A polynomial approach to the blind multichannel deconvolution problem

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Abstract— **We propose an algebraic approach to the blind single-input multiple-outputs deconvolution problem. The approach uses an input-output system identification, and then solves directly the Bezout equation which underlies the deconvolution problem. Important enough, the proposed approach allows one to solve the Bezout identity based on the single knowledge of the channel's output and, this holds true even in a noisy setting. The system identification step admits a straightforward interpretation in terms of spatial and temporal linear prediction. An important feature of the approach is that the number of nonzero terms in the channel-equalizer combined response does not exceed the difference between the channel and the equalizer orders. For the overmodeled case this combined response vanishes. Moreover, a method to blindly identify the channel as the minimum polynomial basis of a certain subspace may be deduced from this approach. A link with subspace and least squares methods is thus straightforward.**

I. Introduction

Let $\{s(n)\}$, an unobservable sequence of symbols emitted by a digital source, be sent through a single-input, L-outputs communication channel as depicted in Figure 1. The observed signal $Y(n) \in \mathbb{C}^L$, is modelled as

$$\underbrace{\begin{bmatrix} y_1(n) \\ \vdots \\ y_L(n) \end{bmatrix}}_{Y(n)} = \underbrace{\begin{bmatrix} h_1(z) \\ \vdots \\ h_L(z) \end{bmatrix}}_{H(z)} s(n) + \underbrace{\begin{bmatrix} w_1(n) \\ \vdots \\ w_L(n) \end{bmatrix}}_{W(n)}, \qquad (1)$$

where we interpret z as the unit delay operator: $zs(n) = s(n-1)$. In this model, $H(z)$ represents the L dimensional vector-valued transfer function of the channel and, $W(n) \in \mathbb{C}^L$ is an additive noise. Each subchannel, $h_i(z)$, corresponds to the propagation path from the source to one of an array of $L > 1$ sensors at the receiver. The same model also arises from a fractional-space setting in which, the subchannels are virtual and represent the polyphase components of the over sampled channel, see [1] and references therein.

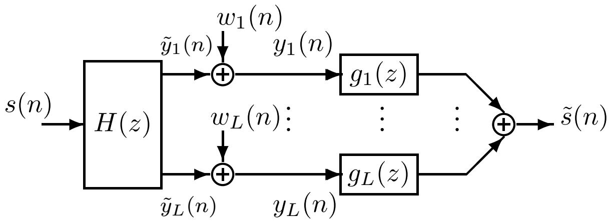

Fig. 1. Multichannel equivalent system model.

We are interested in the blind deconvolution/equalization of such a multichannel, *i.e.* the reconstruction of the source symbols $s(n)$, up to a scalar factor and an integer delay, from the sole observation sequence $\{Y(n)\}$. The purpose is to solve the following problem.

Problem 1: Given $\{Y(\cdot)\}$ a sequence of observations obtained from model (1), find an L-row vector-valued polynomial $G(z) = [g_1(z) \cdots g_L(z)]$ such that

$$r(z) \stackrel{\triangle}{=} G(z)H(z) = \sum_{\ell=1}^{L} g_\ell(z)h_\ell(z) = \alpha z^\nu, \qquad (2)$$

where α is a scalar scale factor and ν is an integer.

In the noiseless case ($W(n)$ is absent), any solution (if one exists) of Problem 1 yields a perfect reconstruction of a scaled and delayed version of the original source signal: $\tilde{s}(n) = \alpha s(n-\nu)$. In a noisy setting, however, a solution of this problem does not allows one to solve directly the corresponding equalization problem.

The channel may have a finite or infinite impulse response support. In the former case, the channel's transfer function is of the form

$$H(z) = \sum_{k=0}^{\kappa+1} H_k z^k, \; H_k \in \mathbb{C}^L.$$

Each subchannel transfer function, $h_i(z), i = 1, L$, is therefore a scalar polynomial of degree not exceeding $\kappa + 1$. Equation (2) is thus the well-known Bezout equation, which is known to be solvable provided the vector valued polynomial $H(z)$ is irreducible. The irreducibility of $H(z)$ means that the polynomials $h_i(z), i = 1, L$ are mutually prime (so-called no common zeros condition [2]).

