Finite-State Markov Modeling of Flat Fading Channels

Cecilio Pimentel, Tiago Falk and Luciano Lisbôa
Communications Research Group - CODEC
Department of Electronics and Systems
Federal University of Pernambuco
P.O. Box 7800 - 50711-970 - Recife - Brazil
E.mail: cecilio@npd.ufpe.br

Abstract—**The stochastic properties of the binary channel that describe the successes and the failures of the transmission of a modulated signal over a time correlated flat fading channel is considered for investigation. This analysis is employed to develop K-th order Markov models for such a burst channel. The order of the Markov model that generates accurate analytical models is estimated for a broad range of fading environments. The parameterization and the accuracy of an important class of hidden Markov models known as the Gilbert-Elliott channel are also investigated. Fading rates are identified in which the K-th order and the Gilbert-Elliott channel model approximate the fading channel with similar accuracy. The latter model is useful for approximating slowly fading processes, since it provides a far more compact parameterization.**

I. Introduction

In a typical mobile communication channel the transmitted signal undergoes attenuation and distortion caused by multipath propagation and shadowing. The non-frequency selective (flat) fading channel imposes multiplicative narrow-band complex Gaussian noise (referred to as *fading* process) on the transmitted signal. As a consequence, abrupt changes in the mean received signal level may occur, and the autocorrelation function of the fading process may lead to the occurrence of a burst of bit errors. The analytical analysis of a communication system operating over such a correlated fading process is difficult and there are no analytical expressions for several statistics relevant to system performance evaluation.

Finite state channel (FSC) models have been widely accepted as an effective approach to characterize the correlation structure of the fading process [1]-[11]. An FSC is described by a deterministic or probabilistic function of a first-order Markov chain, where each state may be associated with a particular channel quality. The strategy adopted by many researchers to design FSC models for fading channels consists in representing each state of a first-order Markov chain by a non-overlapping interval of the received instantaneous signal to noise ratio [2]-[7]. Criteria to partitioning the signal to noise ratio are discussed in [2], [5], [7]. The modulation and demodulation schemes are incorporated into the model through the crossover probability of the binary symmetric channel associated to each fading state. An information theoretic metric was proposed in [3] to validate the first-order model. The limitations of this criterion and the applicability of the first-order assumption have been discussed in [7]. Other model structures have also been proposed to represent the quantized signal to noise ratio, including, higher-order Markov models [8], general Hidden Markov models [11], and Gilbert-Elliott channels [10].

This work received partial support from the Brazilian National Council for Scientific and Technological Development (CNPq) under Grant 300987/96-0.

This paper concerns the development of FSC models for a discrete communication system composed by a modulator, a time correlated flat fading channel, and a hard quantized demodulator. The FSC model describes the successes and the failures of the symbol transmitted over a fading channel, which is represented mathematically as a binary error sequence. We consider two classes of FSC models commonly used to characterize fading channels: The Kth-order Markov model and the Gilbert-Elliott channel (GEC) model. We first describe a methodology to estimate the parameters of these models directly from the binary error sequence. A method recently proposed in [7] to judge model accuracy is applied to identify the order of the Markov model that satisfactorily approximates the channel, for several fading regimes. A similar study evaluates the accuracy of the GEC model. The results presented here allow us to study coding performance on correlated fading channels using the analytical techniques developed to analyze burst channels represented as specific FSC models [10]-[15].

II. The Channel Model

We consider a communication system that employs M-ary FSK modulation, a time correlated flat Rician fading channel and non-coherent demodulation. The complex envelope of the received signal at the input to the demodulator is corrupted by a multiplicative Rician fading and by an additive white Gaussian noise with zero mean and autocorrelation function $\frac{1}{2}\mathbf{E}\{\tilde{N}(t+\tau)\tilde{N}^\star(t)\} = N_0\delta(\tau)$. The complex envelope of the fading process, $\tilde{G}(t) = \tilde{G}_I(t) + \jmath\,\tilde{G}_Q(t)$, is a complex, wide-sense stationary, Gaussian process with real constant mean m, where $\jmath = \sqrt{-1}$, and the quadrature components $\tilde{G}_I(t)$ and $\tilde{G}_Q(t)$ are mutually independent Gaussian process with the same covariance function, named $C(\tau)$. Although the analysis carried out here can be applied to a fading process with arbitrary covariance function, we adopted the Clarke's model [16], [17] for $C(\tau)$:

$$C(\tau) = \frac{1}{2}\,\mathbf{E}\{[\tilde{G}(t+\tau) - m]\,[\tilde{G}^\star(t) - m]\} = \sigma_g^2\, J_0(2\pi f_D \tau),$$

where $J_0(x)$ is the zero-order Bessel function of the first kind, f_D is the maximum Doppler frequency, and σ_g^2 is the variance of $\tilde{G}(t)$. For a fixed time instant, the fading envelope $A_k \triangleq$

$\sqrt{\tilde{G}_I^2(kT) + \tilde{G}_Q^2(kT)}$ (where T is the symbol interval) has the Rician probability density function given by:

$$p_A(a) = \frac{a}{\sigma_g^2} e^{-(m^2+a^2)/2\sigma_g^2} I_0(\frac{ma}{\sigma_g^2}), \quad (1)$$

where $I_0(x)$ is the zero-order modified Bessel function of the first kind. When the in-phase process is zero-mean ($K_R = 0$), the fading envelope follows the Rayleigh probability density function:

$$p_A(a) = \frac{a}{\sigma_g^2} \exp\left(\frac{-a^2}{2\sigma_g^2}\right). \quad (2)$$

