ON THE CAPACITY OF NOISY CHANNEL WITH M-ARY RUNLENGTH-LIMITED INPUTS

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ABSTRACT

Motivated by forthcoming of new materials that supports multi-amplitude data storage, McLaughlin and Luo [1] present some results on the capacity of noiseless M-ary runlength-limited codes. In this paper we tackle with a M-ary noisy channel which inputs are M-ary runlength-limited sequences. We obtain a lower bound on the capacity of that a channel and calculate the power spectral density of its input sequences. The results presented here can be viewed as a generalization of that given in [2] for the binary case.

1. INTRODUCTION

Magnetic and optical recording systems for M-level recording are now available [1]. The material employed in that systems keep linearity of the channel, but M-ary data sequences should obey runlength-limited (RLL) or $(d,\ k)$ constraints. Shannon capacity of a noiseless M$(d,\ k)$ code is given by base-2 logarithm of the largest real root of the equation [3]:

$$x^{k+2} - x^{k+1} - (M-1)\, x^{k-d+1} + M - 1 = 0 \tag{1}$$

In this paper is presented a lower bound on the capacity of a noisy channel whose input is by M-ary RLL constrained sequences. The noisy channel is modeled as a generalization of the binary symmetric channel (BSC) shown in Figure 1. For the sake of simplicity we assume constant crossover probabilities, despite of its simplicity, the channel model adopted captures the mechanism of errors generation in recovering $M(d,\ k)$ recorded information.

The analysis of power spectrum density (PSD) of the sequences used in the $M(d,k)$ codes can provides information as DC level and other spectral desired characteristics. In this work we obtain the PSD of $M(d,\ k)$ codes using a technique proposed by Bilardi and Pierobon in [4].

A M-ary (d,k) code is a set of sequences of symbols from an alphabet $\mathcal{A} = \{0, 1, 2, ..., M-1\}$ where at least d and at most k zeroes can be accepted between nonzero symbols. Often we represent RLL $M(d,k)$ constraint by means a finite-state sequential machine (FSSM) shown in Fig. 2. Any RLL $M(d,\ k)$ sequence corresponds to a sequence of labels through paths off the FSSM. There are $(k+1)$ states represented by random variable $S \in \{0, 1, \ldots, k\}$. We denote by $P\,(i,\ i+1) = (1-p_i)\,,\ i = 0, \ldots, k-1$ the probability of transition from the state i to the state $i+1$. The probability of transition from state $i,\ i = d,\ d+1, \ldots,\ k$ to state 0 is given by

$$P\,(i,\ 0) = \frac{p_i}{M-1}. \tag{2}$$

The transition probability matrix corresponding to the Markov chain of the FSSM shown in Figure 3 is given by:

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & ... & 0 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \ldots & 0 & 0 & \ldots & 0 \\ p_d & 0 & 0 & ... & q_d & 0 & ... & 0 \\ p_{d+1} & 0 & 0 & ... & 0 & q_{d+1} & ... & 0 \\ ... & \ldots & ... & ... & ... & ... & ... & ... \\ p_{k-1} & 0 & 0 & ... & 0 & 0 & ... & q_{k-1} \\ p_k & 0 & 0 & ... & 0 & 0 & ... & 0 \end{bmatrix}, \tag{3}$$

where probabilities $p_i,\ i = 1, \ldots, k$ sum up transition probabilities of arcs labeled by nonzero symbols.

2. PROBABILITY FOR M-ARY SEQUENCES

In [2] the binary (d,k) sequences was analyzed using a division in phrases. In that case given a stream of bits, the begin of a phrase is the first zero symbol and his end it is marked at the first 1 appear in the sequence, the phrase length is the number of bits. It was created a random variable X_i related with the phrases lengths. In that work the probability distribution of X, was given by

$$P(X=i) = 2^{-iC},\ i = d+1, ..., k+1 \tag{4}$$

where C is obtained from equation 1 with $M = 2$.

Using the same form to analyze a $M(d,k)$ sequence, considering the end of a phrase when the first symbol different of zero appear, we obtain the distribution of L the random variable of the phrases lengths in the M-ary case as

$$P(L=i) = \frac{2^{-i(H_P(X)+\log(M-1))}}{\sum_{j=d+1}^{k+1} 2^{-j(H_P(X)+\log(M-1))}}. \tag{5}$$

where $H_P(X)$ is the entropy of the variable X for the binary sequence with the same d and k obtained using Equation 4.