In the sequel, we assume that $H(z)$ is an irreducible vector-valued polynomial of degree $\kappa+1$. In these settings, blind multichannel deconvolution/equalization reduces to a *blind* resolution of the Bezout equation as stated in Problem 1. The different approaches to the blind equalization problem may be split between the direct and the indirect equalization methods.

The basic principle of almost all direct methods is implicitly the search of a solution of Equation (2) as a global minimum/maximum of a high order statistics cost function, given in terms of the equalizer output. Examples of very relevant cost functions are given by the family of minimum entropy criteria [5] which, as shown in [7], [8], underlies the popular Godard algorithm [6] as well as the Shalvi-Weinstein [3] and the Super-exponential algorithms [4].

The starting point of the indirect methods is to observe that, if $H(z)$ were available then one could easily solve (2). Now, it has been shown that the second order statistics of the data are sufficient to identify the channel $H(z)$, under the so-called *no common zeros condition* (see [2]). Based on this statement,

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several second order blind identification methods have been developed. These include the subspace methods, the cyclostationarity based methods and the least squares methods (see [1] and the references therein).

The no common zeros condition, or equivalently the irreducibility of $H(z)$, is a generic condition for all second order methods, including subspace, cyclostationarity and least squares methods, all based on this observation. They proceed to first identify the channel and next equalize it by solving (2).

In this paper, we propose a polynomial approach to Problem 1. The approach uses, as an intermediary step, an adaptive input-output system identification of a certain polynomial matrix. This identification which may be interpreted as a spatial and temporal linear prediction problem, represents the cornerstone of our approach. Actually the polynomial $G(z)$, that solves the Bezout equation which underlies the deconvolution problem, is obtained in terms of this polynomial matrix. An estimation of the channel order (using *e.g.* SVD) which is generic to all second order methods is however necessary to solve (2). We show that the number of nonzero terms in the channel-equalizer combined response does not exceed the difference between the orders of the channel and $G(z)$. For the overmodeled case this combined response vanishes. The procedure proposed can be used for blind identification and/or blind equalization, and the Bezout equation can be solved with a minimal and/or a maximal delay. Finally, a connection with subspace and least squares methods is presented.

II. A POLYNOMIAL APPROACH

This section starts with the Bezout equation and describes different transformation steps towards the construction of a solution of Problem 1.

A. Preliminaries

To begin, consider $l+m$ polynomials

$$p_1(z), \ldots, p_l(z), q_1(z), \ldots q_m(z),$$

with complex coefficients. The set of the polynomials

$$\sum_{i=1}^{l} a_i(z)p_i(z) + \sum_{j=1}^{m} b_j(z)q_j(z)\ ;\ a_i(z), b_j(z) \in \mathbb{C}[z]$$

is a principal ideal of the ring $\mathbb{C}[z]$, of polynomials with complex coefficients. If the generators $p_i(z), i = 1, l$ and $q_j(z), j = 1, m$ are mutually prime (*i.e.* their greatest common divisor is one) then the principal ideal contains the identity polynomial and therefore, it coincides with the entire ring $\mathbb{C}[z]$. These elementary facts are known since Bezout. Now let us define

$$Q(z) \triangleq \begin{bmatrix} q_1(z) \\ \vdots \\ q_m(z) \end{bmatrix} \text{ and } P(z) \triangleq \begin{bmatrix} p_1(z) \\ \vdots \\ p_l(z) \end{bmatrix} \tag{3}$$

and assume that the vector valued polynomial $Q(z) \in \mathbb{C}^m[z]$ is irreducible. In this case, one may find a row vector valued polynomial $Q^\sharp(z)$ such that the Bezout identity

$$Q^\sharp(z)Q(z) = 1, \tag{4}$$

holds. With $P(z)$ as defined above, the vector valued polynomial

$$\begin{bmatrix} P(z) \\ \hdashline Q(z) \end{bmatrix} \in \mathbb{C}^{l+m}[z]$$

is also irreducible and likewise, the Bezout equation

$$\underbrace{[a_1(z)\cdots a_l(z)]}_{A(z)} P(z) + \underbrace{[b_1(z)\cdots b_m(z)]}_{B(z)} Q(z) = \alpha z^\nu, \tag{5}$$

where α and ν are a scaling factor and a given degree, respectively, is solvable for $A(z), B(z)$.