At each signaling interval of length T, the demodulator forms the M decision variables and decides which signal was more likely to have been transmitted. We define a binary error process $\{E_k\}_{k=1}^{\infty}$, where $E_k = 0$ indicates no symbol error at the kth interval, and $E_k = 1$ indicates a symbol error. It can be shown that the probability of an error sequence of length n, $\mathbf{e}_n = e_1 e_2 \dots e_n$, may be expressed as [9]:

$$\begin{aligned} P(\mathbf{e}_n) &= \sum_{l_1=e_1}^{M-1} \cdots \sum_{l_n=e_n}^{M-1} \left(\prod_{k=1}^{n} \frac{\binom{M-1}{l_k}(-1)^{l_k+e_k}}{l_k+1} \right) \\ &\frac{\exp\{-\frac{E_s}{N_0} K_R \mathbf{1}^T \mathbf{F}((K_R+1)\mathbf{I} + \overline{\mathbf{C}}^{\star}\mathbf{F})^{-1}\mathbf{1}\}}{\det(\mathbf{I} + \frac{E_s}{N_0}(1+K_R)^{-1}\overline{\mathbf{C}}^{\star}\mathbf{F})}, \end{aligned} \quad (3)$$

where $\overline{\mathbf{C}}^{\star}$ is the complex conjugate of the normalized $n \times n$ covariance matrix of $\tilde{G}(t)$ (the (i,j)th entry of $\overline{\mathbf{C}}^{\star}$ is $J_0(2\pi f_D(i-j)T)$, $\mathbf{F}$ is a diagonal matrix defined as $\mathbf{F} = \text{diag}(\frac{l_1}{l_1+1}, \dots, \frac{l_n}{l_n+1})$, E_s is the energy of the transmitted symbol, $K_R = m^2/2\sigma_g^2$ is the Rician factor, $\mathbf{1}$ is a column vector of ones, and the superscript $[\cdot]^T$ indicates the transpose of a matrix. We will call this discrete fading model from the modulator input to the demodulator output the discrete channel with Clarke's autocorrelation (DCCA) model. Hereafter, we consider binary modulation ($M = 2$), so the DCCA model has three parameters K_R, $f_D T$, and E_s/N_0.

Equation (3) can be used to calculate the probability of any error event relevant to the analysis of the DCCA fading model. For example, the probability the error bit is a 1 is:

$$P(1) = \frac{1+K_R}{2+2K_R+\frac{E_s}{N_0}} e^{-\frac{K_R \frac{E_s}{N_0}}{2+2K_R+\frac{E_s}{N_0}}}, \quad (4)$$

and the probability of two consecutive ones (errors) is:

$$P(11) = \frac{(1+K_R)^2}{\left(2+2K_R+\frac{E_s}{N_0}\right)^2 - \left(\rho\frac{E_s}{N_0}\right)^2} e^{-2\frac{K_R \frac{E_s}{N_0}}{2+2K_R+(\rho+1)\frac{E_s}{N_0}}}, \quad (5)$$

where ρ is the correlation coefficient of two consecutive samples of the fading process $\tilde{G}(t)$:

$$\rho = J_0(2\pi f_D T). \quad (6)$$

Equation (3) will be employed to parameterize an FSC model that accurately reflects the statistical description of the real error process. A brief description of FSC models is given next.

Consider $\{S_k\}_{k=0}^{\infty}$ an N-state, first-order Markov chain with a finite state space $\mathcal{N}_N = \{0, 1, \dots, N-1\}$. Let $\mathbf{P}$ be an $N \times N$ transition probability matrix, whose $(i,j)^{th}$ entry is the transition probability $p_{i,j} = P(S_k = j \mid S_{k-1} = i)$, $i, j \in \mathcal{N}_N$. The FSC model generates an error symbol according to the following probabilistic mechanism. At the k^{th} time interval, the chain makes a transition from state $S_{k-1} = i$ to $S_k = j$ with probability $p_{i,j}$ and generates an output (error) symbol $e_k \in \mathcal{N}_2$ (independent of i), with probability $b_{j,e_k} = P(E_k = e_k \mid S_k = j)$. Conditioned to the state process, the error process is memoryless, that is, $P(\mathbf{e}_n \mid \mathbf{s}_n) = \prod_{k=1}^{n} P(e_k \mid s_k)$. We are assuming that the distribution of the initial state is the stationary distribution $\mathbf{\Pi} = [\pi_0, \pi_1, \dots, \pi_{N-1}]^T$.

The probability of an error sequence generated by the FSC model, conditioned to the initial state, is:

$$\begin{aligned} P(\mathbf{e}_n \mid s_0) &= \sum_{\mathbf{s}_n \in \mathcal{N}_N^n} P(\mathbf{e}_n \mid \mathbf{s}_n, s_0)\, P(\mathbf{s}_n \mid s_0) \\ &= \sum_{\mathbf{s}_n \in \mathcal{N}_N^n} \prod_{k=1}^{n} b_{s_k,e_k}\, p_{s_{k-1},s_k}. \end{aligned}$$

Hence

$$P(\mathbf{e}_n) = \sum_{s_0=0}^{N-1} \pi_{s_0} \sum_{\mathbf{s}_n \in \mathcal{N}_N^n} \prod_{k=1}^{n} b_{s_k,e_k}\, p_{s_{k-1},s_k}. \quad (7)$$

