$L[p(\lambda_i)] = \frac{1}{\sqrt{1+2\sigma^2 s}} \exp\left(-\frac{s_i^2}{1+2\sigma^2 s}\right)$$

where *s* is the complex frequency (Laplace variable). Knowing that ξ_i and ψ_i, i = 1, 2, ..., *n*, are mutually independent, the Laplace transform $L[p(\gamma)]$ of $p(\gamma)$ is found as a $2n$-fold multiplication of $L[p(\gamma)]$. Therefore

$$L[p(\gamma)] = \frac{1}{(1+2s\sigma^2)^n} \exp\left(-\frac{s\sum_{i=1}^{n}(p_i^2 + q_i^2)}{1+2s\sigma^2}\right)$$

whose inverse is given by [6, page 1026, Eq. 29.3.77]

$$p(\gamma) = \frac{1}{2\sigma^2}\left(\frac{\gamma}{\sum_{i=1}^{n}(p_i^2+q_i^2)}\right)^{\frac{n-1}{2}} \exp\left[-\frac{\gamma + \sum_{i=1}^{n}(p_i^2+q_i^2)}{2\sigma^2}\right] I_{n-1}\left(\frac{\sqrt{\gamma\sum_{i=1}^{n}(p_i^2+q_i^2)}}{\sigma^2}\right) \quad (4)$$

It is possible to show that $\hat{r}^2 = E[r^2] = 2n\sigma^2 + \sum_{i=1}^{n}(p_i^2 + q_i^2)$. In the same way $E[r^4] = 4n\sigma^4 + 4\sigma^2\sum_{i=1}^{n}(p_i^2+q_i^2) + \left[2n\sigma^2 + \sum_{i=1}^{n}(p_i^2+q_i^2)\right]^2$. Therefore $Var(r^2) = 4n\sigma^4 + 4\sigma^2\sum_{i=1}^{n}(p_i^2+q_i^2)$.

We define $\kappa = \frac{\sum_{i=1}^{n}\left(p_i^2 + q_i^2\right)}{2n\sigma^2}$. Note that κ is the ratio between the *total* power of the dominant components and the *total* power of the scattered waves. Then

$$\frac{E^2\left(r^2\right)}{Var\left(r^2\right)} = n \times \frac{(1+\kappa)^2}{(1+2\kappa)} \qquad (5)$$

Note from Equation 5 that n may be totally expressed in terms of physical parameters such as mean squared value, variance of the power, and the ratio of the total power of the dominant components and the total power of the scattered waves of the fading signal. Note also that whereas these physical parameters are of a continuous nature, n is of a discrete nature. It is plausible to presume that if these parameters are to be obtained by field measurements, their ratios, as defined in Equation 5, will certainly lead to figures that may depart from the exact n. Several reasons exist for this. One of them, probably the most meaningful one, is that, although the model proposed here is general, it is in fact an approximate solution to the so-called random phase problem, as are approximate solution to the random phase problem all the other well-known fading models. The limitation of the model can be made less stringent by defining μ as

$$\mu = \frac{E^2\left(r^2\right)}{Var\left(r^2\right)} \times \frac{1+2\kappa}{(1+\kappa)^2} \qquad (6)$$

μ being the *real* extension of n. (We note that in derivation of the Nakagami-m model [7], the parameter n, which describes the number of "component signals" [7], therefore *discrete*, is also written in terms of the Nakagami *continuous* parameter m as $m = n/2$.) It has been observed experimentally by Nakagami [7] that $\frac{E^2\left(r^2\right)}{Var\left(r^2\right)} \geq \frac{1}{2}$. Therefore, for the κ-μ Distribution

$$\frac{\mu(1+\kappa)^2}{1+2\kappa} \geq \frac{1}{2} \qquad (8)$$

with $\kappa \geq 0$ and $\mu \geq 0$. Using the definitions and the considerations as above and by means of a transformation of variables and a series of algebraic manipulations, the κ-μ probability density function of the envelope can be written from Equation 4 as

$$\hat{r}p(r) = \frac{2\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)}\left(\frac{r}{\hat{r}}\right)^{\mu}\exp\left(-\mu(1+\kappa)\left(\frac{r}{\hat{r}}\right)^2\right) \times$$
$$\times I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)}\frac{r}{\hat{r}}\right) \qquad (9)$$

In the same way, the probability density function of the power is given as

$$\overline{w}p(w) = \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)}\left(\frac{w}{\overline{w}}\right)^{\frac{\mu-1}{2}} \times$$
$$\times \exp\left(-\mu(1+\kappa)\left(\frac{w}{\overline{w}}\right)\right)I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)\left(\frac{w}{\overline{w}}\right)}\right) \qquad (10)$$

Equations 9 and 10 in their normalized forms are respectively given by Equations 1 and 2.

