An Optimized FXLMS Based Algorithm for Application in Nonlinear Environments

Márcio Holsbach Costa[1] and José Carlos Moreira Bermudez[2]

[1]Lab. de Eng. Biomédica, Escola de Engenharia e Arquitetura, Universidade Católica de Pelotas, Pelotas-RS, Brazil
[2]Department of Electrical Engineering, Universidade Federal de Santa Catarina, Florianópolis-SC, Brazil
E-mails: m.costa@ieee.org, j.bermudez@ieee.org

Abstract - **This work presents a new algorithm based on the Minimum Output Variance Least Mean Square Estimator when the influence of the secondary path can not be neglected and its output is constrained by a saturation nonlinearity. This situation is typical in several adaptive modeling and control systems where the associated hardware and transducers have a finite power handling capability. Analytical expressions are obtained for the behavior of the mean weight vector and for the mean square error for Gaussian inputs and slow learning. The optimum penalty factor is determined. The new algorithm provides an unbiased solution to the associated nonlinear mean square estimation problem for small estimation errors of the secondary path and degree of nonlinearity. Monte Carlo simulations show excellent agreement with the predictions of the theoretical model.**

I. Introduction

Adaptive filters have been successfully and largely used in communications and control systems [1]. The family of stochastic gradient adaptive algorithms (the LMS family) is the most popular in many applications, such as echo cancellation, equalization and active noise or vibration control (ANC). The main reasons for that are its simplicity and robustness [1,2].

In general, these algorithms are designed for maximum performance in linear environments. However, the associated hardware (power amplifiers and transducers) frequently becomes an important source of nonlinear effects [3,4]. In many cases, such nonlinear effects can severely impair the adaptive algorithm performance [3,7,8].

In system identification problems, one of the most common nonlinear effects is saturation at the adaptive filter output [3,5,7,9]. One solution to avoid such nonlinear distortions is to overdesign the system. The signal power is kept sufficiently low compared to existing saturation levels. This solution increases the cost and limits the system's performance. A more effective solution is to embed an automatic control of the nonlinear effects within the adaptive algorithm. This is usually achieved by adding a penalty function to the adaptive algorithm's cost function in order to control the input signal to the nonlinearity. Algorithms that use this approach are called minimum effort adaptive filters. Examples are the Leaky-LMS, which seeks to minimize the norm of the filter's tap weight vector, and the Minimum Output Variance Least Mean Square adaptive estimator (MOV-LMS), which minimizes the adaptive filter output power. From these, the MOV-LMS is the most appropriate to handle saturation nonlinearities at the adaptive filter output, as it directly controls the signal power at the nonlinearity input. However, different choices of the penalty term lead to different steady-state misadjustments [5,6].

Reference [7] studied the behavior of the LMS algorithm with a saturation nonlinearity at the adaptive filter output. It was demonstrated that the mean converged weights correspond to a biased solution with respect to the minimum of the MSE surface. This bias is a multiplicative scalar, which is a function of the system's degree of nonlinearity. A recent work [8] demonstrated the possibility of achieving maximum cancellation in a nonlinear environment by properly designing the penalty factor of the MOV-LMS algorithm.

The results obtained in [7,8] cannot be directly applied when the secondary path (the signal path containing the adaptive filter) significantly modifies the control signal produced by the adaptive filter. Such is the case when there is a filtering operation in the secondary path.

Reference [9] extended the analysis presented in [7] for the LMS algorithm to the Filtered-X LMS (FXLMS) case. The results in [9] show that the FXLMS algorithm performance can be significantly affected even in systems with small degrees of nonlinearity. The steady-state weight vector bias includes a directional change if the estimation of the secondary path is not perfect. Moreover, the multiplicative bias cannot be controlled even with perfect secondary path estimation. Thus, there is a need for an adaptive algorithm that can provide unbiased solutions for system identification problems with a saturation nonlinearity in the secondary path.

This work presents the analysis of a variant of the MOV-LMS adaptive algorithm, called MOV-FXLMS. Like the regular FXLMS algorithm, the MOV-FXLMS algorithm takes into account a linear filtering operation in the secondary path. Unlike the FXLMS algorithm, it also takes into account a memoryless saturation nonlinearity at the linear filter output.

Deterministic nonlinear recursions are derived for the mean weight and mean square error. These recursions can predict the algorithm behavior during transient and in steady-state. The optimum penalty factor is determined as a function of the system's degree of nonlinearity. A simple procedure is suggested for the estimation of the nonlinearity. It is shown that the new algorithm using the optimal penalty factor converges to the minimum of the performance surface (maximum cancellation level).

Finally, comparisons between the theoretical model predictions and the results of Monte Carlo simulations are provided, which show very good agreement. The robustness of the algorithm to errors in the estimation of the degree of nonlinearity is verified.