3. LOWER BOUND ON THE CAPACITY OF THE $M-$ARY $(D,\ K)$ CROSSOVER CHANNEL

Let $\{\mathbf{S}\}$ be the stationary Markovian chain given by the sequence of states of the FSSM shown in Figure 3. Let $\{\mathbf{Y}\}$ be the stationary Markovian process formed by symbols $Y = Y(S_1,\ S_2)$, where S_1 and S_2 are consecutive states from the process $\mathbf{S}$. The noisy channel we assume is shown in Figure 1. The channel has M-ary input Y and its output Z, is equal to the input with probability $1-\alpha$. With probability $\frac{\alpha}{2}$, on the other hand, Z equals a neighbor symbol. We note that if $M=2$ the model is equivalent to well-known BSC channel.

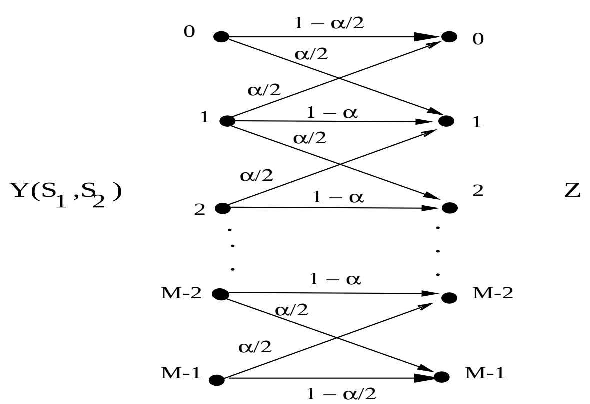

Figure 1: Noisy Channel with Crossover Probability $\frac{\alpha}{2}$, with M-ary alphabet. The input $\mathbf{Y}$ is the stationary Markovian process formed by symbols $Y = Y(S_1,\ S_2)$, where S_1 and S_2 are consecutive states of FSSM shown in Figure 3.

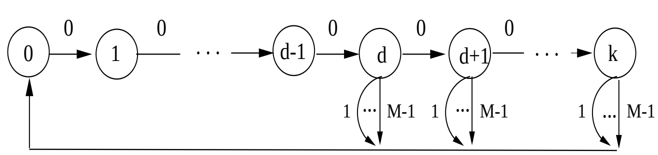

Figure 2: State machine for $M(d,k)$ sequences.

In this section we derive a lower bound to the capacity of the M-ary noisy crossover channel of the Figure 1. In order

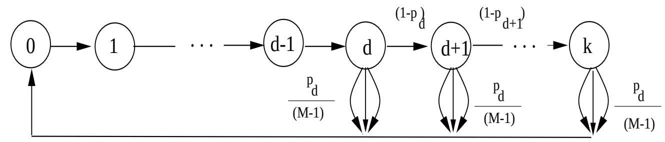

Figure 3: State machine for transition probability.

to perform this derivation we use the following lemma due to Wolf et all in [2]:

Lemma 1 *The capacity of a memoryless channel under the constraint that the input is stationary and Markovian is lower-bounded by*

$$C \geq \sup_{P(S_1,S_2)} I\left(S_2; Z/S_1\right) \tag{6}$$

where S_1 and S_2 are consecutive states of Markovian process that produces the sequence $\{\mathbf{Y}\}$, $P(S_1,S_2)$ the probability of step S_1, S_2 and $Z \in \mathcal{A}$ is the output affected by the noise.

The mutual information of Equation 6 can be written as

$$I(S_2;\ Z/S_1) = H(S_2/S_1) + H(Z/S_1) - H(S_2,\ Z/S_1). \tag{7}$$

From the stationary probabilities the states $P\left[S=j\right]$, $j=0,\ldots,k$, transition probability matrix given by Eq. 3 and Figure 3 we are able to derive the entropies of Eq. 7 (see Appendix). Replacing the entropy values calculated in the Appendix in Eq 7 (after some algebrism) we obtain:

