Viterbi Decoder Simulation on Nonsymmetric and Erasure Binary-Input Channels Using Importance Sampling

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Abstract

Importance sampling (IS) is a modified Monte Carlo (MC) simulation technique that improves computational efficiency by reducing the variance of a given simulation estimator. In this paper IS is combined with error event simulation in order to evaluate efficiently the performance of Viterbi decoding used on a nonsymmetric and on an erasure binary-input channel. Stationary and nonstationary biasing approaches are employed and comparative results are presented.

1 Introduction

Computer simulation is frequently used to estimate the digital communication system performance, particularly when nonlinearities or band-limited systems are present. The bit error probability P_b is one of the most used parameter for evaluating the digital communication system performance, and a number of simulation techniques can be used to estimate this parameter [1]. The Monte Carlo (MC) method is the most general of the techniques, since no *a priori* assumptions are made. On the other hand, it is computationally the most costly of the methods. This cost is related to the number of observations (samples) that must be made in order to obtain a certain reliability of the estimated parameter. In general, it is required $100/P_b$ samples to obtain a 10% precision related to the estimator's standard deviation of the true value P_b [2]. Even for error probabilities around 10^{-4} - 10^{-3}, depending on system complexity, computer run times may be prohibitive. The modified MC technique based on importance sampling (IS) can often reduce significantly the amount of samples for parameter estimating. The basic idea is to choose a "biased" simulation distribution which increases the relative frequence of "important" events, in general, error events. In order to obtain an unbiased estimator, the simulation outputs are weighted by "a posteriori" likelihood ratio. The IS simulation design problem is the selection of the simulation distribution which minimizes the estimator's computational run time. This technique was successful applied in nonlinear and non Gaussian channels, particularly in satellite and optical communication channels. However, few works attempt to apply IS to coded systems. An article by [3] seems to be the first attempt, but the IS efficiency gains were quite modest for large constraint length codes.

In [2], it is shown that the error event method can be extremely powerful when employed in conjunction with IS. Additionally, it is shown that the efficiency gain does not depend on code constraint length of the convolutional code. In this paper this method is reviewed and applied to a nonsymmetric binary-input channel modeling an optical on-off keying (OOK) convolutionally encoded communication system. A binary erasure channel is also modeled. In Section 2 the error event IS simulation is presented. Simulation results are shown in Section 3 and Section 4 concludes the paper.

2 Modified Monte Carlo Method

The modified MC method is based in importance sampling (IS) technique, differently from the conventional MC method in which the occurrence of an relevant event contributes with weight one for the estimate calculation, here, each relevant event is weighted by a value different from one called importance sampling weight. This weighting process is related to the sample statistic distribution changing, in such a way that the relevant (important) events appear more frequently.

2.1 Discrete Memoryless Channel

Given that the input to a discrete memoryless channel (DMC) is a sequence of n symbols $u_0, u_1, ..., u_{n-1}$ selected from the alphabet $\mathcal{X}$ and the corresponding output is a sequence of symbols $v_0, v_1, ..., v_{n-1}$ selected from the alphabet $\mathcal{Y}$, the joint conditional probability is

$$P(Y = v_0, ..., Y = v_{n-1}|X = u_{0,...,}X = u_{n-1}) = \prod_{k=0}^{n-1} P(Y = v_k|X = u_k)$$

This relation mathematically states the memoryless condition of the channel.

2.2 Error Event Simulation Method

Now, we begin a brief review of the Viterbi decoding process fundamentals. More details may be found in [4].

A convenient way to describe the state transitions of a convolutional encoder as a function of time is a trellis diagram. A sequence of connected branches $\mathbf{u} = (u_0, u_1, ...)$ defines a path in the trellis. The code symbols sequence determined by a path $\mathbf{u}$ is defined as $\mathbf{x}(\mathbf{u}) = (x_0, x_1, ...)$, where $x_i = x(u_i)$. The Viterbi decoder output can be viewed as a sequence of correlated decisions on possible branches defined by a trellis path. Decoding errors occur in bursts, called error events, in which a decoded path diverges from the correct path. Formally, an error event $\mathbf{u}'$ is a partial sequence of incorrectly decoded branches, which begins at a correct node, finished at a correct node, and there is no correct node between them. The number of branches in an error event defines its length, and is denoted by $L(\mathbf{u}')$.

