The Rounded Hartley Transform

R. J. de Sobral Cintra, H. M. de Oliveira and C. O. Cintra

Abstract—**A new multiplication-free transformation derived from DHT is introduced: the RHT. Investigations on the properties of the RHT led us to the identification of an approximate inverse. We showed that RHT is not involutional like the DHT, but it exhibits approximate involutional property. Thus instead of using the actual inverse transform, the RHT is taken as an involutional transformation, allowing the use of direct (multiplication-free) to evaluate the inverse. A fast algorithm to compute RHT is presented, which offers zero multiplications and $\mathcal{O}(n \log n)$ additions for power-of-2 blocklengths. This algorithm shows embedding properties. We also extended RHT to the two-dimensional case. This permitted us to perform a preliminary analysis on the effects of RHT on images. Despite of some SNR loss, RHT can be interesting for applications which involve image monitoring associated to decision making, such as military applications or medical imaging.**

I. INTRODUCTION

DISCRETE transforms have a significant role in digital signal processing. A relevant example is the discrete Hartley transform (DHT), which offers many advantages over the more popular discrete Fourier transform. To cite major advantages, (i) DHT is a real-valued transform (no complex arithmetic is needed), (ii) it possesses same formula for forward and inverse transform, (iii) it has a computational equivalence to DFT [1], (iv) DHT shows high symmetry, which is desirable from the implementation point-of-view, and (v) it is mathematically elegant. These characteristics have motivated a lot of research to promote the use of DHT instead of DFT. Thus DHT had hit many applications such as spectral analysis, convolution computation, adaptive filters, interpolation, communication systems and medical imaging [2]. A representative reference list with the literature about the Hartley transform is found in [3].

Another important area of signal processing concerns with the minimal complexity methods. The class of multiplication-free discrete transforms, such as Walsh/Hadamard transform, has attracted much interest, since those transforms provide low computational complexity. The multiplication-free paradigm was adopted by Reed *et alli* in the implementation of the arithmetic Fourier transform [4]. Recently another algorithm of this kind was proposed: the arithmetic Hartley transform [5]. An interesting approach was done by Bhatnagar: using Ramanujan numbers, another multiplication-free transform was invented [6]. Approximation procedures are also being taken in consideration. In a recent paper [7], Dee-Jeoti proposed the approximate fast Hartley transform, though the multiplicative complexity of this procedure is not null.

Seeking for new procedures with the multiplication-free property, we introduced in this paper a new transformation: the rounded Hartley transform (RHT), a transformation with zero multiplicative complexity. Figure 1 places the RHT among other transforms.

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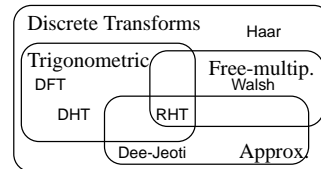

Fig. 1. Discrete transforms and some of their classes. Rounded Hartley transform is placed in the intersection of many classes.

In section II and III, we define the RHT and discuss the philosophy behind its constructs: an approximate inverse. Section IV brings a first approach to devise a fast algorithm for RHT, using the theoretical background found in [8]. We explored a naive example, 16-RHT, and derived arithmetic complexity bounds. Subsequently in section V, the two-dimensional case was analyzed by introducing the 2-D RHT. The effects of 2-D RHT on standard classical images were investigated, particularly we calculated peak signal-noise ratio. We ended this paper establishing a connection between the new RHT and the Walsh/Hadamard transform.

II. THE ROUNDED HARTLEY TRANSFORM

Let $\mathbf{v}$ be an n-dimensional vector with real elements. The discrete Hartley transform establishes a pair of signal vectors $\mathbf{v} \xleftrightarrow{\mathcal{H}} \mathbf{V}$, where the elements of $\mathbf{V}$ are defined by

$$V_k \triangleq \sum_{i=0}^{n-1} v_i \operatorname{cas}\left(\frac{2\pi i k}{n}\right), \quad k = 0, 1, \ldots, n-1, \quad (1)$$

where $\operatorname{cas}(x) \triangleq \cos(x) + \sin(x)$. This transform leads to the definition of Hartley matrix $\mathbf{H}_n$, which elements are on the form $h_{i,k} = \operatorname{cas}\left(\frac{2\pi i k}{n}\right)$.