$$\begin{aligned} I(S_2;\ Z/S_1) &= \sum_{j=0}^{k} P[S=j]h(p_j) \\ &+ \sum_{j=d}^{k-1} P[S=j]\left\{ H\left[(1-p_j)(1-\frac{\alpha}{2}) + \frac{p_j}{M-1}\frac{\alpha}{2},\right.\right. \\ &\left.\frac{p_j}{M-1}(1-\frac{\alpha}{2}) + (1-p_j)\frac{\alpha}{2}\right] \\ &- H\left[(1-\frac{\alpha}{2})(1-p_j),\right. \\ &\left.\left.(1-p_j)\frac{\alpha}{2}, \frac{p_j}{(M-1)}\frac{\alpha}{2}, (1-\frac{\alpha}{2})\frac{p_j}{M-1}\right]\right\}. \end{aligned} \tag{8}$$

The stationary probability of jth state is given by

$$P[S=j] = P[S=0]\prod_{i=1}^{j-1} q_j, \tag{9}$$

where $q_j = 1-p_j$ and $P[S=0] = 1/L_s$, L_s is the average length of the sequences [2].

We note that for an erasure channel model the lower bound given in [5] does not depend on M, the input alphabet size. This in contrast with the result obtained here for crossover channel model.

The lower bound on the capacity is calculated using the Equations 8 and 9. The supreme of the Equation 6 was obtained by a search in the input probability space p_j, $j = d, ..k$. We observe that for $M = 2$ the same values of capacity given in Wolf [2] were obtained with the generalized formulas 8 and 9, as was expected.

For $d = 0$ and $k = 3$, in the Figure 4 shows the capacity for some values of M. The same was made for $d = 1$ and $k = 3$ and the results is showed in the Figure 5.

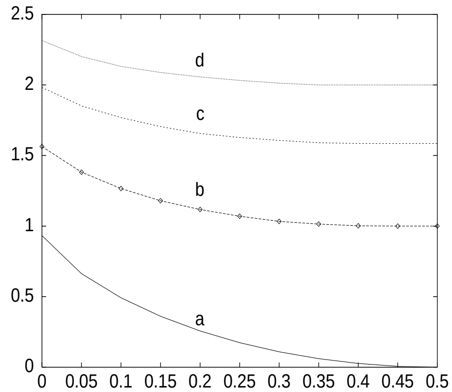

Figure 4: Capacity for $d = 0$ and $k = 3$ a)$M = 2$, b)$M = 3$, c)$M = 4$ and d)$M = 5$

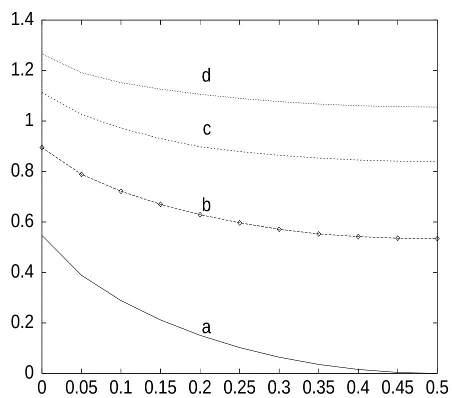

Figure 5: Capacity for $d = 1$ and $k = 3$ a)$M = 2$, b)$M = 3$, c)$M = 4$ and d)$M = 5$

It can be observed the effect of the restriction in the capacity, since when $\alpha = 0$, the channel do not have crossing, the capacity was a value lower than $C = log(M)$.

It can be verified a increase in the capacity at last the probability of crossing be great.

In the next section it is presented some results of power spectrum density of the M-ary sequences.

4. POWER SPECTRUM DENSITY OF $M(D, K)$ SEQUENCES.

In order to apply the method given in [4], we first split the zero state obtaining the equivalent FSSM displayed in Figure 6. The correspondent transition probability matrix is given by Eq. 10:

$$\mathbf{P} = \begin{bmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{p_d}{(M-1)} & \dots & \frac{p_d}{(M-1)} & q_d & 0 & \dots & 0 \\ \frac{p_{d+1}}{(M-1)} & \dots & \frac{p_{d+1}}{(M-1)} & 0 & q_{d+1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{p_{k-1}}{(M-1)} & \dots & \frac{p_{k-1}}{(M-1)} & 0 & 0 & \dots & 0 \\ \frac{p_k}{(M-1)} & \dots & \frac{p_k}{(M-1)} & 0 & 0 & \dots & q_k \end{bmatrix} \tag{10}$$