At instant time j, suppose that the encoder state is correctly decoded. For this event is defined a random variable $N_b(j)$ as the number of erroneously decoded bits due to an error event which begins at time j. Note that from the error event definition, a partial path which begins in a correct node at time j and finished in a correct node at time $j+1$ is an error event. In this case we call the error event as trivial error event. A trivial error event has 0 wrong bits, $N_b(j) = 0$.

Define the bit error probability when the path $\mathbf{u}$ is transmitted beginning at time j as

$$P_b(\mathbf{u}, j) = E[N_b(j)|\mathbf{x}(\mathbf{u})]$$

where $E[.]$ is the expectation operator. In the case of linear convolutional codes operating on memoryless binary symmetric channel (BSC) and with maximum likelihood decoding, $P_b(\mathbf{u}, j) = P_b$, in other words, the bit error probability does not depend on neither the correct path $\mathbf{u}$ or the time j. In general the bit error probability P_b is defined as

$$P_b = E[P_b(\mathbf{U}, j)]$$

where the expectation is with respect to the random information sequence $\mathbf{U}$. If the channel is stationary but perhaps with memory this expectation is still independent of j.

Consider a linear convolutional codes over a memoryless BSC. Without loss of generality, let $\mathbf{u}$ be the all-zero trellis path and $j = 0$, for estimating $P_b = E[P_b(\mathbf{u}, 0)]$.

The error event simulation is based on a sum

$$P_b = \sum_{\mathbf{u} \in \mathcal{E}} n_b(\mathbf{u}, \mathbf{u}') P(\mathbf{u}'|\mathbf{u})$$

where $\mathcal{E} = \mathcal{E}(\mathbf{u}, 0)$ is the set of all error events which occur at time $j = 0$ when $\mathbf{u}$ is the correct path, $n_b(\mathbf{u}, \mathbf{u}')$ is the number of postdecoding error bits caused by decoding the error event $\mathbf{u}'$ instead of $\mathbf{u}$ when $\mathbf{x}(\mathbf{u})$ is transmitted, and $P(\mathbf{u}'|\mathbf{u})$ is the probability of decoding $\mathbf{u}'$ given the correct path is $\mathbf{u}$ and is called the specific error event probability. Two points must be noted in the above sum. First, the error events $\mathbf{u}' \in \mathcal{E}$ have no fixed length, $L(\mathbf{u}')$ is variable. Second, the sum has infinite terms. However, it is possible to evaluate P_b accurately by concentrating only on dominant terms.

In the error event simulation method, each run considers only one error event of the Viterbi decoder. For understanding the method we have to answer the following question: Given the output channel random sequences $\mathbf{Y} = (\mathbf{Y}_0, \mathbf{Y}_1, ...)$, when a decision is made about the jth trellis branch? This random time instant is denoted by $T_D(j) > j$.

To answer this question, it is helpful to see how the Viterbi algorithm works. Let K be the length of the covolutional encoder shift register. At each time i, 2^K survivor path candidates are recorded, each one terminating at one of the 2^K trellis node. These candidates to survivor branches terminate at distinct nodes then they generate distinct candidate paths. However, if we back trace along these trellis paths, eventually they all merge to a "common stem". As a result $T_D(j)$ is the first time instant that all back trace paths from 2^K candidates merge to a common stem at time j.

The channel output sequences $\mathbf{Y}^{(l)}$ for the lth simulation run is generated by the importance sampling technique, explained in the next subsection.

All candidate paths must begin at the correct node at time $j = 0$. This means that all path are survivor candidates until time $i = K + 1$.