5. THE κ-μ DISTRIBUTION AND THE OTHER FADING DISTRIBUTIONS

The κ-μ Distribution is a general fading distribution that includes the best known fading distributions, namely Rice and Nakagami-m distributions. Note that both Rice and Nakagami-m include the Rayleigh distribution and the Nakagami-m includes the One-Sided Gaussian. Therefore, these distributions can also be obtained from the κ-μ Distribution. The Lognormal distribution may also be well-approximated by the κ-μ Distribution.

5.1 Rice and Rayleigh

The Rice distribution describes a fading signal with one cluster of multipath waves in which one specular component predominates over the scattered waves. Therefore, by setting $\mu = 1$ in Equation 1, the κ-μ Distribution reduces to

$$p(\rho) = \frac{2(1+\kappa)}{\exp(\kappa)}\rho\exp\left(-(1+\kappa)\rho^2\right)I_0\left(2\sqrt{\kappa(1+\kappa)}\rho\right) \qquad (11)$$

which is the Rice probability density function for the normalized envelope. In this case, the parameter κ coincides with the well-known Rice parameter k. Now setting $\kappa = 0$ in Equation 11 (therefore, $\mu = 1$ and $\kappa \to 0$ in the κ-μ Distribution) the Rayleigh distribution can be obtained in an exact manner. Moreover, for $\kappa = m - 1 + \sqrt{m(m-1)}$ in Equation 11 (therefore $\mu = 1$ and $\kappa = m - 1 + \sqrt{m(m-1)}$ in the κ-μ Distribution), where m is the Nakagami parameter, the Nakagami-m distribution can be obtained in an approximate manner.

5.2 Nakagami-m, Rayleigh, and One-Sided Gaussian

The Nakagami-m signal can be understood as composed of clusters of multipath waves with no dominant components within any cluster. Therefore, by setting $\kappa = 0$ in the κ-μ Distribution it should be possible to obtain the Nakagami-m distribution. We note, however, that, apart from the case $\mu = 1$, which has been explored in the previous subsection, the introduction of $\kappa = 0$ in the κ-μ Distribution leads to an indeterminacy (zero divided by zero). For small arguments of the Bessel function the relation $I_{\mu-1}(z) \approx (z/2)^{\mu-1}/\Gamma(\mu)$ holds [6, page 375, Eq. 9.6.7]. Using this in Equation 1, and after some algebraic manipulation,

$$p(\rho) = \frac{2\mu^{\mu}(1+\kappa)^{\frac{\mu}{2}}}{\exp(\mu\kappa)\Gamma(\mu)}\rho^{2\mu-1}\exp\left(-\mu(1+\kappa)\rho^2\right) \quad (12)$$

As $\kappa \to 0$ Equation 12 reduces to

$$p(\rho) = \frac{2\mu^{\mu}}{\Gamma(\mu)}\rho^{2\mu-1}\exp\left(-\mu\rho^2\right) \quad (13)$$

which is the *exact* Nakagami-m density function for the normalized envelope. In this case, the parameter μ coincides with the well-known Nakagami parameter m. Now setting $\mu = 1$ in Equation 13 (therefore, $\mu = 1$ and $\kappa \to 0$ in the κ-Distribution) the Rayleigh distribution can be obtained in an exact manner. In the same way, by setting $\mu = 0.5$ in Equation 13 (therefore, $\mu = 0.5$ and $\kappa \to 0$ in the κ-μ Distribution) the One-Sided Gaussian distribution can be obtained in an exact manner. Moreover, for $\mu = (1+k)^2/(1+2k)$ in Equation 13 (therefore $\kappa \to 0$ and $\mu = (1+k)^2/(1+2k)$ in the κ-μ Distribution), where k is the Rice parameter, the Rice distribution can be obtained in an approximate manner. The Lognormal distribution, given as a function of m in Equation 13 of [7], can also be approximated by the κ-μ Distribution for $e^{-1} \le \rho \le e$, and for $\kappa \to 0$ and $\mu = m$.