Equation (7) can be rewritten in a matrix form. Define two $N \times N$ matrices, $\mathbf{P}(0)$ and $\mathbf{P}(1)$, where $\mathbf{P} = \mathbf{P}(0) + \mathbf{P}(1)$. The $(i,j)th$ entry of the matrix $\mathbf{P}(e_k)$, $e_k \in \{0,1\}$, is $P(E_k = e_k, S_k = j \mid S_{k-1} = i) = b_{j,e_k} p_{i,j}$, which is the probability that the output symbol is e_k when the chain makes a transition from state i to j. Equation (7) has a matrix form given by:

$$P(\mathbf{e}_n) = \mathbf{\Pi}^T \left(\prod_{k=1}^{n} \mathbf{P}(e_k) \right) \mathbf{1}. \quad (8)$$

An FSC model is completely specified by the matrices $\mathbf{P}(0)$ and $\mathbf{P}(1)$. We define in the next section some properties of FSC models and discuss the evaluation of its parameters.

III. Parameterization of Specific FSC Models

We consider two classes of FSC models: Kth-order Markov models and the GEC model. Following the ideas introduced in [9], the parameters of each FSC model will be expressed as functions of the probabilities of binary sequences generated by the model. Then, we apply (3) to estimate these probabilities and to parameterize FSC models that approximates the DCCA correlated fading model.

A. Kth-order Markov Models

A discrete stochastic process $\{E_n\}_{n=1}^{\infty}$ is a Markov process of Kth-order if it obeys the relation:

$$P(e_n \mid e_1 e_2 \cdots e_{n-1}) = P(e_n \mid e_{n-K} \cdots e_{n-1}). \quad (9)$$

A first-order binary Markov model is an FSC model with space state $\{0,1\}$. An error symbol is produced when the chain transitions to state 1. Otherwise, if the chain transition is to state 0, a correct symbol is produced. Therefore, $b_{0,0} = 1$ and $b_{1,1} = 1$. Using (7) for error sequences of length 1 and 2, we get, $p_{i,j} = P(ij)/P(i)$, $i,j \in \{0,1\}$. The stationary vector is $\mathbf{\Pi} = [P(0)\ P(1)]^T$, and the matrices $\mathbf{P}(0)$ and $\mathbf{P}(1)$ for the first-order Markov model are:

$$\mathbf{P}(0) = \begin{bmatrix} \frac{P(00)}{P(0)} & 0 \\ \frac{P(10)}{P(1)} & 0 \end{bmatrix}; \quad \mathbf{P}(1) = \begin{bmatrix} 0 & \frac{P(01)}{P(0)} \\ 0 & \frac{P(11)}{P(1)} \end{bmatrix}.$$

In general, the Kth-order Markov model can be represented as a function of a first-order Markov chain [18]. Each state of the Kth-order model is represented by a binary string of length K. Given two states $u = u_1u_2\cdots u_K$ and $v = v_1v_2\cdots v_K$, we say that u and v *overlap progressively* if $u_2u_3\cdots u_K = v_1v_2\cdots v_{K-1}$. If u and v overlap progressively, then, there is a transition from u to v with probability $P(u_1v_1v_2\cdots v_k)/P(u)$. Otherwise, the state transition probability is zero. Given a state $v = v_1v_2\cdots v_K$, $b_{v,e_k} = v_K$. For example, the matrices $\mathbf{P}(0)$, $\mathbf{P}(1)$ and $\mathbf{\Pi}$ for the second-order Markov model are:

$$\mathbf{P}(0) = \begin{bmatrix} \frac{P(000)}{P(00)} & 0 & 0 & 0 \\ 0 & 0 & \frac{P(010)}{P(01)} & 0 \\ \frac{P(100)}{P(10)} & 0 & 0 & 0 \\ 0 & 0 & \frac{P(110)}{P(11)} & 0 \end{bmatrix};$$

$$\mathbf{P}(1) = \begin{bmatrix} 0 & \frac{P(001)}{P(00)} & 0 & 0 \\ 0 & 0 & 0 & \frac{P(011)}{P(01)} \\ 0 & \frac{P(101)}{P(10)} & 0 & 0 \\ 0 & 0 & 0 & \frac{P(111)}{P(11)} \end{bmatrix};$$

$$\mathbf{\Pi} = [P(00)\ P(01)\ P(10)\ P(11)]^T.$$

Clearly, the number of states grows exponentially with the order K.

B. Gilbert-Elliott Channel

The GEC is a two-state FSC model composed of state 0, which produces errors with small probability, $b_{0,1} \triangleq g$, and state 1, where errors occur with higher probability, $b_{1,1} \triangleq b$, where $g << b$. The transition probabilities of the Markov chain are $p_{0,1} \triangleq Q$ and $p_{1,0} \triangleq q$. The matrices $\mathbf{P}(0)$, $\mathbf{P}(1)$, where $\mathbf{P}(0) + \mathbf{P}(1) = \mathbf{P}$, for the GEC model are given by:

$$\mathbf{P}(0) = \begin{bmatrix} (1-Q)(1-g) & Q(1-b) \\ q(1-g) & (1-q)(1-b) \end{bmatrix}; \tag{10}$$

$$\mathbf{P}(1) = \begin{bmatrix} (1-Q)\,g & Q\,b \\ q\,g & (1-q)\,b \end{bmatrix}. \tag{11}$$

We define next the notation required in this subsection. Consider $\boldsymbol{\sigma}$ any binary sequence of finite length, $Q(\boldsymbol{\sigma}, i) \triangleq P(\mathbf{E}_n = \boldsymbol{\sigma}, S_{n+1} = i)$. Let $\boldsymbol{\phi}$ be an empty sequence, i.e, a sequence of length zero that possesses the properties $\boldsymbol{\phi}\boldsymbol{\sigma} = \boldsymbol{\sigma}\boldsymbol{\phi}$ and $P(\boldsymbol{\phi}) = 1$. Let ϵ and δ be binary symbols. If i is a particular state, $i^\star$ indicates the other state of the Markov chain, that is, if $i = 0$, then $i^\star = 1$. The parameterization of the Gilbert-Elliott channel is based on the following lemma.

