DEGREE OF PURITY FOR 2x2 DENSITY MATRIX AND ITS USE IN QUANTUM INFORMATION THEORY

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ABSTRACT

There are many coincidences that relate the treatment of quantum states in a two-dimensional space and the classical polarisation treatment, due to some similarities that occur in coherence matrix and density matrix. We use these similarities to separate the density matrix in a mixture of a pure state and a maximally mixed state. This mixture is controlled by the degree of purity, that measures how much pure is a quantum state, such as usually done with the coherence matrix of a partially polarised beam. After that, we use this degree of purity in quantum information theory. We find out its relationship with the von Neumann entropy and its use in quantum Kolmogorov distance, fidelity and mutual information.

1. INTRODUCTION

Quantum communication is new area of telecommunications in which quantum states are used to carry information. The simplest quantum states are the pure states. These states can be completely characterized by one measurement. For example, the polarisation of one photon, when linearly polarised, can be identified, with 100% of certainty, by a polarisation measurement once there exists one measure that identify it with a probability equal to 1. The quantum state of a linearly polarised photon, with angle θ (between the electric field vector and the horizontal direction), is represented by [1]:

$$|\theta\rangle \equiv \begin{bmatrix} \cos(\theta)e^{i\phi} \\ \sin(\theta) \end{bmatrix} \tag{1}$$

$$\rho = |\theta\rangle\langle\theta| \equiv \begin{bmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta)e^{i\phi} \\ \cos(\theta)\sin(\theta)e^{-i\phi} & \sin^2(\theta) \end{bmatrix} \tag{2}$$

Equation (1) is the vector representation of the pure state and (2) is the density matrix representation. The density matrix is always Hermitean (eigenvalues always real), positive defined (non-negative eigenvalues) and has unitary trace. Quantum states can also be not pure. In this case, it is called a mixed state. There is no measurement able to identify a mixed state with 100% of certainty. A mixed state is represented by:

$$\rho = \sum_i p_i |\theta_i\rangle\langle\theta_i| = \sum_i p_i \rho_i \tag{3}$$

Concerning the photon's polarization, we can say that the photon has the probability p_i to be polarised with angle θ_i. Associated with the eigenvalues of ρ we have the von Newmann entropy [2]:

$$S = -Tr[\rho \log(\rho)] = \sum_i \lambda_i \log(\lambda_i) \tag{4}$$

where the base of the log is assumed to be 2. If the quantum state is a pure state, we have S =0 and, if the state is maximally mixed (all eigenvalues are equal to $1/N$), $S = \log(N)$, where N is the dimension of the space in which the density matrix belongs (or the number of eigenvalues). Let us now suppose the following experiment, shown in Fig. 1: The sender sends polarized photons to the receptor and this last will try to identify which state was sent measuring the photon polarisation.

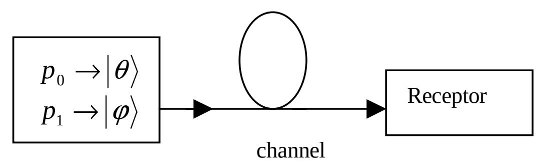

Figure 1 – Quantum communication system with polarized photons.

In Fig. 1, π_0 (π_1) is the *a priori* probability of the sender to send the pure state ρ_0 (ρ_1) = $|\theta\rangle\langle\theta|$ ($|\varphi\rangle\langle\varphi|$). Since the receptor does not know which state will be sent, it "sees" the mixed state:

$$\rho = \pi_0\rho_0 + \pi_1\rho_1 \tag{5}$$

In this case, the receptor will be able to identify the states sent only if ρ_0 and ρ_1 are orthogonal states ($\theta = \varphi \pm 90°$). Pure states are ideal for quantum communication. However, they are hard to preserve. Indeed, due to interaction with the environment, the pure states become mixed states. This process is called decoherence.