6. APPLICATION OF THE κ-μ DISTRIBUTION

The κ-μ Distribution, as implied in its name, is based on two parameters, κ and μ. Its use involves a procedure similar to that of the other distributions, as explained next. From Equation 6, it can be seen that the two parameters κ and μ can be expressed in terms of the ratio between the mean squared value and the variance of the power, which is usually defined as m. In other words

$$m = \frac{\mu(1+\kappa)^2}{1+2\kappa} \quad (14)$$

For a given m, the parameters κ and μ are chosen that yield the best fitting. Note, on the other hand, that, for a given m, the parameter μ shall lie within the range m and 0, obtained for $\kappa = 0$ and $\kappa \to \infty$, respectively. Therefore, for a given m

$$0 \le \mu \le m \quad (15)$$

The parameter μ is then chosen within the range of Equation 15. Given that μ has been chosen, then κ must be calculated as

$$\kappa = \frac{m}{\mu} - 1 + \sqrt{\frac{m}{\mu}\left(\frac{m}{\mu} - 1\right)} \quad (16)$$

so that the relation as in Equation 14 be kept.

7. SAMPLE EXAMPLES OF THE κ-μ DISTRIBUTION

This section shows some plots of the κ-μ Distribution. Figure 1 and Figure 2, respectively, depict a sample of the various shapes of the κ-μ probability density function $p(\rho)$ and probability distribution function $P(\rho)$ as a function of the normalized envelope ρ for the same Nakagami parameter $m = 0.75$. Figure 3 and Figure 4 do the same but for $m = 1.5$. The plots are illustrated for $\kappa \to 0$, $\kappa = 0.69$, 1.37, 2.41, 4.45, 10.48, and 28.49 (which in decibels correspond to approximately -∞, -1.6, 1.4, 3.8, 6.5, 10.2, and 14.6 dB). The corresponding values of μ are respectively 0.75, 0.625, 0.5, 0.375, 0.25, 0.125, and 0.05 for Figure 1 and Figure 2, and 1.5, 1.25, 1.0, 0.75, 0.5, 0. 25, and 0.1 for Figure 3 and Figure 4. The curves for which $\kappa \to 0$ coincide with the Nakagami-m curve, in which case $\mu = m$. The curve for which $\mu = 1$ coincides with the Rice curve.

It can be seen that, although the normalized variance (parameter *m*) is kept constant for each Figure, the curves are substantially different from each other. And this is particularly relevant for the distribution function, in which case the lower tail of the distribution may yield differences in the probability of some orders. This feature renders the κ-μ Distribution very flexible and this flexibility can be used in order to adjust the curves to practical data [9].

8. CONCLUSIONS

This paper presented a general fading distribution – the κ-μ Distribution – that that can be used to better represent the small scale variation of the fading signal. The distribution includes the One-Sided Gaussian, the Rayleigh, and, more generally, the Nakagami-m and the Rice distributions as special cases and offers a higher degree of freedom. Preliminary results show that the κ-μ Distribution provides a very good fitting to experimental data.

9. REFERENCES

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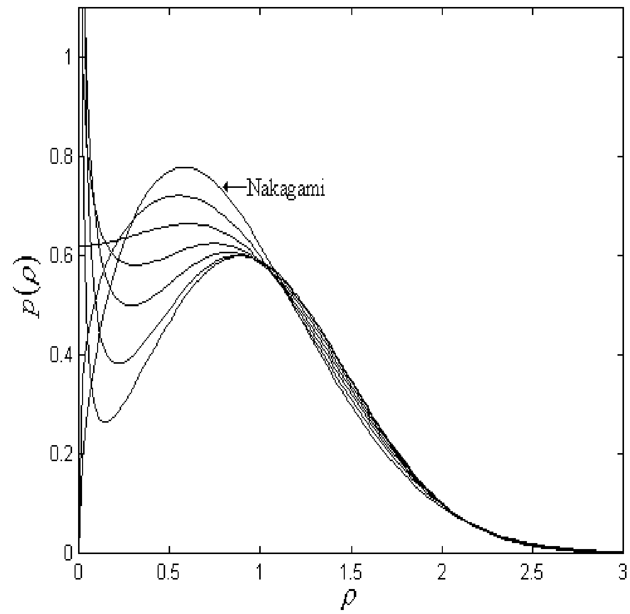

Figure 1: A sample of the various shapes of the κ-μ probability density function for the same Nakagami parameter $m = 0.75$. The Nakagami-m curve is obtained for the condition $\kappa \to 0$ ($\mu = 0.75$).