The roundoff of a matrix is obtained by rounding off its elements. Thus the rounded Hartley matrix elements $\mathsf{h}_{i,k}$ are defined by

$$\mathsf{h}_{i,k} \triangleq \left[\underbrace{\operatorname{cas}\left(\frac{2\pi i k}{n}\right)}_{h_{i,k}}\right], \quad i, k = 0, 1, \ldots, n-1, \quad (2)$$

where $[\cdot]$ denotes the nearest integer function. For the sake of notation, let us denote the rounded Hartley matrix of order n by $\mathbf{H}_n$.

It is easy to see that the elements $\mathsf{h}_{i,k}$ belong to $\{-1, 0, 1\}$, since $|\operatorname{cas}(x)| \leq \sqrt{2}$. Consequently, the rounded Hartley transform can be implemented using only additions, regardless the blocklength. Rounded Hartley transform is a multiplication-free transform, which can be very attractive from the practical point of view.

The first questions to be answered are: (i) Is the spectrum derived from RHT a good estimation of the (true) Hartley spectrum? (ii) Is there an inverse Hartley transform?
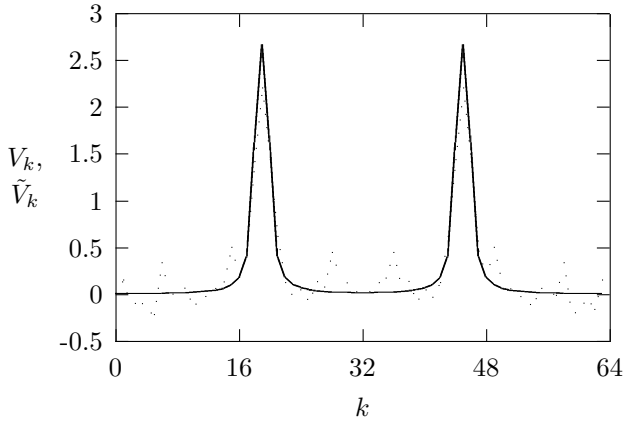
Fig. 2. A Simple example. Hartley spectrum evaluated by discrete Hartley transform **V** (filled line —) and rounded Hartley transform V (dotted line $\cdots$) of a vector **v** with 64 samples of function $f(x) = \cos(90\pi x)(x - \frac{1}{2})^2$, $0 \leq x \leq 1$.

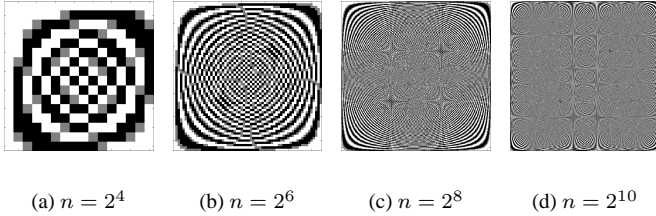| (a) $n = 2^4$ | (b) $n = 2^6$ | (c) $n = 2^8$ | (d) $n = 2^{10}$ |

Fig. 3. Image patterns derived from rounded Hartley matrix for orders $n = 16, 64, 256, 1024$. Each matrix **H** is converted into intensity diagrams by representing their elements in a gray scale. Remark the presence of embedding patterns.

To begin with, we investigated the DHT and the RHT spectra for a few simple signals (HT has real-valued components). Figure 2 shows both spectra for the signal $f(x) = \cos(90\pi x)(x - \frac{1}{2})^2$ sampled by 64 points. A pretty good agreement was observed. A careful analysis of the error, or at least an upper bound, is currently being investigated.

III. APPROXIMATE INVERSE

In order to gain some insight on rounded Hartley transform matrices, intensity diagrams were generated. The value $h_{i,k}$ of each element of $\mathbf{H}_n$ is converted into a gray-scale colormap and the matrix $\mathbf{H}_n$ is then represented by a square with n^2 pixels. Some interesting patterns derived from RHT are show in Figure 3.

