Discriminative Filtering as an Impulse Restoration Problem

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***Abstract*: In image template detection, one wants to build an operator that indicates whether a given template is present in an image, preferably giving its location. In earlier works, we have approached the template detection problem by using two-dimensional discriminative filtering. A discriminative filter maximizes, for a given template, the energy concentration in a single sample of its output. In this paper, we frame discriminative filtering as an impulse restoration problem and propose closed form solutions for it. Simulation results show that the proposed method works well on real images.**

I. Introduction

Image template detection is usually a very important halfway step for a computational vision algorithm. In general, the basic idea of this type of algorithm is to receive an input image and generate a set of lines, borders, edges and other well known geometrical forms as outputs.

Discriminative filtering consists of making a convolution between the image and an operator computed for a specific template. The expected output has samples with large power where the template is present and small power otherwise. It has been proposed by Ben-Arie, et ali [1-5] and extended for the two-dimensional case by Mendonça e da Silva [6].

Discriminative filtering can be considered a particular case of template detection using impulse restoration. In this technique, an image is processed as a template in an unknown position corrupted with additive noise. The objective of the processing is to detect the template as an impulse in the correct location. In this paper, we analyze the discriminative filtering model as an impulse restoration problem.

II. Impulse Restoration: The Classical Problem

In the impulse restoration problem, well approached by Abu-Naser et ali [7,8], an image is modeled as a template in a unknown location corrupted with additive noise. There, this problem was analyzed for one-dimensional signals. For images, a two-dimensional signal is transformed in a column vector after all image columns are concatenated. So, this problem can be stated mathematically as shown in equation 1:

$$g(n) = f(n-n_0) + b(n), \qquad (1)$$

where g(n) is the image, represented as a sequence, $f(n-n_0)$ is the template centered at the sample n_0 and b(n) is the additive noise, that corresponds to the rest of the image.

The term $f(n-n_0)$ of equation 1 can be rewritten as a circular convolution between f(n) and the impulse at location n_0. So:

$$g(n) = f(n) * \delta(n-n_0) + b(n). \qquad (2)$$

Let **F** be the NxN circulant square matrix defined by equation 3.

$$\mathbf{F} = \begin{bmatrix} f(0) & f(1) & \dots & f(N-1) \\ f(1) & f(2) & \dots & f(0) \\ \dots & \dots & \dots & \dots \\ f(N-1) & f(0) & \dots & f(N-2) \end{bmatrix} \qquad (3)$$

With this definition, we can rewrite equation 2 in matrix form:

$$\boldsymbol{g} = \mathbf{F}\,\boldsymbol{\delta} + \boldsymbol{b}, \qquad (4)$$

where $\boldsymbol{g}$, $\boldsymbol{\delta}$ e $\boldsymbol{b}$ are Nx1 vectors.

The impulse restoration problem consists of, given the vector $\boldsymbol{g}$ and the matrix **F**, estimate the vector $\boldsymbol{\delta}$, because we assume that the noise vector $\boldsymbol{b}$ is unknown.

Now assuming that the random noise vector $\boldsymbol{b}$ is gaussian, with zero mean and covariance $\mathbf{C_b}$, and also that $\boldsymbol{\delta}$ is also zero mean with covariance equal to the NxN identity (**I**), the best estimative for $\boldsymbol{\delta}$ (that is, $\hat{\boldsymbol{\delta}}$),which minimizes the mean square error $E[\|\boldsymbol{\delta} - \hat{\boldsymbol{\delta}}\|^2]$, and is linear in relation to the observation $\boldsymbol{g}$ (i.e. $\hat{\boldsymbol{\delta}} = \mathbf{A}\,\boldsymbol{g}$). Using the orthogonality principle, the best estimate $\hat{\boldsymbol{\delta}}$ makes the error $\boldsymbol{\delta} - \hat{\boldsymbol{\delta}}$ be uncorreleted to the observation $\boldsymbol{g}$. So:

$$E \{ (\boldsymbol{\delta} - \hat{\boldsymbol{\delta}}) \boldsymbol{g}^t \} = \mathbf{0} \qquad (5)$$