Lemma 1: The probability of any sequence $\boldsymbol{\sigma}$ generated by the Gilbert-Elliott channel satisfies the following recurrence equation:

$$P(\boldsymbol{\sigma}\epsilon\delta) = c(\epsilon,\delta)P(\boldsymbol{\sigma}\epsilon) + d(\epsilon,\delta)P(\boldsymbol{\sigma}), \tag{12}$$

where

$$\begin{aligned} c(\epsilon,\delta) &= \frac{b_{i,\epsilon}}{b_{i,\epsilon} - b_{i^*,\epsilon}}(p_{i,i^*}b_{i^*,\delta} + p_{i,i}b_{i,\delta}) \\ &- \frac{b_{i^*,\epsilon}}{b_{i,\epsilon} - b_{i^*,\epsilon}}(p_{i^*,i^*}b_{i^*,\delta} + p_{i^*,i}b_{i,\delta}); \end{aligned} \tag{13}$$

$$\begin{aligned} d(\epsilon,\delta) &= \frac{b_{i,\epsilon}b_{i^*,\epsilon}}{b_{i,\epsilon} - b_{i^*,\epsilon}}(p_{i^*,i^*}b_{i^*,\delta} + p_{i^*,i}b_{i,\delta}) \\ &- \frac{b_{i^*,\epsilon}b_{i,\epsilon}}{b_{i,\epsilon} - b_{i^*,\epsilon}}(p_{i,i^*}b_{i^*,\delta} + p_{i,i}b_{i,\delta}). \end{aligned} \tag{14}$$

Proof: The probability of any sequence generated by a GEC model satisfies the relations:

$$P(\boldsymbol{\sigma}) = Q(\boldsymbol{\sigma}, i^*) + Q(\boldsymbol{\sigma}, i). \tag{15}$$

$$P(\boldsymbol{\sigma}\epsilon) = Q(\boldsymbol{\sigma}, i^*)b_{i^*,\epsilon} + Q(\boldsymbol{\sigma}, i)b_{i,\epsilon}. \tag{16}$$

Hence, $Q(\boldsymbol{\sigma}, i)$ and $Q(\boldsymbol{\sigma}, i^*)$ are expressed as:

$$Q(\boldsymbol{\sigma}, i) = \frac{-b_{i^*,\epsilon}}{b_{i,\epsilon} - b_{i^*,\epsilon}}P(\boldsymbol{\sigma}) + \frac{1}{b_{i,\epsilon} - b_{i^*,\epsilon}}P(\boldsymbol{\sigma}\epsilon); \tag{17}$$

$$Q(\boldsymbol{\sigma}, i^*) = \frac{b_{i,\epsilon}}{b_{i,\epsilon} - b_{i^*,\epsilon}}P(\boldsymbol{\sigma}) - \frac{1}{b_{i,\epsilon} - b_{i^*,\epsilon}}P(\boldsymbol{\sigma}\epsilon). \tag{18}$$

The following equation also holds for the GEC model:

$$\begin{aligned} P(\boldsymbol{\sigma}\epsilon\delta) &= Q(\boldsymbol{\sigma}, i^*)b_{i^*,\epsilon}[p_{i^*,i^*}b_{i^*,\delta} + p_{i^*,i}b_{i,\delta}] \\ &+ Q(\boldsymbol{\sigma}, i)b_{i,\epsilon}[p_{i,i^*}b_{i^*,\delta} + p_{i,i}b_{i,\delta}]. \end{aligned} \tag{19}$$

Substituting (17) into (18) and rearranging the terms, yields:

$$\begin{aligned} P(\boldsymbol{\sigma}\epsilon\delta) &= \{\frac{b_{i,\epsilon}b_{i^*,\epsilon}}{b_{i,\epsilon} - b_{i^*,\epsilon}}(p_{i^*,i^*}b_{i^*,\delta} + p_{i^*,i}b_{i,\delta}) \\ &- \frac{b_{i^*,\epsilon}b_{i,\epsilon}}{b_{i,\epsilon} - b_{i^*,\epsilon}}(p_{i,i^*}b_{i^*,\delta} + p_{i,i}b_{i,\delta})\}P(\boldsymbol{\sigma}) \\ &+ \{\frac{b_{i,\epsilon}}{b_{i,\epsilon} - b_{i^*,\epsilon}}(p_{i,i^*}b_{i^*,\delta} + p_{i,i}b_{i,\delta}) \\ &- \frac{b_{i^*,\epsilon}}{b_{i,\epsilon} - b_{i^*,\epsilon}}(p_{i^*,i^*}b_{i^*,\delta} + p_{i^*,i}b_{i,\delta})\}P(\boldsymbol{\sigma}\epsilon) \\ &= c(\epsilon,\delta)P(\boldsymbol{\sigma}\epsilon) + d(\epsilon,\delta)P(\boldsymbol{\sigma}). \end{aligned}$$