Let $J^{(l)} > 0$ denote the time index of the first decoded branch which merges into a correct node, i. e., the time index at which an error event is decoded. Define $T_M^{(l)} = T_D^{(l)}(J^{(l)})$ as the time at which the lth simulation run detect an error event. Each run only must generate the data sequence $\mathbf{Y}^{(l)}$ until this time. This in important point, it is not necessary generate an infinite sequence $\mathbf{Y}^{(l)}$, of course impossible. However, $T_M^{(l)}$ is a random variable, and hence, the simulation has a random length.

2.3 Error Event Simulation and Importance Sampling

When a sequence $\mathbf{x}(\mathbf{u})$ is transmitted, the true conditional joint density of the channel output is $f(\mathbf{y}|\mathbf{x})$. The importance sampling uses a modified joint density, called simulation density, denoted by $f^*(\mathbf{y}|\mathbf{x})$. The importance sampling estimate for the specific error event probability $P(\mathbf{u}'|\mathbf{u})$ is

$$\hat{P}_L^*(\mathbf{u}'|\mathbf{u}) = \frac{1}{L}\sum_{l=1}^{L} w(\mathbf{Y}^{(l)}|\mathbf{x}(\mathbf{u}))\mathbf{I}_{u'}(\mathbf{Y}^{(l)})$$

where $\mathbf{I}_{u'}(\mathbf{y})$ is the indicator function for decoding $\mathbf{u}'$, $\mathbf{I}_{u'}(\mathbf{y}) = 1$ if $\mathbf{u}'$ is decoded, and 0 otherwise, $w(\mathbf{Y}^{(l)}|\mathbf{x}(\mathbf{u}))$ is the importance sampling weight function, which must be defined to make the estimator $\hat{P}_L^*(\mathbf{u}'|\mathbf{u})$ unbiased, and the parameter L is the number of simulation runs for this specific error event $\mathbf{u}'$.

The algorithm stopping time, denoted by $T_M^{(l)} = t$, can be determined by verifying the channel output sequence $y_0, ...y_t$, thus $I_{\mathbf{u}'}(.)$ can be decomposed as

$$\mathbf{I}_{\mathbf{u}'}(\mathbf{y}) = \sum_{t=0}^{\infty} \mathbf{I}_{\mathbf{u}',t}(y_0, ..., y_t)$$

where $\mathbf{I}_{\mathbf{u}',t}(y_0, ..., y_t)$ is the indicator function of the channel output sequences set $\left\{\mathbf{y} : \mathbf{u}' \text{ is decoded and } T_M^{(l)} = t\right\}$, thus, the equality $\mathbf{I}_{\mathbf{u}'}(\mathbf{y}) = \mathbf{I}_{\mathbf{u}',t}(y_0, ..., y_t)$ occurs whenever $\mathbf{y}$ belongs to the above set. If $f_t(y_0, ..., y_t|\mathbf{x})$ denotes the true joint density of $(Y_0, ..., Y_t)$, and if $f_t^*(y_0, ..., y_t|\mathbf{x})$ denotes the importance sampling simulation joint density of $(Y_0, ..., Y_t)$, given the sequence $\mathbf{x}$ is transmitted, then the appropriate weight function is

$$w(\mathbf{y}|\mathbf{x}) = \sum_{t=0}^{\infty} w_t(y_0, ..., y_t|\mathbf{x}) I_t(y_0, ..., y_t)$$

where $I_t(y_0, ..., y_t)$ is the indicator function of the channel output set $\left\{\mathbf{y} : T_M^{(l)} = t\right\}$, and $w_t(.)$ is the ratio

$$w_t(y_0, ..., y_t|\mathbf{x}) = \frac{f_t(y_0, ..., y_t|\mathbf{x})}{f_t^*(y_0, ..., y_t|\mathbf{x})}.$$

The weight defined by the above equation is proved unbiased [2].