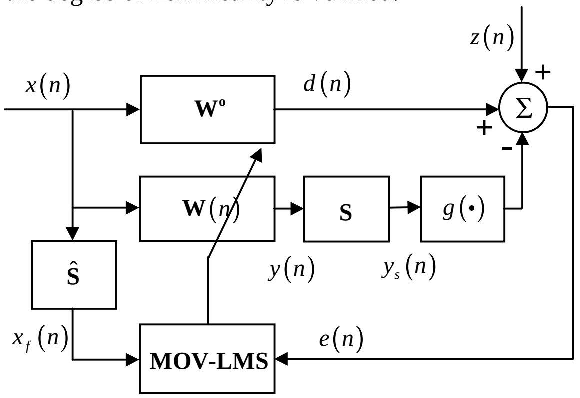

Fig. 1. Block diagram of the system analyzed.

II. The Analysis Model

Consider the system in Fig. 1. $\mathbf{W^o} = \left[w_0^o \quad w_1^o \quad \dots \quad w_{N-1}^o \right]^T$ is the unknown impulse response; $d(n)$ is the primary signal; $z(n)$ is a stationary, white, zero-mean Gaussian measurement noise with variance σ_z^2 and uncorrelated with any other signal. $x(n)$ is stationary, zero-mean and Gaussian. $\mathbf{X}(n) = \left[x(n) \quad x(n-1) \quad \dots \quad x(n-N+1) \right]^T$ is the observed data vector and $e(n)$ is the error signal. $\mathbf{W}(n) = \left[w_0(n) \quad w_1(n) \quad \dots \quad w_{N-1}(n) \right]^T$ is the adaptive weight vector. $\mathbf{S} = [s_0 \quad s_1 \quad \dots \quad s_{M-1}]^T$ is the secondary path impulse response and $\hat{\mathbf{S}} = \left[\hat{s}_0 \quad \hat{s}_1 \quad \dots \quad \hat{s}_{\hat{M}-1} \right]^T$ is the estimate of $\mathbf{S}$. $x_f(n)$ is the filtered reference signal; $\mathbf{X}_f(n) = \left[x_f(n) \quad x_f(n-1) \quad \dots \quad x_f(n-N+1) \right]^T$ is the filtered data vector. $y(n)$ is the adaptive filter output and $y_s(n)$ is $y(n)$ filtered by the secondary path. The nonlinearity $g(\bullet)$ is modeled by the error function [7]:

$$g(y) = \int_0^y e^{-\frac{z^2}{2\sigma^2}} dz \tag{1}$$

The behavior of $g(\bullet)$ can be varied between that of a linear device ($\sigma^2 \to \infty$) and that of a hard limiter ($\sigma^2 = 0$) by changing σ and by using a suitable multiplicative constant (considered unity here for simplicity).

III. MOV-FXLMS Algorithm

The MOV-FXLMS algorithm minimizes the instantaneous cost function:

$$J(n) = e^2(n) + \gamma y_s^2(n) \tag{2}$$

Assuming, for a while, the system is purely linear ($g(y) = y$),

$$y_s(n) = \sum_{i=0}^{M-1} s_i \mathbf{W}^T(n-i)\mathbf{X}(n-i) \tag{3}$$

is the secondary path output signal. Evaluating the gradient of (2) with respect to the weight vector yields the weight update equation:

$$\begin{aligned} \mathbf{W}(n+1) &= \mathbf{W}(n) - \frac{\mu}{2}\nabla J(n) \\ &= \mathbf{W}(n) + \mu\left[e(n) - \gamma \cdot y_s(n)\right]\mathbf{X}_f(n) \end{aligned} \tag{4}$$

where μ is the step size, γ is the penalty factor and

$$\mathbf{X}_f(n) = \sum_{j=0}^{\hat{M}-1} \hat{s}_j \mathbf{X}(n-j) \tag{5}$$

is the input signal filtered by an estimation of **S**.

IV. Mean Weight Behavior

The expected value of (4) is determined assuming that the effect of the correlation between input and weight vectors on the algorithm behavior can be neglected, when compared to the effect of the correlations of lagged input vectors [1,7]. Using this assumption and conditioning the expectation of (4) on

$$\mathcal{W} = \left\{ \mathbf{W}(n) \ , \ \mathbf{W}(n-1) \ , \ \dots \ , \ \mathbf{W}(n-M+1) \ \right\} \tag{6}$$