A partially polarised light can be decomposed in a sum of a completely depolarised beam and a completely polarised beam, $J = J_{np} + J_p$, and its degree of polarisation, that shows how much polarised the light is, is defined as being equal to $Tr(J_p)$ [1], where we assume the condition $Tr(J)$=1, necessary when a single photon light is considered. In the same way, we can split the density matrix, belonging to a two-dimensional space, in the sum of two other matrices, $\rho = \rho_1 + \rho_2$, and we can also define a degree of purity, g_p, equal to $Tr(\rho_2)$. We have:

$$\rho = \rho_1 + \rho_2 = \left(1 - g_p\right)\frac{I}{2} + \begin{pmatrix} 0.5\left(g_p + \Delta\right) & \rho_{12} \\ \rho_{12}^* & 0.5\left(g_p - \Delta\right) \end{pmatrix} \quad (6)$$

where I is the identity matrix and

$$\Delta = \rho_{11} - \rho_{22} \quad (7)$$

$$g_p = \sqrt{1 - 4\det(\rho)} \quad (8)$$

In (6) and (7) ρ_{ij} are the elements of ρ. The matrices ρ_1 and ρ_2 do not represent any quantum state, because they have not unitary traces. To get around this problem, we work with (6) to expand ρ in a mixture of a maximally mixed state and a pure state:

$$\rho = \left(1 - g_p\right)\rho_m + g_p \rho_p = \left(1 - g_p\right)\frac{I}{2} + g_p \begin{pmatrix} 0.5\left(1 + \Delta/g_p\right) & \rho_{12}/g_p \\ \rho_{12}^*/g_p & 0.5\left(1 - \Delta/g_p\right) \end{pmatrix} \quad (9)$$

where the degree of purity now works as a probability. The necessary condition for the decompositions (6) and (9) to be unique is $det(\rho_2) = det(\rho_p) = 0$. g_p is a measure of purity of a quantum state, like the von Neumann entropy. The relationship between them can be found using the ρ's eigenvalues and the result is:

$$S = -\frac{1}{2}\log\left\{\frac{1}{4}\left(1 + g_p\right)^{\left(1+g_p\right)}\left(1 - g_p\right)^{\left(1-g_p\right)}\right\} \quad (10)$$

For a pure state $S = 0$ and $g_p = 1$ and for maximally mixed state $S = 1$ [bit] and $g_p = 0$. In Figure 2 we show the plot of (10).

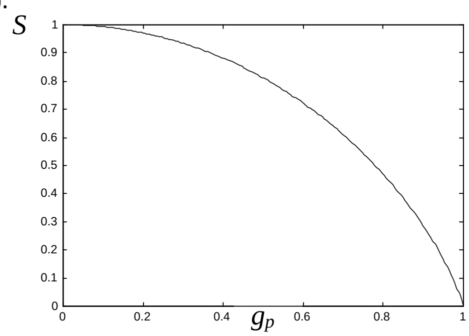

Figure 2 - Entropy x Degree of Purity.

2. QUANTUM KOLMOGOROV DISTANCE USING g_p

For a communication system to work properly, the receptor must be able to distinguish the symbols sent by the sender. Hence, the receptor performs a measurement in the state sent and, according to its results, it infers which state (symbol) was sent. The quantum measurement known as POVM (positive operator value measurement), can be explained suscintly in the following way: For each possible result of a measure, $\{r_1,...,r_i,...r_m\}$, one operator is associated $\{\hat{E}_1,...\hat{E}_i,...,\hat{E}_m\}$, where all operators $\hat{E}_i$ are positive semi-definite and the following condition $\sum_i \hat{E}_i = I$ must be obeyed [2,3]. When we measure a quantum state, represented by the density matrix ρ, using this POVM, we obtain a probability distribution for the possible results, given by:

$$p(r_i \mid \rho) = Tr\left(\rho\hat{E}_i\right) \quad (11)$$

where $p(r_i|\rho)$ is the probability of the value r_i to be obtained as the result of the measurement. Therefore, once the POVM was chosen, for each density matrix we have associated one probability distribution. Hence, we can distinguish two quantum states using the distinguishability measurements applied to distinguish between two probability distributions. One of the most useful is the *error probability*, *PE*. Choosing the POVM that maximizes the distinguishability, we have [3]:

$$PE(\rho_a, \rho_b) = \frac{1}{2} - \frac{1}{4}Tr\left(\left|\rho_a - \rho_b\right|\right) \quad (12)$$