3 Simulation Results

We present now some simulation results. We use a convolutional encoder with code rate $R = 1/2$ and constraint length $K = 5$, with generators $g_0 = 23$ and $g_1 = 35$ in octal. This encoder is sufficiently complex to make a MC simulation, in some situations, almost impossible.

We consider a binary asymmetric channel.

The following transition probabilities values $P(y_i|x_i)$ are typical of an optical channel with on-off keying (OOK), extremely dispersive [5]

$P(y_i\|x_i)$	$y_i = 0$	$y_i = 1$
$x_i = 0$	$0,9999$	$0,01$
$x_i = 1$	$0,0001$	$0,99$

Two models of error event simulation with importance sampling are compared: a stationary model and a nonstationary model.

The simulation was done in two steps, first the source transmits only 0's, and second transmits only 1's. The bit error probability P_b is computed as

$$P_b = P_{b|0}P(0) + P_{b|1}P(1)$$

where $P_{b|i}$ is the bit error probability when bit i is transmitted, $i = 0, 1$. If the source symbols are generated with equal probabilities than $P_b = (P_{b|0} + P_{b|1})/2$

In the stationary model we used the same biasing value at both steps, $P^*(y_i = 1|x_i = 0) = P^*(y_i = 0|x_i = 1) = 0,08$, when 0's and 1's are transmitted respectively. We used 80.000 samples per error event at each step. These values were defined by trial and error, so as to reduce the estimated relative precision . The estimates of the relative precision is defined by the ratio of the estimator's standard deviation and the estimator mean value. To improve the overall efficiency of the simulation, for the first transmitted encoded symbol through the channel $x_0 = (b_{0,0}, b_{0,1})$, where $b_{i,j}$ denotes the jth bit of the ith transmitted symbol, we use a value $1/2$ for the biasing cross probabilities. Formally, this model is not anymore stationary, but we shall continue call it stationary.

In the table below, we present the simulation results for this model. The first column gives the Hamming distance for a given error event $\mathbf{u}'$, and it is the Hamming distance between two symbol sequences $\mathbf{x}(\mathbf{u})$ and $\mathbf{x}(\mathbf{u}')$ until the error event time $L(\mathbf{u}')$. This distance is also called error event distance, and denoted by $d(\mathbf{u}, \mathbf{u}')$. In this simulation we consider error events up to $d(\mathbf{u}, \mathbf{u}') \leq 10$. For a distance d is defined the distance spectrum as the number of error events with Hamming distance d, this parameter is shown in the second column. The information weight is the number of information bits errors $n_b(\mathbf{u}, \mathbf{u}')$ of all error events with $d(\mathbf{u}, \mathbf{u}') = d$, the third column shows these values. The fourth column shows the mean values of the specific error probabilities $P(\mathbf{u}'|\mathbf{u})$ of a specific error event $\mathbf{u}'$ occur given the sequence $\mathbf{x}(\mathbf{u})$ is transmitted, for all error events with $d(\mathbf{u}, \mathbf{u}') = d$. The relative precision is presented percentually in the next column.

In the nonstationary model the biasing value is $1/2$ for each bit of the specific error event $\mathbf{u}'$ which differs from the bits of the transmitted path $\mathbf{u}$. In equation form

$$\begin{aligned} P^*(y_i|x_i) &= P(y_i|x_i) \quad if \;\; b'_{i,j} = b_{i,j} \\ &= 1/2 \qquad\;\; if \;\; b'_{i,j} \neq b_{i,j} \end{aligned}$$

where $b'_{i,j}$ denotes a bit of the specific error event

			Stationary Model		Nonstationary Model	
Hamming Distance	Number of Codewords	Information Weight	$P(\mathbf{u}'\|\mathbf{u})$	Relative Precision(%)	$P(\mathbf{u}'\|\mathbf{u})$	Relative Precision(%)
7	2	4	5.01×10^{-10}	26.8%	$1,04 \times 10^{-9}$	3,7%
8	3	12	1.67×10^{-11}	121.4%	$3,36 \times 10^{-11}$	3,9%
9	4	20	1.05×10^{-10}	154.4%	$2,92 \times 10^{-11}$	11,5%
10	20	72	2.58×10^{-13}	330.1%	$5,27 \times 10^{-13}$	8,9%