yields

$$\begin{aligned} E\left\{\mathbf{W}(n+1)\middle|\mathcal{W}\right\} &= E\left\{\mathbf{W}(n)\middle|\mathcal{W}\right\} + \mu\sum_{j=0}^{\hat{M}-1}\hat{s}_j E\left\{z(n)\mathbf{X}(n-j)\middle|\mathcal{W}\right\} \\ &- \mu\gamma\sum_{i=0}^{M-1}\sum_{j=0}^{\hat{M}-1} s_i\hat{s}_j E\left\{\mathbf{X}(n-j)\mathbf{X}^T(n-i)\mathbf{W}(n-i)\middle|\mathcal{W}\right\} \\ &+ \mu\sum_{j=0}^{\hat{M}-1}\hat{s}_j E\left\{\mathbf{X}(n-j)\mathbf{X}^T(n)\middle|\mathcal{W}\right\}\mathbf{W^o} \\ &- \mu\sum_{j=0}^{\hat{M}-1}\hat{s}_j E\left\{ g\left[\sum_{i=0}^{M-1} s_i\mathbf{W}^T(n-i)\mathbf{X}(n-i)\right]\mathbf{X}(n-j)\middle|\mathcal{W}\right\} \end{aligned} \tag{7}$$

The second expected value on the r.h.s. of (7) is zero since $x(n)$ and $z(n)$ are uncorrelated and zero mean. The third and fourth terms can be easily evaluated using the same assumption above. The last term is evaluated using Bussgang's Theorem as in [7,9]. As a result:

$$\begin{cases} E\left\{z(n)\mathbf{X}(n-j)\middle|\mathscr{W}\right\}=\mathbf{0} \\ E\left\{\mathbf{X}(n-j)\mathbf{X}^T(n-i)\mathbf{W}(n-i)\middle|\mathscr{W}\right\}\cong \mathbf{R}_{i-j}\mathbf{W}(n-i) \\ E\left\{\mathbf{X}(n-j)\mathbf{X}^T(n)\middle|\mathscr{W}\right\}=\mathbf{R}_{-j} \end{cases} \tag{8}$$

where $\mathbf{R}_{i-j}=E\left\{\mathbf{X}(n-j)\mathbf{X}^T(n-i)\right\}$, and

$$E\left\{g\left[\sum_{i=0}^{M-1}s_i\mathbf{W}^T(n-i)\mathbf{X}(n-i)\right]\mathbf{X}(n-j)\middle|\mathscr{W}\right\}$$
$$=\frac{\sum_{i=0}^{M-1}s_i\mathbf{R}_{i-j}\mathbf{W}(n-i)}{\left(\frac{1}{\sigma^2}\sum_{j=0}^{M-1}\sum_{i=0}^{M-1}s_is_j\mathbf{W}^T(n-j)\mathbf{R}_{i-j}\mathbf{W}(n-i)+1\right)^{1/2}} \tag{9}$$

Substituting (8) and (9) in (7) and assuming slow adaptation (μ sufficiently small), the fluctuations of $\mathbf{W}(n-i)$ about $E\left\{\mathbf{W}(n-i)\right\}$ have a negligible effect on the average weight behavior over time [9]. Thus, the expected value of (7) can be approximated by:

$$\begin{aligned} E\left\{\mathbf{W}(n+1)\right\} &\cong E\left\{\mathbf{W}(n)\right\}+\mu\sum_{j=0}^{\hat{M}-1}\hat{s}_j\mathbf{R}_{-j}\mathbf{W}^{\mathbf{o}} \\ &-\mu\gamma\sum_{i=0}^{M-1}\sum_{j=0}^{\hat{M}-1}s_i\hat{s}_j\mathbf{R}_{i-j}E\left\{\mathbf{W}(n-i)\right\} \\ &-\frac{\mu\sum_{j=0}^{\hat{M}-1}\sum_{i=0}^{M-1}\hat{s}_js_i\mathbf{R}_{i-j}E\left\{\mathbf{W}(n-i)\right\}}{\left(\frac{1}{\sigma^2}\sum_{j=0}^{M-1}\sum_{i=0}^{M-1}s_is_jE\left\{\mathbf{W}(n-j)\right\}^T\mathbf{R}_{i-j}E\left\{\mathbf{W}(n-i)\right\}+1\right)^{1/2}} \end{aligned} \tag{10}$$

which is a recursive equation for the mean weight behavior.