With the hypothesis that $\boldsymbol{\delta}$ and $\boldsymbol{b}$ are uncorraleted, we can use the following relations:

$$E \{ \boldsymbol{\delta} \boldsymbol{\delta}^t \} = \mathbf{I}, \qquad (6)$$
$$E \{ \boldsymbol{\delta} \boldsymbol{b}^t \} = \mathbf{0}, \qquad (7)$$
$$E \{ \boldsymbol{b} \boldsymbol{b}^t \} = \mathbf{C_b}. \qquad (8)$$

Consequently,

$$\mathbf{A} = \mathbf{F}^t (\mathbf{F} \mathbf{F}^t + \mathbf{C_b})^{-1}, \qquad (9)$$

or

$$\hat{\boldsymbol{\delta}} = \mathbf{F}^t (\mathbf{F} \mathbf{F}^t + \mathbf{C_b})^{-1} \boldsymbol{g} . \qquad (10)$$

The linear estimator of equation 10 is not in the conventional format, presented in the references [7] and [8]. However, it can be rewritten as:

$$\begin{aligned}
\hat{\boldsymbol{\delta}} &= \mathbf{F}^t (\mathbf{F} \mathbf{F}^t + \mathbf{C_b})^{-1} \boldsymbol{g} = \\
&\mathbf{F}^t (\mathbf{F} \mathbf{F}^t + \mathbf{C_b})^{-1} \mathbf{C_b} (\mathbf{F}^t)^{-1} \mathbf{F}^t \mathbf{C_b}^{-1} \boldsymbol{g} = \\
&\mathbf{F}^t (\mathbf{F} \mathbf{F}^t + \mathbf{C_b})^{-1} (\mathbf{F}^t \mathbf{C_b}^{-1})^{-1} \mathbf{F}^t \mathbf{C_b}^{-1} \boldsymbol{g} = \\
&(\{ \mathbf{F} \mathbf{F}^t + \mathbf{C_b} \} (\mathbf{F}^t)^{-1})^{-1} (\mathbf{F}^t \mathbf{C_b}^{-1})^{-1} \mathbf{F}^t \mathbf{C_b}^{-1} \boldsymbol{g} = \\
&(\mathbf{F} + \mathbf{C_b} (\mathbf{F}^t)^{-1})^{-1} (\mathbf{F}^t \mathbf{C_b}^{-1})^{-1} \mathbf{F}^t \mathbf{C_b}^{-1} \boldsymbol{g} = \\
&(\mathbf{F}^t \mathbf{C_b}^{-1} \{ \mathbf{F} + \mathbf{C_b} (\mathbf{F}^t)^{-1} \})^{-1} \mathbf{F}^t \mathbf{C_b}^{-1} \boldsymbol{g} = \\
&(\mathbf{F}^t \mathbf{C_b}^{-1} \mathbf{F} + \mathbf{I})^{-1} \mathbf{F}^t \mathbf{C_b}^{-1} \boldsymbol{g} , \qquad (11)
\end{aligned}$$

which is the most usual form in the literature.

III. Two-Dimensional Discriminative Filtering

When discriminative filters are used for template detection, one usually wants to maximize the energy of an output sample whenever a match is found. The two-dimensional discriminative signal to noise ratio ($DSNR_2$), defined in [6], is a measure that accounts not only for the maximum energy of a sample, but also considers its energy in relation to the other samples. Thus, for two-dimensional discriminative filters, we need to maximize:

$$DSNR_2 = \frac{c_{i,j}^2}{\left(\sum_m \sum_n c_{m,n}^2\right) - c_{i,j}^2} \qquad (12)$$

The $c_{m,n}$ coefficients are obtained after a two-dimensional convolution between an input image window $u_{m,n}$ and a linear operator Θ having impulse response $\theta_{m,n}$, that may be computed for each template to be matched (see equation 13). The coefficient $c_{i,j}$ is the one where we wish to concentrate the output signal energy.

$$c_{m,n} = \sum_{m'} \sum_{n'} u_{m-m',n-n'} \; \theta_{m',n'} \qquad (13)$$

IV. Parallel Between Discriminative Filtering and Impulse Restoration

As pointed out above, discriminative filtering is a problem equivalent an impulse restoration. We so ask: is there a way to model two-dimensional discriminative filtering as the one presented in section 2 ?