Equation (12) allows us to express $c(\epsilon,\delta)$ and $d(\epsilon,\delta)$, and consequently, b, g, q e Q as functions of the probabilities of error sequences. Substituting $\boldsymbol{\sigma} = \phi$ and $\boldsymbol{\sigma} = \epsilon$ in (12) yields, respectively:

$$P(\epsilon\delta) = c(\epsilon,\delta)P(\epsilon) + d(\epsilon,\delta); \quad (20)$$
$$P(\epsilon\epsilon\delta) = c(\epsilon,\delta)P(\epsilon\epsilon) + d(\epsilon,\delta)P(\epsilon). \quad (21)$$

Solving this linear system, we obtain:

$$c(\epsilon,\delta) = \frac{P(\epsilon\epsilon\delta) - P(\epsilon\delta)P(\epsilon)}{P(\epsilon\epsilon) - P^2(\epsilon)}, \quad (22)$$

and

$$d(\epsilon,\delta) = \frac{P(\epsilon\delta)P(\epsilon\epsilon) - P(\epsilon\epsilon\delta)P(\epsilon)}{P(\epsilon\epsilon) - P^2(\epsilon)}. \quad (23)$$

The following proposition expresses the parameters of the GEC model in terms of $c(\epsilon,\delta)$ and $d(\epsilon,\delta)$, or consequently, in terms of the probability of error sequences of length, at most, 3.

Proposition 1: If $P(01) \neq P(0)\,P(1)$, the parameters of the Gilbert-Elliott channel are uniquely determined by the four probabilities $P(0), P(00), P(000), P(111)$. The parameters b and g are the roots of the quadratic equation:

$$[-1 + c(1,1) + c(0,0)]x^2$$
$$+[1 - c(1,1) - c(0,0) + d(1,1) - d(0,0)]x - d(1,1) = 0, \quad (24)$$

and the parameters Q and q are given by:

$$Q = \frac{c(0,0)b - c(1,1)(1-b) + (g-b)}{g-b};$$
$$q = \frac{c(0,0)g - c(1,1)(1-g) + (b-g)}{b-g}. \quad (25)$$

Proof: From (13) and (14), we have:

$$c(0,0) = (1-g)(1-Q) + (1-b)(1-q); \quad (26)$$
$$c(1,1) = g(1-Q) + b(1-q); \quad (27)$$
$$d(0,0) = -(1-q-Q)(1-g)(1-b); \quad (28)$$
$$d(1,1) = -(1-q-Q)gb, \quad (29)$$

From (26) and (27), and from (28) and (29), we get, respectively:

$$1 - c(0,0) - c(1,1) = -\mu, \quad (30)$$
$$g\,b\left(\frac{d(0,0)}{d(1,1)} - 1\right) = 1 - b - g, \quad (31)$$

where $\mu \triangleq 1 - q - Q$. Combining (29) and (31) results in the quadratic equation:

$$-\mu b^2 + (\mu + d(0,0) - d(1,1))b + d(1,1) = 0, \quad (32)$$

where the same equation holds for g. So, substituting (30) into (32) we conclude that b and g are the roots of (24). Once we have determined b and g, we use (26) and (27) to obtain (25).

IV. Model Evaluation

This section evaluates the accuracy in which the FSC models described in the previous section approximate the DCCA correlated fading channel. In general, it is difficult to define a unique measure to judge if a particular model approximates better the fading channel when compared to other candidates. The criteria commonly used to make this decision include the minimization of a distance measure between the probability of error sequences generated by the model and by the fading channel (e.g. variational, normalized divergence), the information theoretic metric[3], and the comparison of certain statistics of the models, such as, autocorrelation function, and packet error rate [7].

Motivated by the results presented in [7], we compare next the autocorrelation function (ACF) of the DCCA fading model with the ACF of FSC models. The ACF of a binary stationary process $\{E_k\}_{k=1}^{\infty}$ is given by:

$$R[m] = \mathbf{E}\{E_i E_{i+m}\} = P(E_i = 1, E_{i+m} = 1), \quad (33)$$

where $\mathbf{E}\{X\}$ denotes the expected value of a random variable X. A closed-form expression for the ACF of the DCCA model is given by (5), where the correlation coefficient ρ given by (6) is replaced by $\rho(m) = J_0(2\pi m f_D T)$. Then, it follows from (5) that for the special case of Rayleigh fading $(K_R = 0)$:

$$R[m] = \frac{1}{(2 + \frac{E_s}{N_0})^2 - \left(\rho(m)\frac{E_s}{N_0}\right)^2}. \quad (34)$$

The ACF of an FSC model described by the matrices $\mathbf{P}(0)$ and $\mathbf{P}(1)$ is expressed as [13]:

$$R[m] = \boldsymbol{\Pi}^T \left(\mathbf{P}(1)\mathbf{P}^{m-1}\mathbf{P}(1)\right)\mathbf{1}, \quad (35)$$

for $m \geq 1$.