$\mathbf{u}'$ which is being evaluated, and $b_{i,j}$ is a bit of the transmitted path $\mathbf{u}$. In the simulation we used $1,000$ samples per error event per step. Again, we considered error events up to $d(\mathbf{u},\mathbf{u}') \leq 10$. The last two columns in the table shows the obtained results for this model.

The estimated bit error probability for the first model is $P_b = 4.320 \times 10^{-9}$ with $2 \times 25 \times 80,000 = 4,000,000$ runs,, whereas for the nonstationary model $P_b = 5.371 \times 10^{-9}$. and takes $2 \times 25 \times 1.000 = 50.000$ runs. A MC model for estimating a BER on the order of 10^{-9} requires a number between 10 to 100 billions runs to obtain a 10% precision. In order to corroborate the above results, we changed the convolutional encoder by another simple one, with $R = 1/2$, $K = 4$, $g_0 = 13$, and $g_1 = 15$. A theoretical analysis showed an upper bound $P_b < 1.94 \times 10^{-6}$ [5]. We considered now error events up to $d(\mathbf{u},\mathbf{u}') \leq 8$, and we found $P_b = 2.99 \times 10^{-7}$ with 1.92×10^6 runs for the stationary model (0.08 biasing probability), $P_b = 2.63 \times 10^{-7}$ with $24,000$ runs for the nonstationary model, and $P_b = 1.5 \times 10^{-7}$ with 10^8 runs for MC simulation.

The method was also applied to a high level interference channel with a strong signal-to-noise rate. This channel may be modeled by a pure binary erasure channel with no bit errors, in which all interference is detected and blanked. We used the later convolutional encoder in the simulation. If 10% of the total bits are blanked, the estimated BER for the stationary model was $P_b = 1.424 \times 10^{-6}$ with $250000 \times 12 = 3.0 \times 10^6$ runs (0.2 biasing probability), for the nonstationary model $P_b = 2.058 \times 10^{-6}$ and it takes 12000 runs. A MC model was also used and we found $P_b = 2.080 \times 10^{-6}$ and 10^8 runs.

4 Conclusion

It is difficult to quantify the "computational efficiency gain". We coul estimate the ratio of the numbers of simulation runs L_{MC}/L_{IS} required for a specified precision. However we must take care with these figures, because they do not account for a number of factors. First, IS requires additional computing of the IS weight. Second, IS will tend to produce long nontrivial errors events, and back tracing overhead may be significant. Nonetheless, the astronomically high L_{MC}/L_{IS} gains far outweigh these other factors.

IS technique can be effectively applied to a very broad range of systems, but as the system complexity grows, the IS technique must be more finely tuned to the specific problem at hand. The error event simulation method in conjunction with IS proved to be extremely efficient as a tool for evaluate the Viterbi decoder.

Nonstationary model's performance can be seen, in most cases, superior to the stationary model. Conceptually the stationary model handle all transmited code word components $x(u_i)$ in the same way, i. e., all components are biased with the same biasing simulation distribution, hence the biasing distribution is identical in all directions. This is precisely the case of the conventional variance scaling method. On the other hand nonstationary model bias these components in the direction of the specific error decision region, and can be viewed as a mean translation method. So we see that only the error probabilities for error events with relatively low distances are accurately estimated.

As an extension of this work, we intend to use IS to evaluate digital communication systems which use iterative decoding [6]. A first work was pro-

posed recently in [7], where iterative decoding is analyzed for product codes using IS. The IS technique presented there is essentially the same idea presented here for the nonstationary case. When convolutional codes are used as components codes we may use Viterbi decoders as component decoder, and the method presented in this paper may be a natural base to evaluate these schems.

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