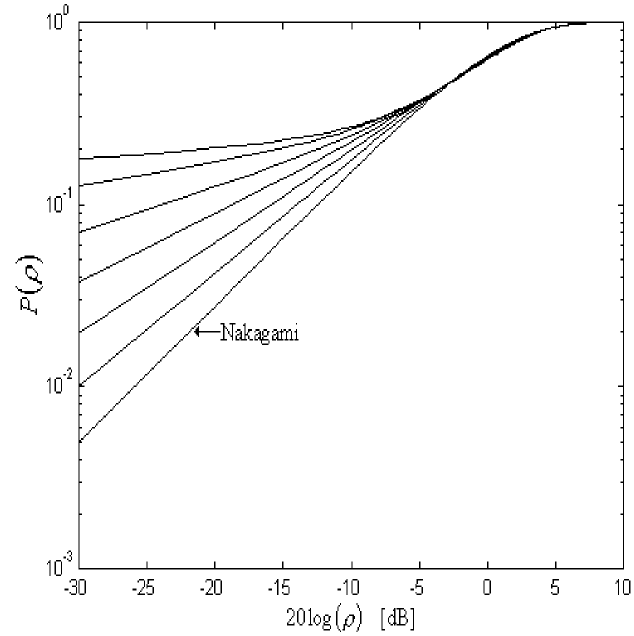

Figure 2: A sample of the various shapes of the κ-μ probability distribution function for the same Nakagami parameter $m = 0.75$. The Nakagami-m curve is obtained for the condition $\kappa \to 0$ ($\mu = 0.75$).

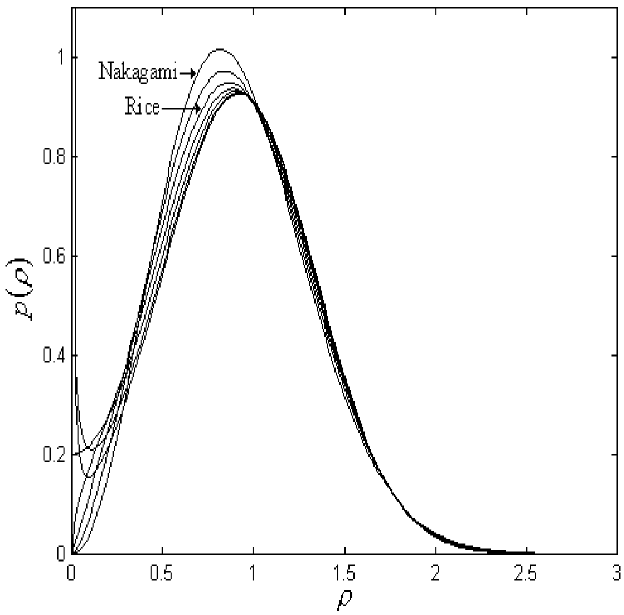

Figure 3: A sample of the various shapes of the κ-μ probability density function for the same Nakagami parameter $m = 1.5$. The Nakagami-m curve is obtained for the condition $\kappa \to 0$ ($\mu = 1.5$). The Rice curve is obtained for the condition $\mu = 1.0$ ($\kappa = 1.37$).

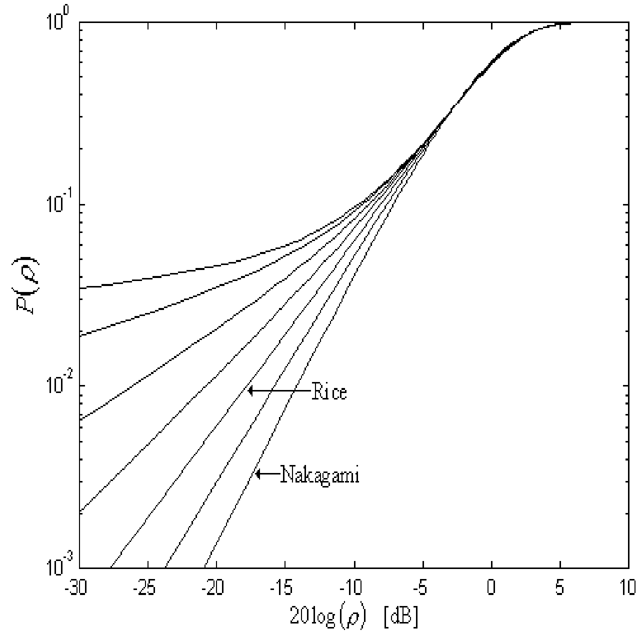

Figure 4: A sample of the various shapes of the κ-μ probability distribution function for the same Nakagami parameter $m = 1.5$. The Nakagami-m curve is obtained for the condition $\kappa \to 0$ ($\mu = 1.5$). The Rice curve is obtained for the condition $\mu = 1.0$ ($\kappa = 1.37$).