The ACF over twenty values of m of the DCCA and the FSC models are compared in Fig. 1. The parameters of the DCCA are $K_R = 0$, $E_s/N_0 = 15$ dB, and $f_D T$ = 0.1 (a), $f_D T$ = 0.02 (b), $f_D T$ = 0.001 (c). Markov models of order up to 6 have been considered. It is observed in Fig. 1(a) that there is a significant gain in accuracy when the order of the Markov model is increased from $K = 0$ (memoryless) to $K = 1$, a little gain is obtained for $K = 2$ and no further gain is observed for $K = 3$. Also, the ACF's of the second-order Markov and the GEC models are very alike. The curves indicate that the first-order Markov model approximates satisfactorily the DCCA fading channel for $f_D T$ = 0.1. It is worth mentioning that we could have chosen either the second-order or the GEC model to approximate the DCCA model, since the ACF's of these three models can barely be distinguished in Fig. 1(a). However, we want to obtain as simple an analytical model as possible at an acceptable complexity level. This tradeoff between accuracy and complexity makes this decision somewhat arbitrary. When the fading rate gets slower the order of the Markov model increases, as expected. For example, we observe (curves not shown) that the second-order Markov is satisfactory for $f_D T = 0.05$. However, we notice that the ACF of the third-order Markov model is a bit closer to that of the DCCA model, but this strictness may not compensate the doubling of the number of states. Again, the ACF's of the third-order and the GEC are very similar. When $f_D T < 0.03$, the ACF of the GEC model diverges from the ACF of the DCCA model. This fact is illustrated in Fig. 1(b), where the curves of the forth and the fifth-order Markov models approximate better the ACF of the DCCA model than that of

TABLE I

ORDER OF THE MARKOV MODEL THAT APPROXIMATES THE DCCA RAYLEIGH FADING CHANNEL FOR SEVERAL VALUES OF $f_D T$.

$f_D T$	K ($E_s/N_0 = 15$ dB)	K $E_s/N_0 = 25$ dB)
0.1	1	1
0.05	2	2
0.03	4	3
0.02	5	4
0.01	6	5
0.001	> 6	> 6

the GEC model. These values of K can be considered as satisfactory approximations of the DCCA model for $f_D T = 0.02$. These conclusions hold for greater sample separations m. In fact, the ACF's of the Kth-order Markov and the DCCA models matches perfectly over an interval of length K. The curves demonstrate that the ACF criterion is reasonably accurate at indicating the order of the Markov model that approximates the DCCA fading model. Markov models may not be practical for very slowly fading channels since the number of states grows exponentially with K and large data sizes are necessary to parameterize the model. Fig. 1(c) illustrates that the GEC model with non-observable states appears to be useful for approximating very slowly varying fading, since it provides a far more compact parameterization. Fig. 2 displays a similar comparison for the case $E_s/N_0 = 25$ dB, $f_D T = 0.02$. It is observed that the ACF of the DCCA model decreases more rapidly with m when compared to Fig. 1 (b), indicating a potential to reduce the order of the Markov approximation.

In order to verify the order K indicated by the ACF method using a different perspective, we calculate the variational distance between the n-dimensional target measure $P(\mathbf{e}_n)$ given by (3) and the measure obtained by the Kth-order Markov model, namely, $P^{(K)}(\mathbf{e}_n)$, which is calculated using (8). The matrices $\mathbf{P}(0)$ and $\mathbf{P}(1)$ are described in Section III. The variational distance is defined as:

$$d_v\left(P(\mathbf{e}_n), P^{(K)}(\mathbf{e}_n)\right) = \sum_{\mathbf{e}_n} |P(\mathbf{e}_n) - P^{(K)}(\mathbf{e}_n)|.$$

Figure 3 reports the variational distance versus the order K for several values of $f_D T$, for $E_s/N_0 = 15$dB (a), $E_s/N_0 = 25$dB (b). Note that a lower distance value indicates a more accurate model. We say that the order of the Markov chain is K_0, when the distance converge to approximately a constant value (roughly zero) for $K \geq K_0$. The orders indicated by the convergence of the variational distance, for the range of fading environments investigated, are consistent with those obtained by the ACF method. The choice of value K_0, as mentioned before, is somewhat arbitrary and Table I summarizes the order indicated by Fig. 3. We notice that, for slow and medium rate fading, the increase of the signal to noise ratio from 15 to 25 dB reduces the order of the Markov model K by 1.

V. CONCLUSIONS

We have developed FSC models that characterize the error sequence of a communication system operating over a fading