A. Mean weight steady-state behavior

Assuming convergence and defining $\mathbf{W}_\infty=\lim_{n\to\infty}E\left\{\mathbf{W}(n)\right\}$, (10) yields, for $E\{\mathbf{W}(n+1)\}=E\{\mathbf{W}(n)\}=\mathbf{W}_\infty$:

$$\mathbf{W}_\infty\cong v\tilde{\mathbf{R}}_{\hat{S}S}^{-1}\tilde{\mathbf{R}}_{\hat{S}}\mathbf{W}^{\mathbf{o}} \tag{11}$$

where

$$v=\frac{1}{\gamma+\frac{1}{\left(v^2\eta^2+1\right)^{1/2}}} \tag{12}$$

and

$$\eta^2=\frac{1}{\sigma^2}\mathbf{W}^{\mathbf{o}T}\tilde{\mathbf{R}}_{\hat{S}}^T\tilde{\mathbf{R}}_{\hat{S}S}^{-T}\tilde{\mathbf{R}}_{SS}\tilde{\mathbf{R}}_{\hat{S}S}^{-1}\tilde{\mathbf{R}}_{\hat{S}}\mathbf{W}^{\mathbf{o}} \tag{13}$$

where

$$\tilde{\mathbf{R}}_{\hat{S}S}=\sum_{j=0}^{\hat{M}-1}\sum_{i=0}^{M-1}\hat{s}_js_i\mathbf{R}_{i-j},\quad \tilde{\mathbf{R}}_{\hat{S}}=\sum_{j=0}^{\hat{M}-1}\hat{s}_j\mathbf{R}_{-j},\quad \tilde{\mathbf{R}}_{SS}=\sum_{j=0}^{M-1}\sum_{i=0}^{M-1}s_is_j\mathbf{R}_{i-j} \tag{14}$$

The scalar v is the real positive solution of the biquadratic equation obtained from (12):

$$\left(\gamma^2\eta^2\right)v^4-\left(2\gamma\eta^2\right)v^3+\left(\eta^2+\gamma^2-1\right)v^2-(2\gamma)v+1=0 \tag{15}$$

It can be shown that the expression for the weight vector corresponding to the minimum mean square error (MSE) is given by [10]

$$\tilde{\mathbf{W}}=\sqrt{1-\frac{1}{2\beta^2}+\sqrt{\frac{1}{4\beta^4}+1}}\cdot\tilde{\mathbf{R}}_{SS}^{-1}\tilde{\mathbf{R}}_S\mathbf{W}^{\mathbf{o}}=k\cdot\tilde{\mathbf{R}}_{SS}^{-1}\tilde{\mathbf{R}}_S\mathbf{W}^{\mathbf{o}} \tag{16}$$

where $\beta^2=\mathbf{W}^{\mathbf{o}T}\tilde{\mathbf{R}}_S^T\tilde{\mathbf{R}}_{SS}^{-1}\tilde{\mathbf{R}}_S\mathbf{W}^{\mathbf{o}}/\sigma^2$.

Comparison of (11) and (16) shows that the MOV-FXLMS algorithm provides a biased solution in general, which differs from (16) in both magnitude and direction. The direction change depends on the quality of the secondary path estimation $\hat{\mathbf{S}}$. The multiplicative bias depends on both the estimation of **S** and the nonlinearity parameter σ. However, differently from the FXLMS algorithm, the converged mean weight vector for the MOV-LMS algorithm can be adjusted by properly designing the penalty factor γ in (4).

B. Estimation of the optimum penalty factor

Assuming perfect secondary path estimation, $\hat{\mathbf{S}}=\mathbf{S}$. Thus, $\tilde{\mathbf{R}}_{\hat{S}S}\cong\tilde{\mathbf{R}}_{SS}$ and $\tilde{\mathbf{R}}_{\hat{S}}\cong\tilde{\mathbf{R}}_S$ and the converged mean weight vector is in the same direction of $\tilde{\mathbf{W}}$ in (16), which minimizes the MSE. Moreover, η^2 in (13) is equal to β^2 in (16).

For the MOV-FXLMS algorithm to achieve the minimum MSE, the penalty factor $\gamma=\gamma_{opt}$ must be designed so that v in (11) is equal to k in (16). Making $v=k$ and $\gamma=\gamma_{opt}$ in (15) and rearranging it yields:

$$k^2\gamma_{opt}^2-2k\gamma_{opt}+1-\frac{k^2}{\left(\eta^2k^2+1\right)}=0 \tag{17}$$

The possible solutions of (17) are:

$$\gamma_{opt_{1,2}}=\frac{1}{k}\pm\frac{1}{\sqrt{k^2\eta^2+1}} \tag{18}$$

From (13), note that η^2 varies from 0 to infinity as σ^2 varies from infinity to zero. Using this property, the fact that $\eta^2\cong\beta^2$ and equations (12) and (16), it can be shown that $0<\gamma_{opt}<1/\sqrt{2}$ and that only the solution with the minus sign in (18) satisfies the problem.