PE is not a distance measure. For a distance measure we can use the Kolmogorov distance, *K*, which is related to *PE* by:

$$K(\rho_a, \rho_b) = 1 - 2PE(\rho_a, \rho_b) = \frac{1}{2}Tr\left(\left|\rho_0 - \rho_1\right|\right) \quad (13)$$

When the states are indistinguishable K=0 (PE = 0.5), and when the states are perfectly distinguishable K=1 (PE = 0). Let us now suppose two density matrices written in the form (9):

$$\rho_a = \left(1 - g_p^a\right)\frac{I}{2} + g_p^a \begin{pmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin^2(\theta) \end{pmatrix} \quad (14)$$

$$\rho_b = \left(1 - g_p^b\right)\frac{I}{2} + g_p^b \begin{pmatrix} \cos^2(\varphi) & \cos(\varphi)\sin(\varphi) \\ \cos(\varphi)\sin(\varphi) & \sin^2(\varphi) \end{pmatrix} \quad (15)$$

where, for simplification, the pure parts of (14) and (15) have no complex values. The Kolmogorov distance between these density matrices can be obtained applying (14) and (15) in (13):

$$K = \sqrt{0.25\left(g_p^b - g_p^a\right)^2 + g_p^b g_p^a sin^2(\varphi - \theta)} \quad (16)$$

For example, the Kolmogorov distance between any pure state (g_p = 1) and the maximally mixed state (g_p = 0) is 0.5. In Fig. 3 we show a simulation of $K(\rho_a,\rho_b)$, using (14) and (15), when the density matrices go from pure states to mixed states. The values used in the simulations were $\theta = \pi/4$ and $\varphi = \pi/3$.

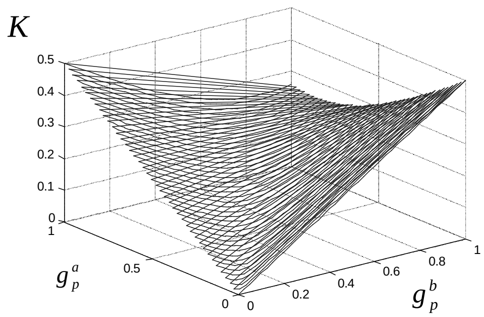

Figure 3 - $K(g_p^a, g_p^b)$.

3. NOISE CHANNEL AND MUTAL INFORMATION USING g_p

The effect of noise in the channel is to transform a pure state in a mixed state. Let us take a quantum communication system where the source sends the quantum pure state $|\alpha_i\rangle$ with probability p_i. After passing through the noisy channel, the state $|\alpha_i\rangle$ evolves to one of the states $\left\{ |\beta_j^i\rangle \right\}_{j=1}^{N}$ with probabilities $\left\{ p_j^i \right\}_{j=1}^{N}$, respectively [4], as shown in Fig. 4.

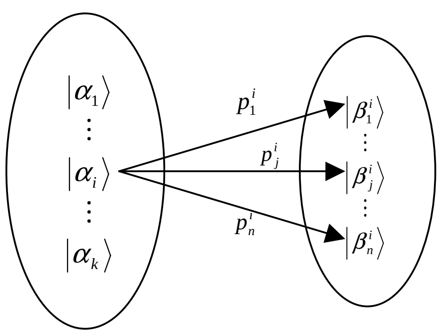

Figure 4 – Noisy channel

Therefore, each initial pure state $|\alpha_i\rangle$ evolve to the mixture:

$$\rho_i = \sum_{j=1}^{N} p_j^i |\beta_j^i\rangle\langle\beta_j^i| = \tag{17}$$
$$(1-g_p^i)\rho_m + g_p^i\rho_p^i = (1-g_p^i)\rho_m + g_p^i|\Phi_i\rangle\langle\Phi_i|$$