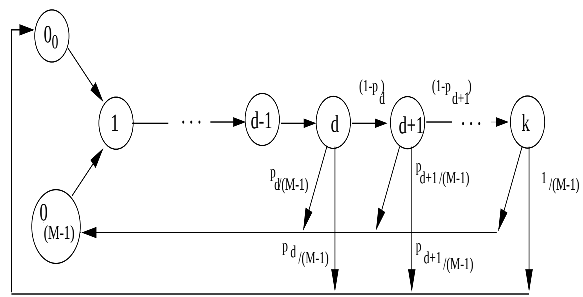

Figure 6: State machine after state splitting.

We calculate the PSD for two recording schemes. First we assume a NRZI scheme shown in Figures 7(binary case) and 8(M-ary case.) Let x_n be the n-th symbol of the $M(d,\ k)$ sequence, in the NRZI recording scheme, if $x_{n+1} = x_n$, the output recording level y_{n+1} is hold. Otherwise, it is updated with $y_{n+1} = \pm x_{n+1}$, inverting the signal of the previous level (see Figure 8.) The second recording scheme uses the rule $y_{n+1} = y_n + x_n \pmod{M}$ to define the output level.

In [2] was proposed a NRZI coding to be applying in sequences (d, k). In that case the result sequence has values +1 e -1 and, if the input is the symbol 0 the output maintain its value, and if an input symbol is 1 the output change its signal. For example observe the Figure 7, in this case the initial state is +1 and the level only change when the input symbol is 1.

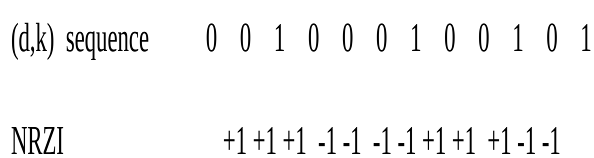

Figure 7: NRZI for the binary case.

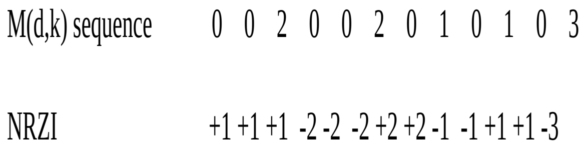

Figure 8: NRZI for the M-ary case.

The power spectral density results for $(d,k) = (1,3)$ and some values of M using the proposed NRZI are presented in the Figure 9. The curves have the same format but with different levels because with the grow of M there are more levels and more power is need.

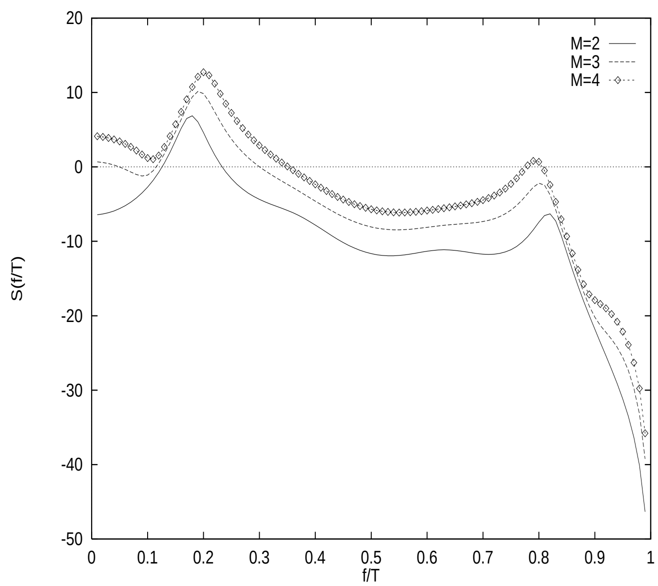

Figure 9: PSD for $(d,k) = (1,3)$ and some values of M.

In [1] another NRZI code was proposed. In this case the $M(d,k)$ sequence is passed throw a pre-coder that generates the output symbol $y_j = y_{j-1} + x_j(modM)$.

For $(d,k) = (1,3)$ and some values of M, in the Figure 10 is shown the PSD of the sequences generated by the second scheme. In this case is observed the presence of DC level and their grow with the increase of M.