Since in practice η^2 is not known, it must be estimated. If $\hat{\eta}^2$ is the estimate of η^2, (18) yields the solution

$$\hat{\gamma}_{opt}=\frac{1}{\sqrt{1-\frac{1}{2\hat{\eta}^2}+\sqrt{\frac{1}{4\hat{\eta}^4}+1}}}-\frac{1}{\sqrt{\hat{\eta}^2+\frac{1}{2}+\sqrt{\hat{\eta}^4+\frac{1}{4}}}} \tag{19}$$

If good estimations of **S** and η^2 are possible, the converged weight vector of the MOV-LMS approaches the point of minimum of the MSE surface:

$$\lim_{n\to\infty} E\{\mathbf{W}(n)\} \cong \sqrt{1-\frac{1}{2\beta^2}+\sqrt{\frac{1}{4\beta^4}+1}} \cdot \tilde{\mathbf{R}}_{SS}^{-1}\tilde{\mathbf{R}}_S \mathbf{W}^{\mathbf{o}} \tag{20}$$

In contrast, the FXLMS mean weight vector for $\hat{\mathbf{S}} = \mathbf{S}$ contains a multiplicative scalar bias that cannot be eliminated.

C. Estimation of η^2

A practical design problem in determining $\hat{\gamma}_{opt}$ is that η^2 is usually unknown. However, it was shown in [7,10] that η^2 corresponds to the degree of nonlinearity of the system and can be estimated by the relation:

$$\hat{\eta}^2 \cong \frac{\pi}{2} \frac{E_{est}\{y_s^2(n)\}\Big|_{\substack{n\to\infty \\ \sigma^2\to\infty}}}{\max_{est}\{g^2(y_s)\}} \tag{21}$$

where $E_{est}\{y_s^2(n)\}$ is an estimate for the variance of $y_s(n)$ and $\max_{est}\{g^2(y_s)\}$ is the estimate of the maximum squared amplitude at the nonlinearity output. Both quantities can be estimated from practical measurements. Using $\hat{\eta}^2$ for η^2 results in a deviation from the minimum MSE. The results in Section VI show that the performance of the MOV-FXLMS is very robust to such estimation errors, especially for large degrees of nonlinearity.

V. MSE Behavior

For sufficiently small μ, a simplified model for the MSE behavior can be obtained in [9]. Assuming small fluctuations about the mean weights [Eq. (16), 9] can be approximated by:

$$\begin{aligned}\xi(n) &\cong \mathbf{W}^{\mathbf{o}T}\mathbf{R}_0\mathbf{W}^{\mathbf{o}} + \sigma_z^2 \\ &- \frac{2\sum_{i=0}^{M-1} s_i \mathbf{W}^{\mathbf{o}T}\mathbf{R}_{-i} E\{\mathbf{W}(n-i)\}}{\sqrt{\frac{1}{\sigma^2}\sum_{j=0}^{M-1}\sum_{i=0}^{M-1} s_j s_i E\{\mathbf{W}(n-j)\}^T \mathbf{R}_{j-i} E\{\mathbf{W}(n-i)\}+1}} \\ &+ \sigma^2 \operatorname{arcsen}\left(\frac{\sum_{j=0}^{M-1}\sum_{i=0}^{M-1} s_j s_i E\{\mathbf{W}(n-j)\}^T \mathbf{R}_{j\text{-}i} E\{\mathbf{W}(n-i)\}}{\sum_{j=0}^{M-1}\sum_{i=0}^{M-1} s_j s_i E\{\mathbf{W}(n-j)\}^T \mathbf{R}_{j\text{-}i} E\{\mathbf{W}(n-i)\}+\sigma^2}\right)\end{aligned} \tag{22}$$

where $E\{\mathbf{W}(n)\}$ is obtained from (10).

A. Steady-State MSE

Assuming weight convergence and using (11) in (22):

$$\begin{aligned}\lim_{n\to\infty}\xi(n) &\cong \mathbf{W}^{\mathbf{o}T}\mathbf{R}_0\mathbf{W}^{\mathbf{o}} + \sigma_z^2 - \frac{2\nu \mathbf{W}^{\mathbf{o}T}\tilde{\mathbf{R}}_S^T\tilde{\mathbf{R}}_{\hat{S}S}^{-1}\tilde{\mathbf{R}}_{\hat{S}}\mathbf{W}^{\mathbf{o}}}{\sqrt{\nu^2\eta^2+1}} \\ &+ \frac{1}{\eta^2}\mathbf{W}^{\mathbf{o}T}\tilde{\mathbf{R}}_{\hat{S}}^T\tilde{\mathbf{R}}_{\hat{S}S}^{-T}\tilde{\mathbf{R}}_{SS}\tilde{\mathbf{R}}_{\hat{S}S}^{-1}\tilde{\mathbf{R}}_{\hat{S}}\mathbf{W}^{\mathbf{o}} \operatorname{arcsen}\left(\frac{\nu^2\eta^2}{\nu^2\eta^2+1}\right)\end{aligned} \tag{23}$$