To search for this answer, we begin writing the image as the result of a two-dimensional circular convolution between the template u(m,n) and the impulse at an unknown location, corrupted with additive noise:

$$g(m,n) = u(m,n) * \delta(m-m_0,n-n_0) + b(m,n). \qquad (14)$$

or

$$\boldsymbol{G} = \boldsymbol{U} * \boldsymbol{\delta} + \boldsymbol{B}. \qquad (15)$$

Here, our interest is to put the equations 14 and 15 in a matrix form resembling equation 4. Therefore, we rewrite the equation 13 (adding noise) as:

$$g_{m,n} = \left\{ \sum_{m'} \sum_{n'} u_{m-m',n-n'} \; \delta_{m',n'} \right\} + b_{m,n} \qquad (16)$$

$$-t \le m',n',m,n \le t.$$

Below, we write some vectors and matrices defined as in the one-dimensional case, but extended for the two-dimensional problem. We note that the vectors are constructed from the image representation as a concatenation of its transposed rows. The images were chosen to be square. Note that matrices **F** and **A** are block circulant in this case

$$\mathbf{g} = \begin{bmatrix} g_{-t,-t} \\ g_{-t,-t+1} \\ \cdots \\ g_{-t,+t} \\ \cdots \\ g_{+t,+t} \end{bmatrix} \qquad (17)$$

$$\boldsymbol{\delta} = \begin{bmatrix} \delta_{-t,-t} \\ \delta_{-t,-t+1} \\ \cdots \\ \delta_{-t,+t} \\ \cdots \\ \delta_{+t,+t} \end{bmatrix} \qquad (18)$$

$$\mathbf{F} = \begin{bmatrix} u_{0,0} & u_{0,-1} & \cdots & u_{0,+1} & u_{-1,0} & \cdots & u_{+1,+1} \\ u_{0,+1} & u_{0,0} & \cdots & u_{0,+2} & u_{-1,+1} & \cdots & u_{+1,+2} \\ & \cdots & & & \cdots & & \\ u_{-1,-1} & u_{-1,-2} & \cdots & u_{-1,0} & u_{-2,-1} & \cdots & u_{0,0} \end{bmatrix} \quad (19)$$

$$\mathbf{A} = \begin{bmatrix} \theta_{0,0} & \theta_{0,-1} & \cdots & \theta_{0,+1} & \theta_{-1,0} & \cdots & \theta_{+1,+1} \\ \theta_{0,+1} & \theta_{0,0} & \cdots & \theta_{0,+2} & \theta_{-1,+1} & \cdots & \theta_{+1,+2} \\ & \cdots & & & \cdots & & \\ \theta_{-1,-1} & \theta_{-1,-2} & \cdots & \theta_{-1,0} & \theta_{-2,-1} & \cdots & \theta_{0,0} \end{bmatrix} \quad (20)$$

where $\theta_{i,j}$ are the coefficients of the desired circular convolutor filter.

To clarify the structure of **F** and **A**, we will use a notation based on blocks. Let $\mathbf{H_r}$ be the operator that transforms the row **r** of a generic window 2 N+1 x 2 N+1 **v** in a 2 N+1 x 2 N+1 circulant matrix, according to the following rule (equation 21):

$$\mathbf{H_r}(\mathbf{v}) = \begin{bmatrix} \mathbf{v}_{r,0} & \mathbf{v}_{r,-1} & \mathbf{v}_{r,-2} & \cdots & \mathbf{v}_{r,+3} & \mathbf{v}_{r,+2} & \mathbf{v}_{r,+1} \\ \mathbf{v}_{r,+1} & \mathbf{v}_{r,0} & \mathbf{v}_{r,-1} & \cdots & \mathbf{v}_{r,+4} & \mathbf{v}_{r,+3} & \mathbf{v}_{r,+2} \\ & \cdots & & & & \cdots & \\ \mathbf{v}_{r,+N} & \mathbf{v}_{r,+N-1} & \mathbf{v}_{r,+N-2} & \cdots & \mathbf{v}_{r,-N+2} & \mathbf{v}_{r,-N+1} & \mathbf{v}_{r,-N} \\ \mathbf{v}_{r,-N} & \mathbf{v}_{r,+N} & \mathbf{v}_{r,+N-1} & \cdots & \mathbf{v}_{r,-N+3} & \mathbf{v}_{r,-N+2} & \mathbf{v}_{r,-N+1} \\ & \cdots & & & & \cdots & \\ \mathbf{v}_{r,-1} & \mathbf{v}_{r,-2} & \mathbf{v}_{r,-3} & \cdots & \mathbf{v}_{r,+2} & \mathbf{v}_{r,+1} & \mathbf{v}_{r,0} \end{bmatrix} \quad (21)$$