Throughout the paper, we shall assume that

$$\deg P(z) = \deg Q(z) = \kappa + 1$$

and we consider a solution $Q^\sharp(z)$ of the Bezout identity (4) having $\deg Q^\sharp(z) \le \kappa$. Consequently, we are looking for a solution of (5) of the form

$$[\, A(z) \,\vdots\, B(z) \,] = \sum_{i=0}^{\kappa} [\, A_i \,\vdots\, B_i \,] z^i.$$

To proceed, let us write the Bezout equation (5) using the identity (4) as

$$A(z)P(z)\underbrace{Q^\sharp(z)Q(z)}_{=1} + B(z)Q(z) = \alpha z^\nu \underbrace{Q^\sharp(z)Q(z)}_{=1}, \tag{6}$$

or, equivalently, as

$$\{A(z)P(z)Q^\sharp(z) + B(z)\}Q(z) = \{\alpha z^\nu Q^\sharp(z)\}Q(z). \tag{7}$$

Now, if the pair $\{A(z), B(z)\}$ is chosen such that an equality holds between the two bracketed terms in (7), *i.e.*,

$$A(z)P(z)Q^\sharp(z) + B(z) = \alpha z^\nu Q^\sharp(z), \tag{8}$$

then this pair must solve the Bezout equation (5). Starting from this observation, we show below how to construct a solution of Problem 1.

B. A solution of Problem 1

To begin, let $F(z)$ denotes the $(l \times m)$-matrix valued polynomial

$$F(z) \triangleq P(z)Q^\sharp(z) = \sum_{i=0}^{2\kappa+1} F_i z^i, \tag{9}$$

which appears in (8). As we shall see, this matrix valued polynomial will play a prominent part in the developments that follow. The next theorem shows how, in the case when $F(z)$ is square, $l = m$, one can express a solution of Problem 1, only in terms of the coefficients matrices F_i.

Theorem 1: Let $Q(z)$ and $P(z)$ be two $\mathbb{C}^m$-polynomials with

$$\deg Q(z) = \deg P(z) = \kappa + 1$$

and such that $Q(z)$ is irreducible and $P_0 = P(0) \neq \mathbf{0}_m$. Let $Q^\sharp(z)$ be a $\mathbb{C}^{1\times m}$-polynomial of degree $M \le \kappa$, which fulfills

$$Q^\sharp(z)Q(z) = 1.$$

As in (9), define by $F(z)$, the square polynomial matrix

$$F(z) = P(z)Q^\sharp(z) = \sum_{i\geq 0} F_i z^i, \text{ with } F_i \in \mathbb{C}^{m\times m}.$$

Finally, let

$$A(z) = \sum_{k=0}^{M} A_k z^k \text{ and } B(z) = \sum_{k=0}^{M} B_k z^k$$

be two $\mathbb{C}^{1\times m}$-polynomials obtained from the systems of equations

$$\underbrace{\begin{bmatrix} F_1^t & F_2^t & \cdots & F_{M+1}^t \\ F_2^t & F_3^t & \cdots & F_{M+2}^t \\ \vdots & \iddots & \iddots & \vdots \\ F_{M+1}^t & F_{M+2}^t & \cdots & F_{2M+1}^t \end{bmatrix}}_{\Gamma_{M+1}} \begin{bmatrix} A_M^t \\ \vdots \\ A_1^t \\ A_0^t \end{bmatrix} = \begin{bmatrix} \mathbf{0}_m \\ \vdots \\ \mathbf{0}_m \\ \mathbf{\Psi} \end{bmatrix} \quad \text{(10a)}$$

$$\begin{bmatrix} A_M & A_{M-1} & \cdots & A_0 \\ A_{M-1} & A_{M-2} & \cdots & \mathbf{0}_{1\times m} \\ \vdots & \iddots & \iddots & \vdots \\ A_0 & \mathbf{0}_{1\times m} & \cdots & \mathbf{0}_{1\times m} \end{bmatrix} \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_M \end{bmatrix} = - \begin{bmatrix} B_M \\ \vdots \\ B_1 \\ B_0 \end{bmatrix} \quad \text{(10b)}$$

for some $\mathbf{\Psi} \in \mathbb{C}^m$.