where ν can be obtained through the solution of (15). Assuming $\hat{\mathbf{S}} \cong \mathbf{S}$ and $\hat{\eta}^2 \cong \eta^2 \cong \beta^2$, (16) and (19) lead to:

$$\begin{aligned}\lim_{\substack{n\to\infty \\ \gamma_{opt}}}\xi(n) &\cong \mathbf{W}^{\mathbf{o}T}\mathbf{R}_0\mathbf{W}^{\mathbf{o}} + \sigma_z^2 \\ &+ \mathbf{W}^{\mathbf{o}T}\tilde{\mathbf{R}}_S^T\tilde{\mathbf{R}}_{SS}^{-1}\tilde{\mathbf{R}}_S\mathbf{W}^{\mathbf{o}}\left[-\frac{2k}{\sqrt{k^2\beta^2+1}}+\frac{1}{\eta^2}\operatorname{arcsen}\left(\frac{k^2\beta^2}{k^2\beta^2+1}\right)\right]\end{aligned} \tag{24}$$

where k is defined in (16). Eq. (24) is an approximation for the mimimum of the MSE surface [10].

VI. Simulations

This section presents analytical and simulation results to verify the properties and accuracy of the theoretical models given by (10), (11), (19), (22) and (23). Consider the system in Fig. 1 with the following parameters: $x(n)$ with $\sigma_x^2 = 1$, eigenvalue spread ($\lambda_{\max}/\lambda_{\min}$) of $\mathbf{R_{xx}}$ equal to 27.33, $\mathbf{W}^{\mathbf{o}} = [0.2756 \quad 0.3675 \quad 0.6890 \quad 0.4593 \quad 0.3215]^T$, $\sigma_z^2 = 10^{-6}$, $\mathbf{W}(0) = \mathbf{0}$, $\mathbf{W}^{\mathbf{o}T}\mathbf{W}^{\mathbf{o}} = 1$, $\mathbf{S}^T\mathbf{S} = 1$, $\hat{\mathbf{S}}^T\hat{\mathbf{S}} = 1$, $\mathbf{S} = [0.9264 \quad 0.3369 \quad 0.1684]^T$, $\hat{\mathbf{S}} = [0.9670 \quad 0.2417 \quad 0.0806]^T$, $\mu = 0.005$.

Figs. 2 and 3 show the MSE and the mean weight behavior of the second coefficient for four different degrees of nonlinearity. There is an excellent agreement between theory (Eqs. (10) and (22)) and simulation results (averaged over 1000 runs). Only one every 100 iterations is plotted for better visualization.

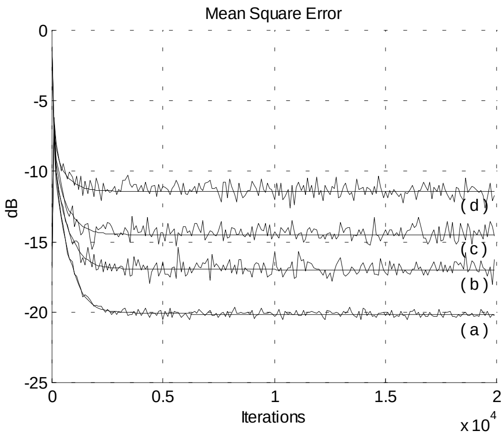

Fig. 2. MSE. Comparisons between the analytical model (continuous) and simulations (ragged) for (a) $\eta^2 = 0.001$; (b) $\eta^2 = 0.3$; (c) $\eta^2 = 0.5$; (d) $\eta^2 = 0.9$; where $\hat{\eta}^2 = 0.8 \cdot \eta^2$ and $\hat{\gamma}_{opt}$.

Fig. 4 presents the optimum penalty factor as a function of the degree of nonlinearity. Note that it varies between zero (linear case, $\eta^2 = 0$) and $\sqrt{2}/2$ (hard limiter, $\eta^2 \to \infty$).

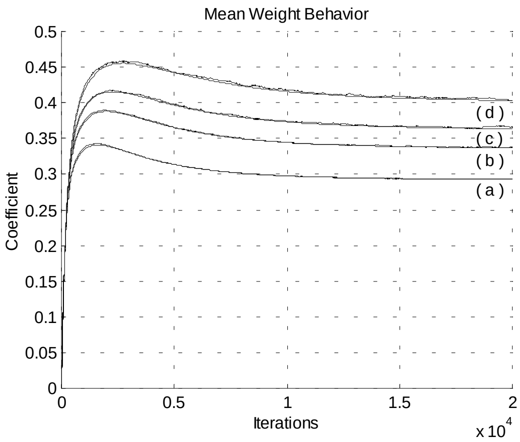

Fig. 3. Evolution of the second coefficient. Comparisons between the analytical model (continuous) and simulations (ragged) for (a) $\eta^2 = 0.001$; (b) $\eta^2 = 0.3$; (c) $\eta^2 = 0.5$; (d) $\eta^2 = 0.9$; where $\hat{\eta}^2 = 0.8 \cdot \eta^2$ and $\hat{\gamma}_{opt}$.