With the definition of operator $\mathbf{H_r}$, we can write **F** and **A**:

$$\mathbf{F} = \begin{bmatrix} \mathbf{H_0(U)} & \mathbf{H_{-1}(U)} & \cdots & \mathbf{H_{-t}(U)} & \mathbf{H_{+t}(U)} & \cdots & \mathbf{H_{+1}(U)} \\ \mathbf{H_{+1}(U)} & \mathbf{H_0(U)} & \cdots & \mathbf{H_{-t+1}(U)} & \mathbf{H_{-t}(U)} & \cdots & \mathbf{H_{+2}(U)} \\ & \cdots & & & \cdots & & \\ \mathbf{H_{+t}(U)} & \mathbf{H_{+t-1}(U)} & \cdots & \mathbf{H_0(U)} & \mathbf{H_{-1}(U)} & \cdots & \mathbf{H_{-t}(U)} \\ \mathbf{H_{-t}(U)} & \mathbf{H_{+t}(U)} & \cdots & \mathbf{H_{+1}(U)} & \mathbf{H_0(U)} & \cdots & \mathbf{H_{-t+1}(U)} \\ & \cdots & & & \cdots & & \\ \mathbf{H_{-1}(U)} & \mathbf{H_{-2}(U)} & \cdots & \mathbf{H_{+t}(U)} & \mathbf{H_{+t-1}(U)} & \cdots & \mathbf{H_0(U)} \end{bmatrix} \quad (22)$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{H_0(\Theta)} & \mathbf{H_{-1}(\Theta)} & \cdots & \mathbf{H_{-t}(\Theta)} & \mathbf{H_{+t}(\Theta)} & \cdots & \mathbf{H_{+1}(\Theta)} \\ \mathbf{H_{+1}(\Theta)} & \mathbf{H_0(\Theta)} & \cdots & \mathbf{H_{-t+1}(\Theta)} & \mathbf{H_{-t}(\Theta)} & \cdots & \mathbf{H_{+2}(\Theta)} \\ & \cdots & & & \cdots & & \\ \mathbf{H_{+t}(\Theta)} & \mathbf{H_{+t-1}(\Theta)} & \cdots & \mathbf{H_0(\Theta)} & \mathbf{H_{-1}(\Theta)} & \cdots & \mathbf{H_{-t}(\Theta)} \\ \mathbf{H_{-t}(\Theta)} & \mathbf{H_{+t}(\Theta)} & \cdots & \mathbf{H_{+1}(\Theta)} & \mathbf{H_0(\Theta)} & \cdots & \mathbf{H_{-t+1}(\Theta)} \\ & \cdots & & & \cdots & & \\ \mathbf{H_{-1}(\Theta)} & \mathbf{H_{-2}(\Theta)} & \cdots & \mathbf{H_{+t}(\Theta)} & \mathbf{H_{+t-1}(\Theta)} & \cdots & \mathbf{H_0(\Theta)} \end{bmatrix} \quad (23)$$

Inspecting equations 22 and 23, we note that **F** and **A** are block circulant, and each $\mathbf{H_i(\Theta)}$ and $\mathbf{H_i(U)}$ is also block circulant.

Another problem remains: how to guarantee that $\mathbf{F}^{-1}$ has the appropriate format of equation 20, so that the solution $\mathbf{A} = \mathbf{F}^{-1}$ (when $\mathbf{C_b} = \mathbf{0}$) can be transformed in a convolutor operator ? In fact, it is not a problem, because the inverse matrix of a block circulant matrix is also block circulant.