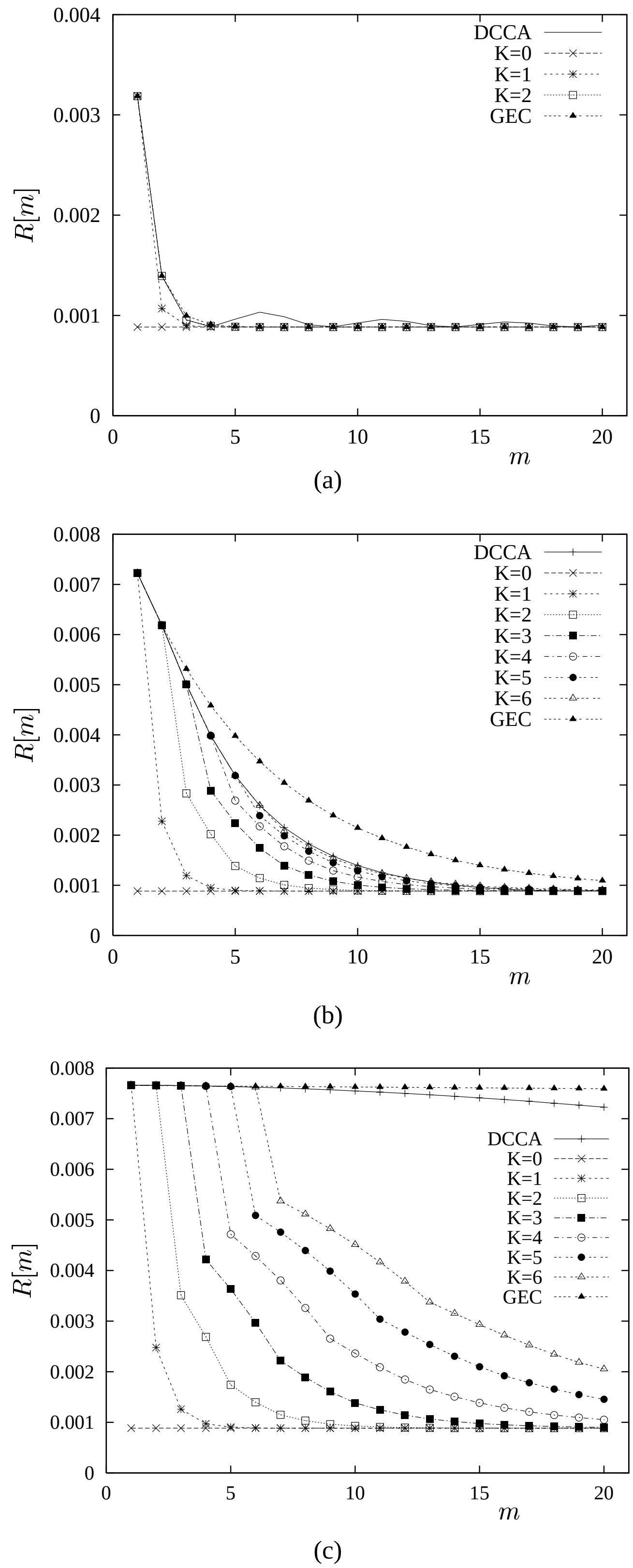

Fig. 1. Comparison of the autocorrelation functions of the DCCA fading model, the Kth-order Markov model ($K = 0, 1, \cdots, 6$), and the GEC model. The DCCA model is Rayleigh fading ($K_R = 0$), with $E_s/N_0 = 15$ dB, and $f_D T$ = 0.1 (a), $f_D T$ = 0.02 (b), $f_D T$ = 0.001 (c).

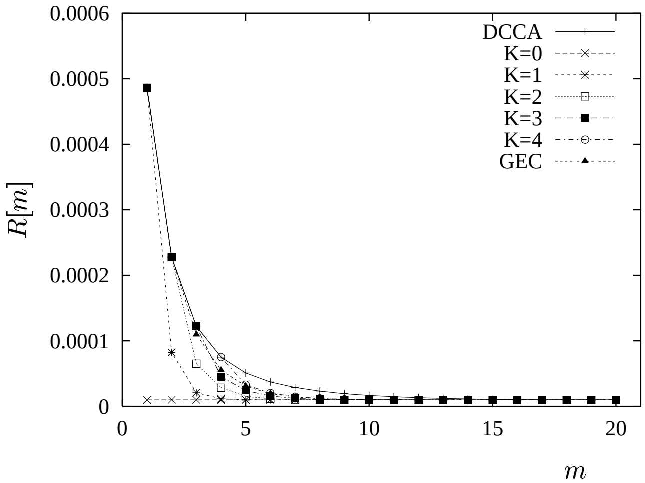

Fig. 2. Comparison of the autocorrelation functions of the DCCA fading model, the Kth-order Markov model ($K = 0, 1, \cdots, 6$), and the GEC model. The DCCA model is Rayleigh fading ($K_R = 0$) with $E_s/N_0 = 25$ dB, $f_DT = 0.02$.