In (17), once more, we suppose that there are not complex values and $\langle\Phi_i|\Phi_i\rangle = 1$. We can find g_p^i from p_j^i evaluating $\langle\Phi_i|\rho_i|\Phi_i\rangle$ in both sides of (17):

$$g_p^i = 2\sum_{j=1}^{N} p_j^i \left|\langle\Phi_i|\beta_j^i\rangle\right|^2 - 1 \tag{18}$$

If $|\Phi_i\rangle \in \left\{ |\beta_j^i\rangle \right\}_{j=1}^{N}$, then (18) reduces to:

$$g_p^i = 2p_n^i - 1 \tag{19}$$

in which $|\Phi_i\rangle = |\beta_n^i\rangle$ was supposed. To identify the quantum state sent, the receiver will perform a measure in the state received. Let's suppose that, for this, it uses the operator $\hat{A}$, with eigenvalues $\left\{ |\varepsilon_k\rangle \right\}_{k=1}^{M}$, to perform the measurement. Thus, for the noisy channel described earlier, the classical mutual information per symbol, $I = H(\rho)-H(\rho|\varepsilon)$, where H is the Shannon entropy, is given by:

$$I_i = -\sum_{k=1}^{M}\left[\frac{(1-g_p^i)}{2}+g_p^i\sigma\right]\log_2\left[\frac{(1-g_p^i)}{2}+g_p^i\sigma\right] - \frac{(1-g_p^i)M}{2} + g_p^i\sum_{k=1}^{M}\sigma\log_2(\sigma) \tag{20}$$

$$\sigma = \left|\langle\varepsilon_k|\Phi_i\rangle\right|^2 \tag{21}$$

The average mutual information and the Kholevo upper bound [2,5] are found to be:

$$I = \sum_{i=1}^{N} p_i I_i \tag{22}$$

$$I_i \le -\frac{1}{2}log_2\left\{\frac{(1+g_p^i)^{(1+g_p^i)}(1-g_p^i)^{(1-g_p^i)}}{4}\right\} - (1-g_p^i) \tag{23}$$

As an example, we will analyse the quantum communication system shown in Fig. 5.

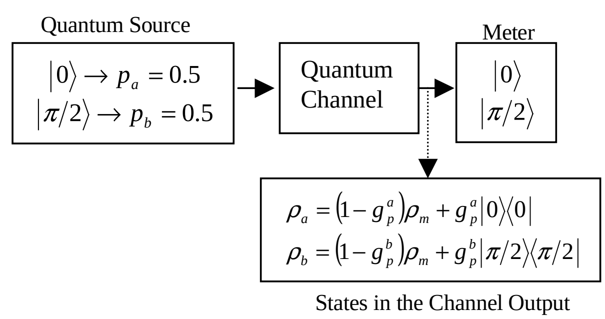

Figure 5– Binary quantum communication system.

In this figure, the emitter sends the pure states |0⟩ and |π/2⟩, with probabilities p_a = p_b = 0.5. After passing through the channel, |0⟩ and |π/2⟩ evolve to the mixed states ρ_a and ρ_b, respectively. These states are measured by the operator whose eigenvalues are the states |0⟩ and |π/2⟩. Using these data in (14) we find, for the classical mutual information per symbol, the following equations:

$$I_a = -\frac{1}{2} log_2 \left[\frac{\left(1+g_p^a\right)^{\left(1+g_p^a\right)} \left(1-g_p^a\right)^{\left(1-g_p^a\right)}}{4} \right] - \left(1-g_p^a\right) \quad (24)$$

$$I_b = -\frac{1}{2} log_2 \left[\frac{\left(1+g_p^b\right)^{\left(1+g_p^b\right)} \left(1-g_p^b\right)^{\left(1-g_p^b\right)}}{4} \right] - \left(1-g_p^a\right) \quad (25)$$

$$I = \frac{1}{2}\left(I_a + I_b\right) \quad (26)$$

Comparing (24) and (25) with (23), we observe that the Kholevo's limit was achieved due to correct choice of the measurement basis. In Fig. 6 we can see a plot of the average information, (26).