To keep rigorous with Hartley-Bracewell definition of the Hartley transform, in this section, the Hartley matrix $\mathbf{H}_n$ and the rounded Hartley transform $\mathbf{H}_n$ are scaled by $1/\sqrt{n}$. Without any kind of conceptual loss, this scaling does not interfere with the results hereafter derived and brings a greater elegance and harmony to the following constructs.

One of the most appealing properties of the classical DHT is the fact that discrete Hartley matrix $\mathbf{H}_n$ is an involution, which implies $\mathbf{H}_n^{-1} = \mathbf{H}_n$ (self-inverse). However, after the round operation, $\text{cas}(\cdot)$ kernel loses this characteristic and RHT is not an involution, since $\mathbf{H}_n^{-1} \neq \mathbf{H}_n$.

We found out by explicit computation that the inverse of $\mathbf{H}_n$ does exist for order $n \leq 1024$. Unfortunately $\mathbf{H}_n^{-1}$ is not as in-
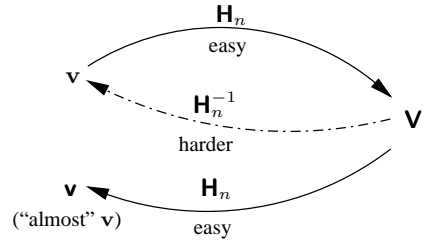

Fig. 4. Approximate inverse. This diagram show the main idea behind our work. We are concerned with matrices that "almost" invert. Filled arrows represent low computational complexity, while the dashed one stands for a higher computational complexity.

teresting as $\mathbf{H}_n$, since it is computationally more intensive. This fact was the key point that led us to a greater concern on inverse matrices. We are particularly interested in finding out matrices which have the properties of (i) being *almost* the inverse of a given matrix and (ii) being computationally more interesting than the actual inverse.

That is, given a matrix $\mathbf{A}$ we are looking for a matrix $\tilde{\mathbf{A}}$, such as:

$$\mathbf{A} \cdot \tilde{\mathbf{A}} \approx \mathbf{I}, \tag{3}$$

where $\mathbf{I}$ is the identity matrix. This is called an *approximate inverse*.

After further examination on $\mathbf{H}_n^{-1}$, we observed that it resembles $\mathbf{H}_n$ itself. In fact, $\mathbf{H}_n^{-1}$ is *almost* $\mathbf{H}_n$. Since $\mathbf{H}_n$ is defined from $\mathbf{H}_n$, we verify that $\mathbf{H}_n$ is, in some sense, *almost* involutional The qualitative idea was exposed, Figure 4 may elucidate it.

A. An Approximate Inverse of Rounded Hartley Matrix

Definition 1 (Matrix Period) *The period of a matrix* $\mathbf{A}$ *is the smallest positive integer k such that* $\mathbf{A}^{k+1} = \mathbf{A}$. $\qquad\square$

For example, an idempotent transformation $\mathbf{A}$ satisfies $\mathbf{A}^2 = \mathbf{A}$, since it has period $k = 1$. A linear transformation which has period $k = 2$ is an involution.

Definition 2 *The n-norm of a matrix* $\mathbf{A}$ *is defined by*

$$\bar{\mu}(\mathbf{A}) = \frac{\|\mathbf{A}\|}{n}, \tag{4}$$

where n is the order of $\mathbf{A}$ *and* $\| \cdot \|$ *represents Frobenius norm of a matrix:*

$$\|\mathbf{A}\| = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{i,j}|^2 \right)^{1/2},$$

where $a_{i,j}$ are the elements of matrix $\mathbf{A}$. $\qquad\square$

Now let us examine the rounded Hartley matrix $\mathbf{H}_n$. Evaluating the n-norm of $\mathbf{H}_n^2 - \mathbf{I}_n$ for $n = 2, 3, \ldots, 1024$, one can plot the graph depicted in Figure 5. After a data analysis of these points, we fitted them to a Freundlich model:

$$\bar{\mu}\left(\mathbf{H}_n^2 - \mathbf{I}_n\right) \approx an^b, \tag{5}$$
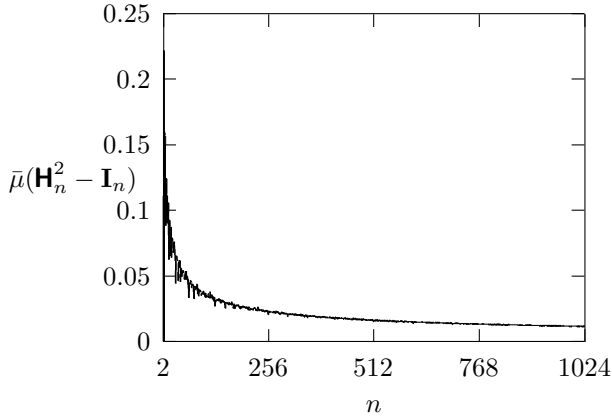
where $a \approx 0.35167$ and $b \approx -0.49324$.

Fig. 5. The n-norm of $\left(\mathbf{H}_n^2 - \mathbf{I}_n\right)$ for $n = 2, 3, \ldots, 1024$.

These observations allowed us to infer on the asymptotical behavior of $\bar{\mu}\left(\mathbf{H}_n^2 - \mathbf{I}_n\right)$ and state the following conjecture:

Conjecture 1

$$\lim_{n \to \infty} \bar{\mu}\left(\mathbf{H}_n^2 - \mathbf{I}_n\right) = 0. \tag{6}$$

$\square$

An interpretation of this conjecture is the following. In a sense, the "distance" between $\mathbf{H}_n^2$ and $\mathbf{I}_n$ is decreasing as we have larger blocklengths n.

Definition 3 *In the scope of this work, an approximate involution will be defined as a transformation* $\mathbf{A}$ *which satisfies*

$$\mathbf{A}^2 \approx \mathbf{I}. \tag{7}$$

Alternatively, it can be viewed as a transformation with approximate period of 2. $\square$

Thus we can stated that the rounded Hartley matrices are approximate involutions.

B. General Comments

In this subsection, we state some initial observations about the rounded Hartley transform without further derivations or proofs.

B.1 Error

Since an approximate inverse is not precisely the inverse of a given matrix, this approach of retrieving data from an approximate inverse introduces some degradation, as expected. The RHT is given by $\mathbf{V} = \mathbf{H}_n \mathbf{v}$. We shall use $\mathbf{v} = \mathbf{H}_n \mathbf{V} = \mathbf{H}_n^2 \mathbf{v}$ to compute the inverse, instead of the exact inverse RHT $\mathbf{v} = \mathbf{H}_n^{-1} \mathbf{V}$. Thus this procedure introduces an error by the use of the approximate inverse. The error is therefore

$$\mathbf{v} - \mathbf{v} = \left(\mathbf{H}_n^2 - \mathbf{I}_n\right) \mathbf{v}. \tag{8}$$

As we see, the error $\mathbf{v} - \mathbf{v}$ depends on $\mathbf{H}_n^2 - \mathbf{I}_n$, as well as on the original message $\mathbf{v}$.

B.2 Fractal

Since the measure $\bar{\mu}\left(\mathbf{H}_n^2 - \mathbf{I}_n\right)$ presents a fractional exponent (Equation 5), objects $\mathbf{H}_n^2 - \mathbf{I}_n$ could be associated with some fractal. The patterns displayed in Figure 6 show a kind of self-similar behavior, as expected.
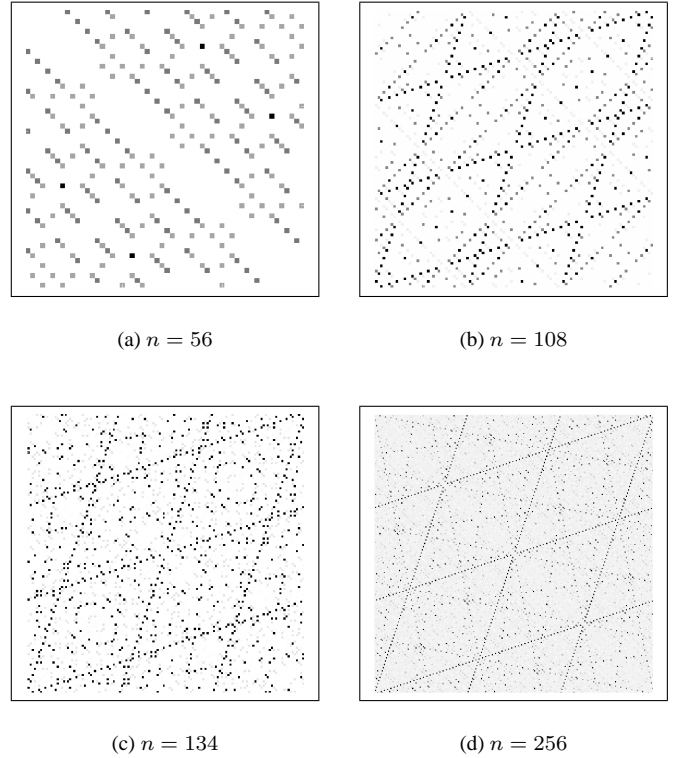