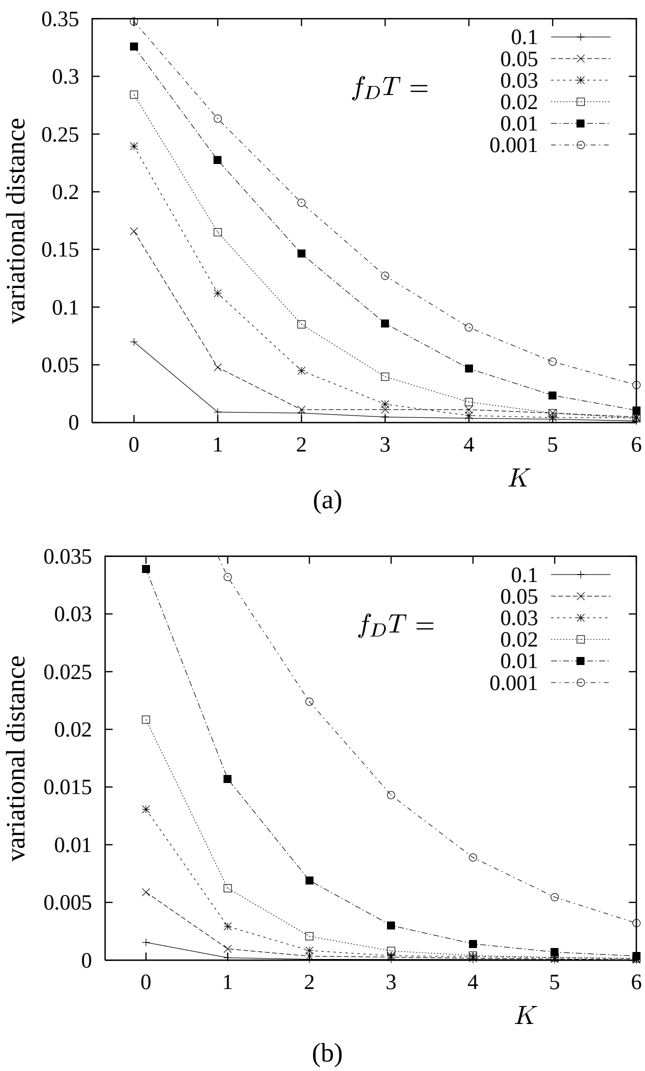

Fig. 3. Variational distance versus the order K having f_DT as a parameter. Rayleigh fading ($K_R = 0$), $E_s/N_0 = 15$ dB (a), $E_s/N_0 = 25$ dB (b).

channel. Markov models of order up to 6 have been proposed as an approximation to the DCCA model for a broad range of fading environments. We have used two criteria to estimate the order of the Markov process: The autocorrelation function, and the variational distance. Both criteria lead to a similar conclusion that the Kth-order Markov model is a good approximation to the DCCA model. This analysis reinforces the results in [7] regarding the effectiveness of ACF criterion to estimate the order of Markov models. It is observed that the first-order approximation is satisfactory for values of f_DT around 0.1. For fast and medium fading rates ($f_DT > 0.03$), the Kth-order (for judiciously selected K) is as accurate as the GEC model. For slower fading rates ($f_DT < 0.02$) Markov models of order greater than 6 are required.

References

[1] F. Swarts and H. C. Ferreira, “Markov characterization of digital fading mobile VHF channels”, *IEEE Transactions on Vehicular Technology*, Vol. 43, pp. 977-985, Nov. 1994.

[2] H. Wang and N. Moayeri, “Finite-State Markov channel - A useful model for radio communication channels,” *IEEE Transactions on Vehicular Technology*, vol. 44, pp. 163-171, Feb 1995.

[3] H. Wang and P. Chang, “On verifyng the first-order Markovian assumption for a Rayleigh fading channel model,” *IEEE Transactions on Vehicular Technology*, vol. 45, pp. 353-357, May 1996.

[4] M. Zorzi, R. Rao, and L. B. Milstein, “ARQ error control for fading mobile radio channels,” *IEEE Transactions on Vehicular Technology*, vol. 46, pp. 445-455, May 1997.

[5] Q. Zhang and S. Kassam, “Finite-State Markov model for Rayleigh fading channels,” *IEEE Transactions on Communications*, vol. 47, pp. 1688-1692, Nov. 1999.

[6] F. Babich and G. Lombardi, “A Markov model for the mobile propagation channel,” *IEEE Transactions on Vehicular Technology*, vol. 49, pp. 63-73, Jan. 2000.

[7] C. Tan and N. C. Beaulieu, “On first-order Markov Modeling for the Rayleigh Fading Channel,” *IEEE Transactions on Communications*, vol. 48, pp. 2032-2040, Dec. 2000.

[8] F. Babich, O. Kelly, and G. Lombardi “Generalized Markov Modeling for Flat Fading,”*IEEE Transactions on Communications*, vol. 48, pp. 547-551, Apr. 2000.

[9] C. Pimentel and I. F. Blake, “Modeling burst channels using partitioned Fritchman’s Markov models,” *IEEE Transactions on Vehicular Technology*, Vol. 47, No. 3, pp.885-899, Aug. 1998.

[10] L. Wilhelmsson e L. B. Milstein, “On the effect of imperfect interleaving for the Gilbert-Elliott channel”, *IEEE Transactions on Communications*, Vol. 47, pp. 681-688, May 1999.

[11] H. Turin and R. van Nobelen, “Hidden Markov Modeling of Flat Fading Channels,” *IEEE Journal on Selected Areas in Communications*, vol. 16, pp. 1809-1817, Dec. 1998.

[12] M. Zorzi, “Some results on error control for burst-error channels under delay constraints,” *IEEE Transactions on Vehicular Technology*, Vol. 50, pp.12-24, Jan. 2001.

[13] C. Pimentel and I. F. Blake, “Enumeration of Markov chains and burst error statistics for finite state models, ” *IEEE Transactions on Vehicular Technology*, Vol. 48, pp. 415-428, Mar. 1999.

[14] C. Pimentel and I. F. Blake, “Concatenated coding performance for FSK modulation on time-correlated Rician fading channels, ”*IEEE Transactions on Communications*, vol. 46, pp. 1610-1618, Dec. 1998.

[15] G. Sharma, A. Hassan, and A. Dholakia, “Performance evaluation of burst-error correcting codes on a Gilbert-Elliott channel, ” *IEEE Transactions on Communications*, vol. 46, pp. 846-849, Jul. 1998.

[16] R. H. Clarke, “A statistical theory of mobile-radio reception,” *Bell System Tech. Journal*, vol. 47, pp. 957-1000, 1968.

[17] M. Gans, “A power-spectral theory of propagation in the mobile radio environment, ” *IEEE Transactions on Vehicular Technology*, vol. 21, pp. 27-38, Feb. 1972.

[18] T. W. Anderson and L. A. Goodman, “Statistical Inference about Markov Chains,” *Annals of Mathematical Statistics*, vol. 28, pp. 89-110, Mar. 1957.