Then, for $M = \kappa$, the pair $\{A(z), B(z)\}$ solves the Bezout equation (5), where $\nu = 2\kappa + 1$ and the scaling factor is given by $\alpha = \mathbf{\Psi}^t Q_0$.

Proof: Given a polynomial

$$X(z) = X_0 + X_1 z + \cdots + X_K z^K,$$

we define its reciprocal as

$$\hat{X}(z) = z^K X(z^{-1}) = X_K + X_{K-1}z + \cdots + X_0 z^K.$$

If

$$X(z) = \ldots + X_{-1}z^{-1} + X_0 + X_1 z + X_2 z^2 + \ldots$$

is in $L_2^{p\times q}$, we define its strictly causal projection as

$$[X(z)]_+ = X_1 z + X_2 z^2 + \ldots$$

and its anticausal projection as

$$[X(z)]_- = \ldots + X_{-1}z^{-1} + X_0.$$

Now, a direct verification shows that equation (10a) is obtained from the Laurent expansion of $\hat{A}(z^{-1})F(z)$, by equating to zero the coefficients of z^k, for $k = 1, M$ and by equating the coefficient of z^{M+1} to $\mathbf{\Psi}^t$. In a same way, equation (10b) sets the coefficient of z^{-k} to $-B_{M-k}$, for $k = 1, M$. Since $\deg F(z) = 2\kappa + 1$ and $\deg A(z) = \deg B(z) = M \leq \kappa$, the strictly causal and anticausal projections of $\hat{A}(z^{-1})F(z)$ read as

$$[\hat{A}(z^{-1})F(z)]_+ = z^{M+1}R(z) \quad \text{(11a)}$$

$$[\hat{A}(z^{-1})F(z)]_- = -\hat{B}(z^{-1}), \quad \text{(11b)}$$

where $R(z)$ is some $\mathbb{C}^{1\times m}$-polynomial of degree $2\kappa - M$ such that $R(0) = \mathbf{\Psi}^t$. As these projections are complementary, we therefore deduce

$$\hat{A}(z^{-1})F(z) + \hat{B}(z^{-1}) = z^{M+1}R(z).$$

Multiplying on the right both sides of this equality by $z^M Q(z)$ yields

$$A(z)P(z) + B(z)Q(z) = z^{2M+1}R(z)Q(z), \quad \text{(12)}$$

so that $R(z)Q(z)$ must be a scalar polynomial of degree $\kappa - M$. For $M = \kappa$, this polynomial reduces to $\mathbf{\Psi}^t Q_0$. This then proves that the pair $\{A(z), B(z)\}$ obtained from (10) solves the Bezout equation (5), where the scaling factor α and the delay ν are given by $\alpha = \mathbf{\Psi}^t Q_0$ and $\nu = 2\kappa + 1$. ∎

Remark 1: When $Q(z)$ and $P(z)$ are scalar polynomials, *i.e.* when $m = l = 1$, equation (4) admits the trivial non polynomial solution $Q^\sharp(z) = 1/Q(z)$. $F(z)$ then becomes a rational function. In this case, Theorem 1 reduces to [10, Theorem 8.8.1] due to Fuhrmann.