(a) $n = 56$

(b) $n = 108$

(c) $n = 134$

(d) $n = 256$

Fig. 6. Some interesting pictorial matrix patterns for $\mathbf{H}_n^2$, $n = 56, 108, 134$ and 256. A gray scale is used to plot the intensity of the elements: the darker the element, the greater its magnitude (white denotes zeroes). Main diagonal omitted for better visualization, since the magnitude of the diagonal elements is much greater than the other elements'.

IV. A FAST ALGORITHM FOR RHT

In order to derive a fast algorithm, we use a naive example: 16-point RHT, which transform matrix $\mathbf{H}_{16}$ is shown below:

$$\mathbf{H}_{16} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & 1 \\ 1 & 1 & 1 & & - & - & - & & 1 & 1 & 1 & & - & - & - & \\ 1 & 1 & & - & - & 1 & 1 & 1 & - & - & & 1 & 1 & - & - & \\ 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & 1 & - & 1 & 1 & - & & 1 & - & - & 1 & & - & 1 & & - \\ 1 & & - & 1 & - & & 1 & - & 1 & & - & 1 & - & & 1 & - \\ 1 & - & & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & & - & 1 \\ 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & - & 1 & - & 1 & & - & 1 & - & 1 & - & 1 & - & 1 & & - \\ 1 & - & 1 & & - & 1 & & - & 1 & - & & - & 1 & 1 & - & 1 \\ 1 & - & & 1 & - & & 1 & - & 1 & & - & 1 & & - & 1 & \\ 1 & - & & 1 & 1 & - & - & 1 & 1 & - & & 1 & 1 & - & - & 1 \\ 1 & - & - & & 1 & 1 & & - & 1 & 1 & - & - & & 1 & 1 & \\ 1 & - & - & - & 1 & 1 & 1 & & - & - & - & & 1 & 1 & 1 & \\ 1 & 1 & & & - & - & - & - & - & - & & & 1 & 1 & 1 & 1 \end{bmatrix},$$

where "$-$" represents -1 and blank spaces are zeroes.

Using methods described in [9], [8], we obtained the implementation diagram displayed in Figure 7.

The algorithm turns out to have embedding properties: shorter transforms are found in major ones. In the 16-point RHT, the following transforms are embedded: 2-, 4- and 8-point RHT. By zeroing some inputs, a shorter transform is promptly available (e.g. let $v_8 = \cdots = v_{15} = 0$ to have an 8-point RHT). This feature makes the algorithm particularly flexible to a much larger range of applicabilities [9], [8], [10].

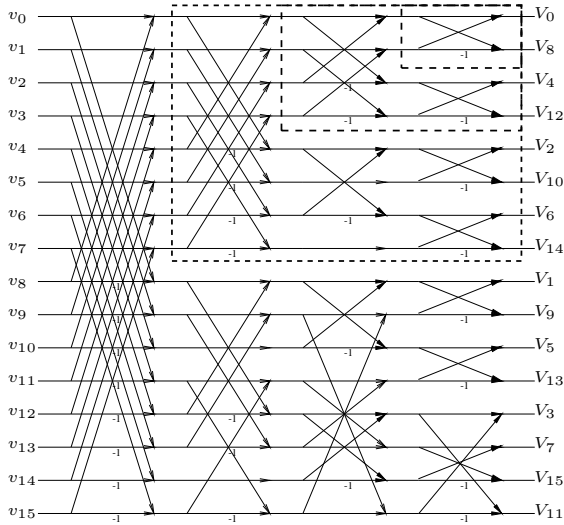For blocklengths which are power of two, one can find out the