Consider the following example, using equations 22 and 23, where **U** is the template of equation 24 (90^0 corner).

$$\mathbf{U} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (24)$$

$$\mathbf{F} = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad (25)$$

$$\mathbf{A} = 0{,}25 \left[\begin{array}{ccc|ccc|ccc} 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ \hline 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ \hline -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \end{array}\right] \quad (26)$$

$$= \left[\begin{array}{ccc|ccc|ccc} \theta_{0,0} & \theta_{0,-1} & \theta_{0,+1} & \theta_{-1,0} & \theta_{-1,-1} & \theta_{-1,+1} & \theta_{+1,0} & \theta_{+1,-1} & \theta_{+1,+1} \\ \theta_{0,+1} & \theta_{0,0} & \theta_{0,-1} & \theta_{-1,+1} & \theta_{-1,0} & \theta_{-1,-1} & \theta_{+1,+1} & \theta_{+1,0} & \theta_{+1,-1} \\ \theta_{0,-1} & \theta_{0,+1} & \theta_{0,0} & \theta_{-1,-1} & \theta_{-1,+1} & \theta_{-1,0} & \theta_{+1,-1} & \theta_{+1,+1} & \theta_{+1,0} \\ \hline \theta_{+1,0} & \theta_{+1,-1} & \theta_{+1,+1} & \theta_{0,0} & \theta_{0,-1} & \theta_{0,+1} & \theta_{-1,0} & \theta_{-1,-1} & \theta_{-1,+1} \\ \theta_{+1,+1} & \theta_{+1,0} & \theta_{+1,-1} & \theta_{0,+1} & \theta_{0,0} & \theta_{0,-1} & \theta_{-1,+1} & \theta_{-1,0} & \theta_{-1,-1} \\ \theta_{+1,-1} & \theta_{+1,+1} & \theta_{+1,0} & \theta_{0,-1} & \theta_{0,+1} & \theta_{0,0} & \theta_{-1,-1} & \theta_{-1,+1} & \theta_{-1,0} \\ \hline \theta_{-1,0} & \theta_{-1,-1} & \theta_{-1,+1} & \theta_{+1,0} & \theta_{+1,-1} & \theta_{+1,+1} & \theta_{0,0} & \theta_{0,-1} & \theta_{0,+1} \\ \theta_{-1,+1} & \theta_{-1,0} & \theta_{-1,-1} & \theta_{+1,+1} & \theta_{+1,0} & \theta_{+1,-1} & \theta_{0,+1} & \theta_{0,0} & \theta_{0,-1} \\ \theta_{-1,-1} & \theta_{-1,+1} & \theta_{-1,0} & \theta_{+1,-1} & \theta_{+1,+1} & \theta_{+1,0} & \theta_{0,-1} & \theta_{0,+1} & \theta_{0,0} \end{array}\right] \quad (27)$$

With a fast analysis, we see that **A**, the inverse of **F**, has the form of a convolutor operator **Θ**, where

$$\mathbf{\Theta} = 0{,}25 \begin{bmatrix} -1 & -1 & +1 \\ +1 & +1 & -1 \\ +1 & +1 & -1 \end{bmatrix} \quad (28)$$

As a conference:

$$\Rightarrow \quad \mathbf{\Theta * U} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (29)$$

which offered a perfect impulse at the center (infinite $DSNR_2$).

V. Alternative Approach To The Impulse Restoration

With the formulation of the previous section, we find $\boldsymbol{\Theta}$ that maximizes the $DSNR_2$ for a given template. However, that computation does not avoid that another template offers a better $DSNR_2$, when filtered with $\boldsymbol{\Theta}$, when noise is present. This could cause some false detections.

A possible solution for this problem is to consider $DSNR_2$ as a function of $\boldsymbol{\Theta}$ and $\mathbf{U}$ and look for a Θ which maximizes the $DSNR_2$ when $\mathbf{U}$ varies. This is the same as the "alternative approach" of the discriminative filtering, described in [6], but modeled as an impulse restoration problem.