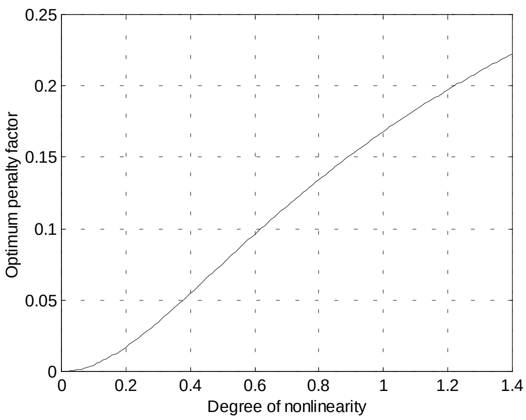

Fig. 4. Optimum penalty factor *versus* the system's degree of nonlinearity ($\hat{\gamma}_{opt} \times \eta^2$).

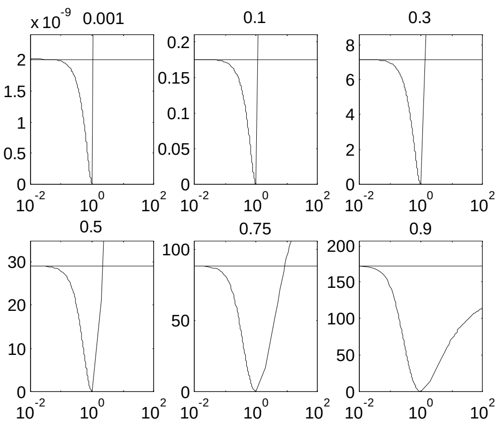

Fig. 5. Comparisons between the steady-state misadjustments of the conventional FXLMS (horizontal line) and the MOV-FXLMS algorithm (curved line) for a perfect estimation of the secondary path ($\hat{\mathbf{S}} = \mathbf{S}$) and deviations of $\hat{\eta}^2$.

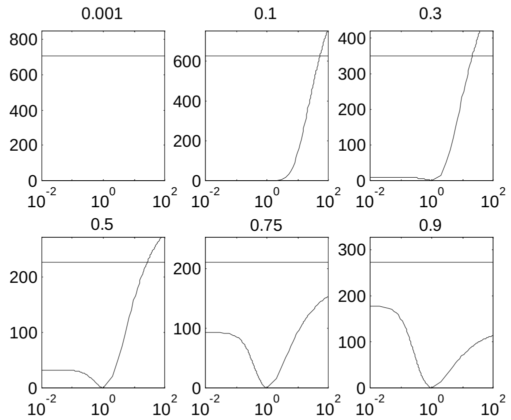

Fig. 6. Comparisons between the steady-state misadjustments of the conventional FXLMS (horizontal line) and the MOV-FXLMS algorithm (curved line) for a imperfect estimation of the secondary path ($\hat{\mathbf{S}} \neq \mathbf{S}$) and deviations of $\hat{\eta}^2$.

Figs. 5 and 6 compare the theoretical steady-state misadjustments achieved by the conventional FXLMS algorithm (upper horizontal line) [9] and by the MOV-FXLMS algorithm (curved line) as a function of the ratio $\hat{\eta}^2/\eta^2$. The results in Fig. 5 were obtained for $\hat{\mathbf{S}} = \mathbf{S}$, while those in Fig. 6 are for $\hat{\mathbf{S}} \neq \mathbf{S}$. The title on top of each plot corresponds to the value of η^2. The vertical axes give the misadjustment value. The horizontal axes show the values of the ratio $\hat{\eta}^2/\eta^2$. Note that the robustness of the MOV-FXLMS algorithm increases with η^2.

Table 1. Comparisons between the minimum possible MSE (ξ_{MIN}), the steady-state FXLMS MSE, Eq. (23) (ξ_{THEO}) and simulations (ξ_{SIM}) of the MOV-FXLMS (in dB) with a perfect estimation of the secondary path for a highly nonlinear system ($\eta^2 = 0.9$).