Fig. 7. Flow graph for the fast algorithm of 16-point rounded Hartley transform. Note the complete absence of multipliers. The dashed boxes denote shorter transforms embedded in 16-point RHT.

the following arithmetic complexity:

$$A(n) = \mathcal{O}(n \log_2 n), \tag{9}$$

$$M(n) = 0, \tag{10}$$

where $\mathcal{O}(\cdot)$ is the Landau symbol.

V. 2-D ROUNDED HARTLEY TRANSFORM

Original two-dimensional Hartley transform of an $n \times n$ image is defined by

$$b_{u,v} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} a_{i,j} \cdot \mathrm{cas}\left(\frac{ui + vj}{n}\right), \tag{11}$$

where $a_{i,j}$ are the elements of an image $\mathbf{A}$ and $b_{u,v}$ are the elements of the Hartley transform of $\mathbf{A}$.

Since $\mathrm{cas}(\cdot)$ kernel is not separable, we cannot express the two-dimensional transform in terms of a single matrix equation, like the 2-D discrete Fourier transform. Thus, we defined the two-dimensional rounded Hartley transform similarly to Bracewell's method for two-dimensional discrete Hartley transform [11].

Let $\mathbf{A}$ be the $n \times n$ image matrix. We start the procedure by calculating a temporary matrix T, as follows:

$$\mathsf{T} = \mathbf{H}_n \cdot \mathbf{A} \cdot \mathbf{H}_n, \tag{12}$$

where $\mathbf{H}_n$ is the rounded Hartley matrix of order n. This is equivalent to take one-dimensional Hartley transform of the rows, and then transform the columns [12].

Establishing that the elements of $\mathbf{T}$ are represented on the form $\mathsf{t}_{i,j}$, $i, j = 0, \ldots, n-1$, we can consider three new matrices built from the temporary matrix T: $\mathbf{T}^{(c)}$, $\mathbf{T}^{(l)}$ and $\mathbf{T}^{(cl)}$ which elements are $\mathsf{t}_{i,n-j \pmod n}$, $\mathsf{t}_{n-i \pmod n,j}$, $\mathsf{t}_{n-i \pmod n,n-j \pmod n}$, respectively. These different indexes flip matrix $\mathbf{T}$ in left-right direction, except from the first column ($\mathbf{T}^{(c)}$); in up-down direction, except from the first line ($\mathbf{T}^{(l)}$); and both operations at same time ($\mathbf{T}^{(cl)}$).

Using these constructs, the rounded Hartley transform $\mathbf{B}$ of an $n \times n$ image $\mathbf{A}$ is defined as

$$\mathbf{B} \triangleq \mathbf{T} + \mathbf{T}^{(c)} + \mathbf{T}^{(l)} - \mathbf{T}^{(cl)}. \tag{13}$$

This definition derives directly from the $\mathrm{cas}(\cdot)$ property: $\mathrm{cas}(a+b) = \mathrm{cas}(a)\,\mathrm{cas}(b) + \mathrm{cas}(a)\,\mathrm{cas}(-b) + \mathrm{cas}(-a)\,\mathrm{cas}(b) - \mathrm{cas}(-a)\,\mathrm{cas}(-b)$.