Expressing the maximization of the $DNSR_2$, as an impulse restoration problem, we have to minimize $E[\| \boldsymbol{\delta} - \hat{\boldsymbol{\delta}} \|^2]$, that is, the estimation mean square error. The solution for the alternative approach is found after determining the $\mathbf{A}$ matrix so that $E[\| \boldsymbol{\delta} - \hat{\boldsymbol{\delta}} \|^2]$, that is a function $\zeta(\underline{\mathbf{F}},\mathbf{A},\mathbf{C_b})$ ($\underline{\mathbf{F}}$ is the circulant matrix constructed for a generic template), is minimized when $\underline{\mathbf{F}} = \mathbf{F}$, where $\mathbf{F}$ is the template whose discrimination is desired.

$E[\| \boldsymbol{\delta} - \hat{\boldsymbol{\delta}} \|^2]$ can be expressed as:

$$
\begin{aligned}
& E[\| \boldsymbol{\delta} - \hat{\boldsymbol{\delta}} \|^2] = \zeta(\underline{\mathbf{F}},\mathbf{A},\mathbf{C_b}) = E\{ (\boldsymbol{\delta} - \mathbf{A}\boldsymbol{g})^t (\boldsymbol{\delta} - \mathbf{A}\boldsymbol{g}) \} = \\
& E\{ (\boldsymbol{\delta} - \mathbf{A}\underline{\mathbf{F}}\boldsymbol{\delta} - \mathbf{A}\boldsymbol{b})^t (\boldsymbol{\delta} - \mathbf{A}\underline{\mathbf{F}}\boldsymbol{\delta} - \mathbf{A}\boldsymbol{b}) \} = \\
& E\{ \boldsymbol{\delta}^t\boldsymbol{\delta} - \boldsymbol{\delta}^t\mathbf{A}\underline{\mathbf{F}}\boldsymbol{\delta} - \boldsymbol{\delta}^t\underline{\mathbf{F}}^t\mathbf{A}^t\boldsymbol{\delta} + \boldsymbol{\delta}^t\underline{\mathbf{F}}^t\mathbf{A}^t\mathbf{A}\underline{\mathbf{F}}\boldsymbol{\delta} + \boldsymbol{b}^t\mathbf{A}^t\mathbf{A}\boldsymbol{b} \} = \\
& E\{ \boldsymbol{\delta}^t\boldsymbol{\delta} + \boldsymbol{b}^t\mathbf{A}^t\mathbf{A}\boldsymbol{b} \} + \\
& \qquad E\{ \boldsymbol{\delta}^t (\underline{\mathbf{F}}^t\mathbf{A}^t\mathbf{A}\underline{\mathbf{F}} - \mathbf{A}\underline{\mathbf{F}} - \underline{\mathbf{F}}^t\mathbf{A}^t)\boldsymbol{\delta} \} \qquad (30)
\end{aligned}
$$

As we can see, only the second term of the above result depends on $\underline{\mathbf{F}}$. It is necessary that its minimum occurs when $\underline{\mathbf{F}} = \mathbf{F}$. We then rewrite this term as:

$$
\begin{aligned}
& E\{ \boldsymbol{\delta}^t (\underline{\mathbf{F}}^t\mathbf{A}^t\mathbf{A}\underline{\mathbf{F}} - \mathbf{A}\underline{\mathbf{F}} - \underline{\mathbf{F}}^t\mathbf{A}^t)\boldsymbol{\delta} \} = \\
& E\{ \boldsymbol{\delta}^t (\mathbf{I} - \mathbf{A}\underline{\mathbf{F}})^t (\mathbf{I} - \mathbf{A}\underline{\mathbf{F}})\boldsymbol{\delta} \} - 1 \qquad (31)
\end{aligned}
$$

Since,

$$
\begin{aligned}
& E\{ \boldsymbol{\delta}^t (\mathbf{I} - \mathbf{A}\underline{\mathbf{F}})^t (\mathbf{I} - \mathbf{A}\underline{\mathbf{F}})\boldsymbol{\delta} \} = \\
& E\{ \| (\mathbf{I} - \mathbf{A}\underline{\mathbf{F}})\boldsymbol{\delta} \|^2 \} \geq 0 \quad , \qquad (32)
\end{aligned}
$$

the minimum error is obtained when inequation 32 becomes zero for $\underline{\mathbf{F}} = \mathbf{F}$. For this case, it is:

$$\mathbf{A} = \mathbf{F}^{-1} \quad . \qquad (33)$$

An interesting characteristic of this result is that the linear estimator obtained with the alternative approach does not depend on any noise statistics.