5. CONCLUSIONS

We have obtained a lower bound on the capacity for a noisy channel with M-ary RLL constrained. The new lower bound generalizes that proposed by Wolf et all [2] for the BSC channel with binary RLL sequences as its input. It was observed that the capacity lower bound for $M > 2$ holds for relatively high noise level (high crossover probability).

For the noiseless M-ary RLL channel we have obtained the probability distribution of the length, L, of phrases of M-

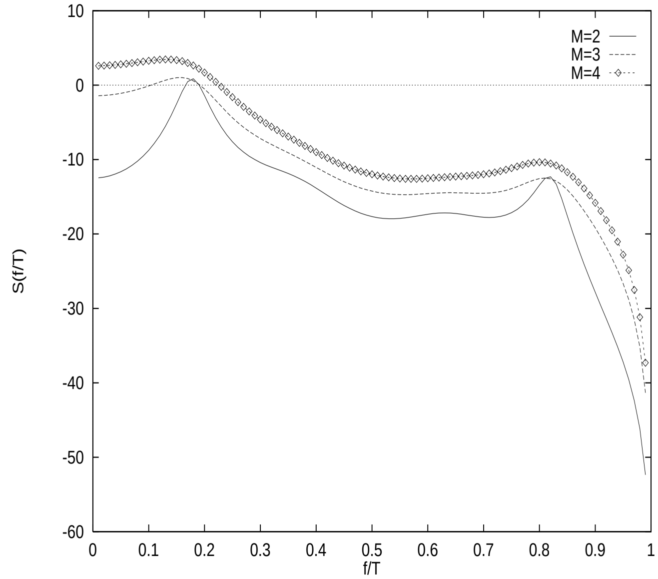

Figure 10: NRZI mod M scheme.

ary symbols, defined similarly to the binary RLL sequences. This is another generalization of a result given in Wolf [2]. Power spectral densities for RLL M-ary sequences are presented. The results are displayed for two recording schemes, NRZI and NRZI modulo M.

A. CALCULATION OF THE MUTUAL INFORMATION $I\left(S_2; Z \mid S_1\right)$

In this Appendix we derive the mutual information of Eq. 7 [6], by means calculating each of entropies $H(S_2 \mid S_1)$, $H(Z \mid S_1)$ and, $H(S_2,\ Z \mid S_1)$. In the following we denote

$$h(x) = -x\log x - (1-x)\log(1-x),$$

the binary entropy.

From Figure 3 we obtain:

$$H(S_2/S_1) \quad = \quad \sum_{j=d}^{k-1} P[S=j]h(p_j) \qquad (11)$$

From Figures 1 and 3 we take, after some reasoning, the probability distributions used to obtain the last two entropies conditioned on S_1.

$$\begin{aligned} H(Z/S_1) &= \sum_{j=0}^{d-1} P[S=j] h(\alpha/2) \\ &+ \sum_{j=d}^{k-1} P[S=j] H\left[(1-p_j)(1-\alpha/2)\right. \\ &+ \frac{p_j}{M-1}\alpha/2, \\ &\quad (1-p_j)\alpha/2 + \frac{p_j}{M-1}(1-\alpha/2), \\ &\quad \left.(M-2) H\left(\frac{p_j}{M-1}\right)\right] \\ &+ P[S=k] H\left[\frac{1}{M-1}\alpha/2\ ,\right. \\ &\quad \left.\frac{1}{M-1}(1-\alpha/2), \frac{M-2}{M-1}\right]; \end{aligned} \tag{12}$$

and

$$\begin{aligned} H(S_2;Z/S_1) &= \sum_{j=0}^{d-1} P[S=j] h(\alpha/2) \\ &+ \sum_{j=d}^{k-1} P[S=j] H\left[(1-\alpha/2)(1-p_j),\right. \\ &\quad (1-p_j)\frac{\alpha}{2}, \frac{p_j}{M-1}(1-\alpha/2), \\ &\quad \left.\frac{p_j}{M-1}\alpha/2, \frac{(M-2)}{M-1}p_j\right] \\ &+ P[S=k] H\left[\frac{1}{M-1}\alpha/2,\right. \\ &\quad \left.\frac{1}{M-1}(1-\alpha/2), \frac{(M-2)}{M-1}\right]; \end{aligned} \tag{13}$$

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