Program 1 A simple MATLAB program to compute 2-D RHT, its approximate inverse and the PSNR.

```
function Z = rcas(N)

i = 0:(N-1);
j = 0:(N-1);
[I,J] = meshgrid(i,j);
Z = round ( cas ( 2 * pi / N * I .* J) );
```

```
function [B, AA, PSNR] = twodrht(file)

A = imread(file ,'bmp');
A = double(A);
[M, N] = size(A);
if M ~= N end;
colormap(gray(256));
K = rcas(N);
TEMP = K * A * K;
TEMPFLIPCOL = [TEMP(:,1),fliplr(TEMP(:,2:N))];
TEMPFLIPROW = [TEMP(1,:);flipud(TEMP(2:N,:))];
TEMPFLIPRC = [TEMPFLIPCOL(1,:);flipud(TEMPFLIPCOL(2:N,:))];
B = (1/2)*(TEMP+TEMPFLIPCOL+TEMPFLIPROW-TEMPFLIPRC);
temp = (1/N) * (1/N) * K * B * K;
tempFLIPCOL = [temp(:,1),fliplr(temp(:,2:N))];
tempFLIPROW = [temp(1,:);flipud(temp(2:N,:))];
tempFLIPRC = [tempFLIPCOL(1,:);flipud(tempFLIPCOL(2:N,:))];
AA  = (1/2)*(temp+tempFLIPCOL+tempFLIPROW-tempFLIPRC);
MSE = (1/N^2) * sum(sum((AA-A).^2));
RMSE = sqrt(MSE);
PSNR = 20 * log10(255/RMSE);
```

Aiming to investigate such degradation (which follows from the use of the approximate inverse), standard images from Signal and Image Processing Institute Image Database [13] at University of Southern California were analyzed. Figures 8 and 9 present original images and their respective recovered images using the approximate inverse transform instead of the (exact) inverse transform. Program 1 lists a naive implementation of 2-D RHT using MATLAB.

Table I brings PSNR (Peak Signal-Noise Ratio) of the standard images after a direct RHT and an approximate inverse RHT. Observe that the PSNR is image dependent: the quantization noise due to the rounding depends on the original image characteristics, such as shape, contrast, dimension etc.

## VI. CONNECTION WITH FOURIER AND WALSH/HADAMARD TRANSFORM

As final comments on RHT, we present some relationship between this new transform and other well-known transforms such as Fourier and Walsh/Hadamard transforms.

Since DHT can be used to compute de DFT [10], and the RHT furnishes a estimate for DHT, we can use RHT to derive a rough — but fast — evaluation of Fourier spectrum.
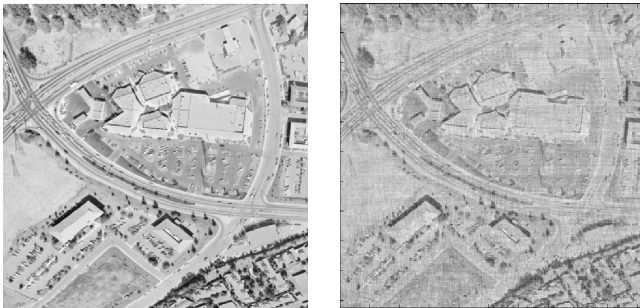
de Oliveira and co-workers [14] found a relationship between discrete Hartley transform and Hadamard transform. Such link was exploited to derive new fast algorithms [8], [14], [15], [16]. In the present framework, we are led to following conjecture:
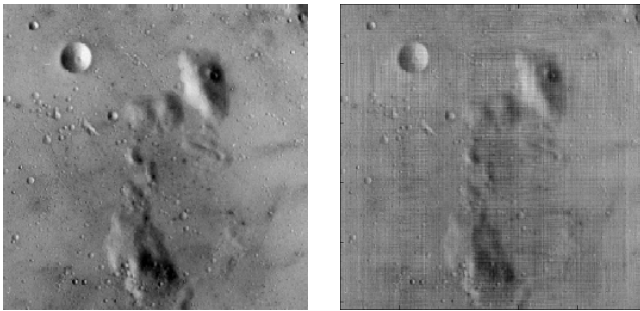
(a) Moon surface
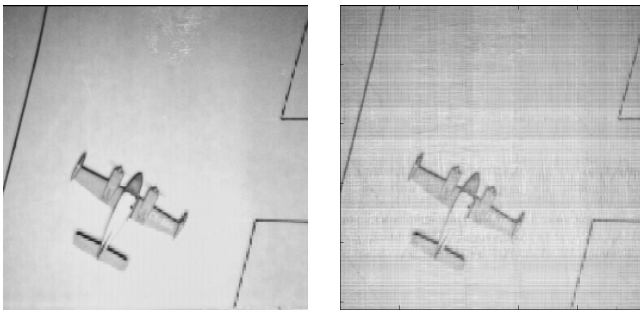
(b) Airplane