$\hat{\eta}^2/\eta^2$	ξ_{MIN}	ξ_{FXLMS}	ξ_{THEO}	ξ_{SIM}
0.01	-11.5	-7.1	-7.1	-7.0
0.1	-11.5	-7.1	-7.6	-7.4
1	-11.5	-7.1	-11.5	-11.5
10	-11.5	-7.1	-9.1	-9.1
100	-11.5	-7.1	-8.2	-8.2

Tables 1 and 2 compare the minimum MSE for the linear case ($\tilde{\mathbf{W}}$ given by (16)), the steady-state MSE for the FXLMS algorithm [9], the theoretical MSE for the MOV-FXLMS algorithm (Eq. (23)) and the simulated MSE obtained using the MOV-LMS algorithm with the estimated optimum penalty factor. The results in Table 1 are for a perfect estimation of the secondary path. Table 2 repeats the results of Table 1 for an imperfect estimation of the secondary path. Eq. (15) is solved numerically in both cases.

Table 2. Comparisons between the minimum possible MSE (ξ_{MIN}), the steady-state FXLMS MSE, Eq. (23) (ξ_{THEO}) and simulations (ξ_{SIM}) of the MOV-FXLMS (in dB) with a imperfect estimation of the secondary path with $\eta^2 = 0.5$.

$\hat{\eta}^2/\eta^2$	ξ_{MIN}	ξ_{FXLMS}	ξ_{THEO}	ξ_{SIM}
0.01	-14.6	-9.5	-13.4	-13.2
0.1	-14.6	-9.5	-13.4	-13.4
1	-14.6	-9.5	-14.6	-14.4
10	-14.6	-9.5	-10.5	-10.6
100	-14.6	-9.5	-8.8	-8.9

VII. Summary

This work presented an analysis of the MOV-FXLMS adaptive algorithm when its adaptive signal path is constrained by a memoryless saturation.

Deterministic equations were derived to the mean weight and MSE behavior, for Gaussian inputs and slow learning. Results were derived for both the transient and steady-state phases of adaptation. Properties of the performance surface were used to obtain an estimative to the optimum penalty factor. The use of the optimal penalty factor leads to maximum cancellation for a perfectly estimated secondary path.

Monte Carlo simulations showed excellent agreement with the analytical models. The robustness of the algorithm to errors in the estimation of the degree of nonlinearity was verified.

As a main result, the MOV-FXLMS algorithm with optimized penalty factor is an interesting alternative to the FXLMS algorithm when low-cost nonlinear hardware is employed.

Acknowledgments

Thanks to FAPERGS (Research Foundation of the State of Rio Grande do Sul - Brazil) and to CNPq (Brazilian Ministry of Science and Technology) under Grant No. 352084/92-8 for funding.

References

[1] Haykin, S., *Adaptive Filter Theory*, Prentice-Hall, 2nd edition, 1991.

[2] Kuo, S., and D. Morgan, *Active Noise Control Systems: Algorithms and DSP Implementations*, Wiley, 1996.

[3] C. Hansen, "Active Noise Control – From Laboratory to Industrial Implementation," *Proceedings of Noise-Con 97*, USA, Pennsylvania, pages 3-38, 1997.

[4] W. Klippel, "Nonlinear Large-Signal Behavior of Electrodynamic Loudspeakers at Low Frequencies," *Journal of Audio Engineering Society*, vol. 40, No. 6, pages 483-496, 1992.

[5] P. Darlington, and G. Xu, "Equivalent Transfer Functions of Minimum Output Variance Mean-Square Estimators," *IEEE Trans. on Signal Processing*, vol. 39, No. 7, pages 1674-1677, July, 1991.

[6] K. Mayyas, and A. Tyseer, "Leaky LMS Algorithm: MSE and Analysis for Gaussian Data," *IEEE Trans. on Signal Processing*, vol. 45, No. 4, pages 927-934, April, 1997.

[7] M.H. Costa, J.C.M. Bermudez and N.J. Bershad, "Stochastic Analysis of the LMS Algorithm with a Saturation Nonlinearity Following the Adaptive Filter Output," *IEEE Trans. on Signal Processing*, vol. 49, No. 7, pages 1370-1387, July, 2001.

[8] J.C. Bermudez and M.H. Costa, "Optimum Leakage Factor for the MOV-LMS Algorithm in Nonlinear Modeling and Control Systems", *ICASSP 2002*, USA, Orlando, 2002.

[9] M.H. Costa, J.C.M. Bermudez and N.J. Bershad, "Stochastic Analysis of the Filtered-LMS Algorithm in Systems with Nonlinear Secondary Paths," *to appear in the June 2002 issue of IEEE Trans. on Signal Processing*.

[10] M.H. Costa, *Behavior of the FXLMS algorithm in Systems with a Nonlinear Secondary Path: Application to Active Noise Control*, Ph.D. Thesis (in Portuguese), Federal University of Santa Catarina, Brazil, 2001.