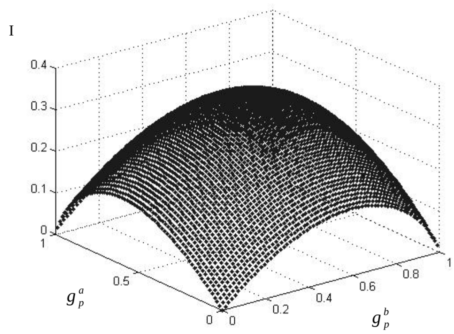

Figure 6 – Average information, $I\left(g_p^a, g_p^b\right)$.

At last, the fidelity, *F*, measures the channel quality for the transmission of quantum states. Suppose that the quantum source transmits the pure state $|\theta_i\rangle$ with probability p_i, and each pure state $|\theta_i\rangle$, after propagation in the channel, evolves to the mixed state ρ_i. The fidelity, *F*, is given by [2,3,5]:

$$F = \sum_i p_i \left\langle \theta_i \middle| \rho_i \middle| \theta_i \right\rangle \quad (27)$$

For the system shown in Fig. 5 we have:

$$F = \frac{1}{2}\left\langle 0 \middle| \rho_a \middle| 0 \right\rangle + \frac{1}{2}\left\langle \pi/2 \middle| \rho_b \middle| \pi/2 \right\rangle \Rightarrow \quad (28)$$

$$F = \frac{1}{2}\left[\begin{array}{l} \left(1-g_p^a\right)\left\langle 0 \middle| \rho_m \middle| 0 \right\rangle + g_p^a \left\langle 0 \middle| \rho_p^a \middle| 0 \right\rangle + \\ \left(1-g_p^b\right)\left\langle \pi/2 \middle| \rho_m \middle| \pi/2 \right\rangle + g_p^b \left\langle \pi/2 \middle| \rho_p^b \middle| \pi/2 \right\rangle \end{array} \right] \Rightarrow \quad (29)$$

$$F = \frac{1}{2} + \frac{g_p^a + g_p^b}{4} \quad (30)$$

If the states at the output of the channel are equal to the states sent, *F* =1. In the worst case, when the states at the channel output are maximally mixed states, *F*=0.5.

4. SUMMARY

We started by splitting the density matrix ρ in a mixture of a maximally mixed state and a pure state. This mixture is controlled by the degree of purity, g_p, that shows how much pure is a quantum state. Then, we applied g_p in quantum information theory, finding its relationship with the von Neumann entropy, and expressing the quantum Kolmogorov distance, classical mutual information and fidelity, as functions of the degree of purity.

5. ACKNOWLEDGMENTS

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6. REFERENCES

[1] Mandel, L., and Wolf, E., 1995, *Optical Coherence and Quantum Optics* (Cambridge University Press).

[2] Charles H. Bennet, "Quantum Information", *Physica Scripta*, Vol. T76, pp. 210-217, 1998.

[3] Graaf, J. V., 1997, Towards a Formal Definition of Security for Quantum Protocols, Ph.D. thesis, Département d'Informatique et de Recherche Opérationnelle, Université de Montréal.

[4] Seth Lloyd, "Capacity of the noise quantum channel", *Physical Review A*, Vol. 55, N° 3, pp. 1613-1622, March 1997.

[5] Benjamin Schumacher, "Information from Quantum Measurements", Complexity, Entropy, and the Physics of Information, *SFI Studies in the Sciences of Complexity*, Vol. VIII, Ed. W. H. Zurek, Addison-Wesley.