To close this section it is worth to note that the polynomials $A(z)$ and $B(z)$ solve the Bezout identity with maximal delay. The polynomials $P^\sharp(z)$ and $Q^\sharp(z)$ allow one to solve the Bezout identity with a null delay. Now, the solution $Q^\sharp(z)$ of the Bezout identity (4) may be obtained from (11a). To see this, recall, from the proof of Theorem 1, that in the exact modelling case, $M = \kappa$, $R(z)Q(z) = \alpha$ reduces to a scalar constant so that we may identify $R(z) = Q^\sharp(z)$. Once equation (10a) is solved, one can use the obtained solution $A(z)$ to find $Q^\sharp(z)$. Indeed, considering in equation (11a) the coefficients of z^k, for $k = M + 2$ to $2M + 1$, one obtain

$$\begin{bmatrix} F_{M+2}^t & F_{M+3}^t & \cdots & F_{2M+1}^t \\ \vdots & \vdots & \iddots & \mathbf{0} \\ F_{2M}^t & F_{2M+1}^t & \cdots & \vdots \\ F_{2M+1}^t & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \begin{bmatrix} A_M^t \\ \vdots \\ A_1^t \end{bmatrix} = \alpha \begin{bmatrix} Q_1^{\sharp t} \\ \vdots \\ Q_M^{\sharp t} \end{bmatrix}$$

C. Identification of a polynomial matrix

Equations (11) show that $A(z)$ and $B(z)$ depend only from the coefficients matrices F_i, $i = 1, \cdots, 2M + 1$. Therefore, with the (pseudo) rational function $F(z)$ its is possible to solve (5). Then we can re-state Problem 1, with maximal delay ν, in terms of the identification of $F(z)$ from the observations $Y(n)$. To this end we claim Lemma 1.

Lemma 1: Let $\{Y(n)\}$ be the observation sequence obtained according to model (1) with $L = 2m$. For convenience we consider the partition of this sequence as

$$Y(n) = \begin{bmatrix} Y_1(n) \\ \hdashline Y_2(n) \end{bmatrix} = \begin{bmatrix} P(z) \\ \hdashline Q(z) \end{bmatrix} s(n) + \begin{bmatrix} W_1(n) \\ \hdashline W_2(n) \end{bmatrix} \quad \text{(13)}$$

If $Q(z)$ is irreducible, then $F(z) = P(z)Q^\sharp(z)$ is the transfer function of the system with input $\tilde{Y}_2(n) = Y_2(n) - W_2(n)$ and output $Y_1(n)$. As $\tilde{Y}_2(n)$ is not observable, save in the case without noise, $F(z)$ is determined with the solution of the input-output identification problem

$$F(z) = \arg\min_{\hat{F}(z)} E\|Y_1(n) - \hat{F}(z)Y_2(n)\|_2^2.$$

Proof: Let

$$\tilde{Y}_2(n) \stackrel{\triangle}{=} Y_2(n) - W_2(n) = Q(z)s(n)$$

be the noiseless (unobservable) lower block channel's output as in (13) and

$$e(n) \stackrel{\triangle}{=} Y_1(n) - \tilde{F}(z)Y_2(n)$$

be the identification error. As $Q(z)$ is irreducible we may write

$$Q^\sharp(z)\tilde{Y}_2(n) = Q^\sharp(z)Q(z)s(n) = s(n).$$

The vector $Y_1(n) = P(z)s(n) + W_1(n)$ can be rewritten, using the above expression for $s(n)$, as

$$Y_1(n) = P(z)Q^\sharp(z)\tilde{Y}_2(n) + W_1(n).$$

Setting $F(z) = P(z)Q^\sharp(z)$ the identification error can be written as

$$e(n) = (F(z) - \tilde{F}(z))\tilde{Y}_2(n) - \tilde{F}(z)W_2(n) + W_1(n).$$

With the zero mean, independent, identically distributed (iid) noise hypothesis, we have $E\{W_2(n)W_2^\dagger(n)\} = \sigma_w^2 I_{2m}$, and the minimization problem becomes

$$\begin{aligned} F(z) = &\arg\min_{\hat{F}(z)}\{\|[F(z) - \hat{F}(z)]\tilde{Y}_2(n)\|_2^2 \\ &+ \sigma_w^2(\|\hat{F}(z)\|_2^2 + m)\}. \end{aligned} \tag{14}$$

■

Minimization of the mean square error (14) can be interpreted as linear prediction of the received signals at the 1 to m sensors from the received signals at the $m+1$ to $2m$ sensors. That is, the system identification $F(z)$ can be interpreted as spatial and temporal linear prediction problem. The function $F(z)$ can be determined using adaptive input-output system identification. In a noisy setting, the corresponding optimization scheme is biased however. Indeed, from (14) we note a noise enhancement in the mean squares error. We may mention that an unbiased estimation can be achieved using (*e.g.*) a unit-norm constraint in the minimization problem [9].