(c) Aerial

Fig. 8. Direct and Approximate inverse Transform. The original pictures displayed on left were direct and approximate inverse transformed via the rounded Hartley transform. Resulting images are seen on the right. Since RHT is an approximate (and not an exact) involution, it introduces noise due to its nature.
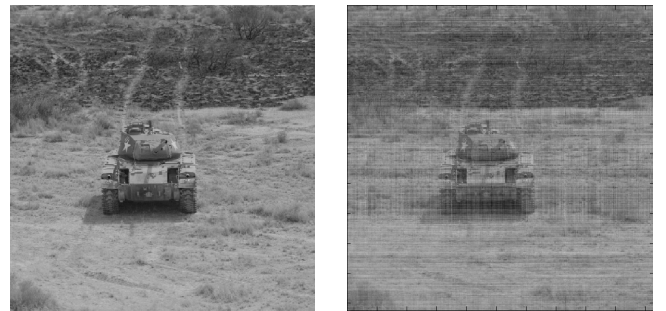
TABLE I

PSNR OF SOME STANDARD IMAGES AFTER A DIRECT AND APPROXIMATE INVERSE ROUNDED HARTLEY TRANSFORM. ALL IMAGES WERE OBTAINED FROM USC-SIPI IMAGE DATABASE. IN PARENTHESIS, IMAGE ID NUMBER.

Image	Dimension (pixels)	PSNR (dB)
Moon surface (`5.1.09`)	256×256	26.5522
Airplane (`5.1.11`)	256×256	25.7277
Aerial (`5.2.09`	512×512	22.2006
APC (`7.1.08`)	512×512	27.3035
Tank (`7.1.09`)	512×512	24.4590
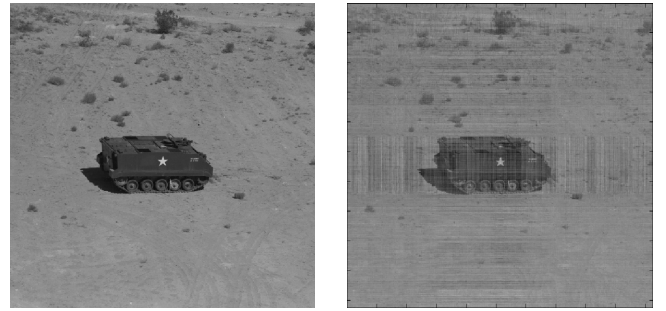
(a) APC

(b) Tank

Fig. 9. Direct and Approximate inverse Transform. Military images.

Conjecture 2 *Let n be a power of 2. The matrix $\mathbf{H}_n$ is identical to Walsh/Hadamard matrix of same order, except for null elements and for a permutation of columns.* ☐

For example, the column permutation that, except for zero elements, converts an 8-point rounded Hartley matrix into a Walsh transformation is $(4\ 8)$ (cyclic notation).

VII. CONCLUSIONS

Discrete Hartley transform has long been used in practical applications. It is real-valued self-inverse transform, more symmetrical than the DFT [10].

A new multiplication-free transform derived from DHT is introduced, the RHT, which kept many properties of discrete Hartley transform. In spite of not being involutional, it is shown that RHT exhibits approximate involutional property, a new concept derived from the periodicity of matrices.

The approximate involutional property was induced from approximate inverse definition. Instead of using the (true) inverse transformation, the RHT is viewed as an involutional transformation, allowing the use of direct (multiplication-free) to evaluate the inverse. Thus, the software/hardware to be used in the computation of both the direct and the inverse RHT becomes exactly the same. The price to be paid by not using the exact inverse transform is some degradation when recovering original signal.

Fast algorithms to compute RHT are presented showing embedded properties. The 2-D RHT is also defined, allowing to analyze the effects of this approach on standard images. Despite of SNR loss, RHT can be very interesting for applications in-

volving image monitoring associated to decision making, such as military applications or medical imaging.

RHT is offered as an efficient way to compute real-time initial estimations of spectral evaluations. Refinement algorithms can be used to improve the image or spectral estimation, when necessary. A class of refinement algorithms for this particularly transform is now our object of investigation.

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