VI. A Mixed Approach

Another alternative is to mix the formulations of the two previous sections, as made in [6].

The mixed approach weights two terms when we look for the operator $\mathbf{A}$. The first term ($E[\| \boldsymbol{\delta} - \hat{\boldsymbol{\delta}} \|^2]$) is such that the smaller it is, the larger $DSNR_2$. The second one ($E\{ \| (\mathbf{I} - \mathbf{A}\underline{\mathbf{F}})\boldsymbol{\delta} \|^2 \}$) is such that the smaller it is, the closest the solution is to the alternative solution.

Equation 34 shows how these terms can be weighed by a constant K ($0 \leq K \leq 1$):

$$
\begin{aligned}
& DF(\mathbf{A}) = \\
& (1-K)\, E[\| \boldsymbol{\delta} - \hat{\boldsymbol{\delta}} \|^2] + K\, E[\| (\mathbf{I} - \mathbf{AF})\boldsymbol{\delta} \|^2] \\
& = (1-K)\, E[\| \boldsymbol{\delta} - \mathbf{AF}\boldsymbol{\delta} - \mathbf{A}\boldsymbol{b} \|^2] + K\, E[\| (\mathbf{I} - \mathbf{AF})\boldsymbol{\delta} \|^2] \quad (34)
\end{aligned}
$$

By hypothesis, the impulse $\boldsymbol{\delta}$ and the noise $\boldsymbol{b}$ are uncorrelated, so the equation 34 can be rewritten using the following form:

$$
\begin{aligned}
& DF(\mathbf{A}) = \\
& \quad (1-K)\, E[\| \boldsymbol{\delta} - \mathbf{AF}\boldsymbol{\delta} - \mathbf{A}\boldsymbol{b} \|^2] + K\, E[\| (\mathbf{I} - \mathbf{AF})\boldsymbol{\delta} \|^2] \\
& \quad = (1-K)\, E[\| \boldsymbol{\delta} - \mathbf{AF}\boldsymbol{\delta} \|^2] + (1-K)\, E[\| \mathbf{A}\boldsymbol{b} \|^2] + \\
& \qquad K\, E[\| (\mathbf{I} - \mathbf{AF})\boldsymbol{\delta} \|^2] \\
& \quad = (1-K)\, E[\| (\mathbf{I} - \mathbf{AF})\boldsymbol{\delta} \|^2] + (1-K)\, E[\| \mathbf{A}\boldsymbol{b} \|^2] + \\
& \qquad K\, E[\| (\mathbf{I} - \mathbf{AF})\boldsymbol{\delta} \|^2] \\
& \quad = E[\| (\mathbf{I} - \mathbf{AF})\boldsymbol{\delta} \|^2] + (1-K)\, E[\| \mathbf{A}\boldsymbol{b} \|^2] \\
& \qquad \text{(because } \boldsymbol{\delta} \text{ and } \boldsymbol{b} \text{ are uncorrelated)} \\
& \quad = E[\| \boldsymbol{\delta} - \mathbf{AF}\boldsymbol{\delta} - \mathbf{A}\{(1-K)^{1/2}\boldsymbol{b}\} \|^2] \\
& \quad = \zeta(\mathbf{F},\mathbf{A},\{1-K\}\mathbf{C_b}). \qquad (35)
\end{aligned}
$$

The result of equation 35 is very significant, because it says that the mixed solution is obtained using the same solution of the one of section 4, changing $\mathbf{C_b}$ by (1-K) $\mathbf{C_b}$. Then, the value of $\mathbf{A}$, using the mixed approach, is:

$$\mathbf{A} = \mathbf{F}^t (\mathbf{F}\mathbf{F}^t + \{1-K\}\mathbf{C_b})^{-1}, \qquad (36)$$

or

$$\mathbf{A} = (\mathbf{F}^t\mathbf{C_b}^{-1}\mathbf{F} + \{1-K\}\mathbf{I})^{-1}\mathbf{F}^t\mathbf{C_b}^{-1} . \qquad (37)$$

Note that, for K = 0, equation 36 is equivalent to equation 9 and, for K = 1, it is equivalent to equation 33.

VII. Simulations

To simulate the algorithm used for the discriminative filter computation, we used the same detection scheme as the one presented in [6]. The chosen templates were 7x7 45^0, 90^0 and 135^0 corners. Figure 1 shows some results presented in references [6] and [2]. Figure 2 shows the simulations obtained with the circular filters computed using the method described in section 5

The computation of the discriminative filters [6] requires the use of some numerical optimization, while the method presented here is analytical. Then, it can be concluded that the computation of the filter using the method described in this paper is more accurate. This can be confirmed after analyzing figures 1 and 2. We can see that the corners detected by the method of this paper were well discriminated and there was a small number of false detections, unlike of the original method.

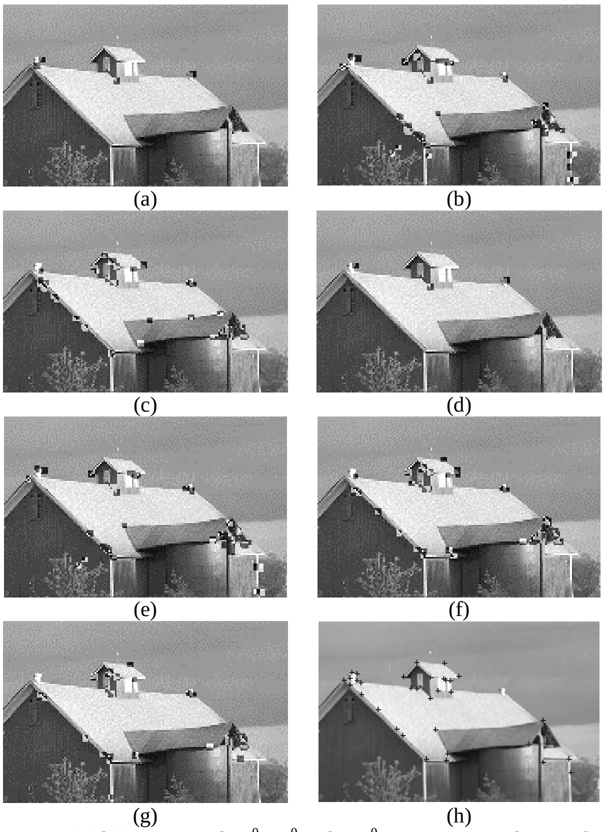

Figure 1: (a)(b)(c) Detected 90^0, 45^0 and 135^0 corners (equivalent method of section 4). (d)(e)(f)(g) Detected corners in mixed method. (h) Results of [2].

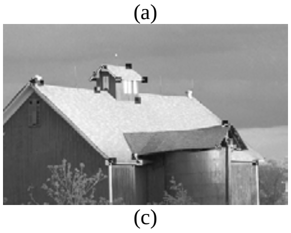

Figure 2: (a)(b)(c) Detected 90^0, 45^0 and 135^0 corners (method of section 4).

As a comparing factor, the threshold used for template detection (figure 2), that is the minimum required $DSNR_2$, is 0.5, while the threshold in [6] was 0.09. It is a very significant improvement.

One difficulty of the method proposed in this paper is that the method requires that **F** be invertible and, in some cases **F** may be ill conditioned. For example, the 90^0 and 135^0 corners do not cause any problems and were constructed as in equation 24 (using a 7x7 format). However, this was not the case for the 45^0 corner, that generates ill conditioned matrices. This problem can be solved by shifting it by one pixel relative to its center.

VIII. Conclusions

In this paper, we present discriminative filtering modeled as an impulse restoration problem. The main objective of this proposed method is to obtain a two-dimensional filter that, when convoluted with the image template, generates as output an image with the energy concerned in only one sample. The advantage of the proposed method in relation to the references [6,2] is that it does not need any optimization algorithm for the filter computation. The simulations show that the method works very well with real images.

IX. References

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