III. Properties-Robustness to order mismatch

In this section we describe some important features of the proposed method as concerning uniqueness and order mismatch.

A. Uniqueness

The existence of a solution of (10a), and by consequence of the Bezout equation (5), is assured by the irreducibility of the pair of polynomials $\{P(z), Q(z)\}$. However, this solution is not unique even in the sufficient order case. To see this, note that the block Hankel matrix Γ_{M+1} in (10a) with dimension $m(M+1) \times m(M+1)$ is singular for $M \geq \kappa$ because $m \geq 2$. In effect, Kronecker's theorem asserts that

$$\begin{aligned} \text{rang}(\Gamma_{M+1}) &= \deg McMillan\ F(z) \\ &= 2\kappa + 1 < m(M+1). \end{aligned} \tag{15}$$

Let $a_i(z) = \sum_{k=1}^{\delta_i} a_i(k)z^k$ where $\delta_i \stackrel{\triangle}{=} \deg a_i(z)$, with $i = 1, m$ and $\delta_{m+1} = M + 1$. When the degrees $\delta_i,\ i = 1, m$ are selected with the constraints

$$\begin{cases} \sum_{i=1}^m \delta_i = 2M - m + 1 \\ M = \delta_1 > \delta_2 > \cdots > \delta_m \end{cases}$$

equation (5) yields a unique solution for $M = \kappa$. This unique solution can be obtained with (10), by absorbing these constraints in the parameterization of the unknown. In another words, we look for a solution $\hat{\mathbf{A}}$ of equation (10a) of the form $[A_M\ A_{M-1}\ \cdots A_0]$ where

$$A_j = \begin{cases} [a_1(j)\ a_2(j)\ \cdots\ a_m(j)] & j \leq \delta_m \\ [a_1(j) \cdots a_k(j)\ 0\ \cdots\ 0] & \delta_{k+1} < j \leq \delta_k \end{cases}$$

B. Undermodeling

Recall from equation (12) that the channel-equalizer combined response is of the form $c(z) = z^{2M+1}r(z)$, where $r(z) = R(z)Q(z)$ is a scalar polynomial of degree $\kappa - M$. Therefore, in the undermodeling case, $M < \kappa$, this combined response has exactly $\kappa - M + 1$ nonzero terms.

C. Overmodeling

Following the preceding arguments it results that in the overmodeled case, the channel-equalizer combined response is zero. Indeed, in this situation the left-hand side of (12) is a polynomial of degree $\kappa + M + 1$ with, by the form of the right hand-side, $2M+1 > \kappa+M+1$ zeros at the origin. This combined response must then vanish.

IV. Link with Subspace and Least-Squares methods

Recall that the basic principle of both the subspace and the least squares blind channel identification is to identify the channel as the kernel of a certain operator. Each method corresponds to a specific choice of the operator though the later is obtained in both cases by exploiting the subspace structure of the observation data [1]. In this section, we show that our method can be cast into this framework. Moreover, in the same way this principle allows one to identify the channel, it also allows us to identify directly the solutions $P^\sharp(z)$ and $Q^\sharp(z)$ of the Bezout identities $P^\sharp(z)P(z) = 1$ and $Q^\sharp(z)Q(z) = 1$ respectively.

To specify the operator in the proposed approach let us suppose that $P(z)$ is also irreducible, such that there exist a $\mathbb{C}^{1\times m}$-polynomial $P^\sharp(z)$ satisfying

$$P^\sharp(z)P(z) = 1, \quad \text{with} \quad \deg P^\sharp(z) \leq \kappa.$$

Setting $S(z) = Q(z)P^\sharp(z)$ we have the equalities

$$F(z)S(z) = P(z)P^\sharp(z) \stackrel{\triangle}{=} T(z)$$

and

$$S(z)F(z) = Q(z)Q^\sharp(z) \stackrel{\triangle}{=} \tilde{T}(z).$$

It is worth to note that $S(z)$ can be found in exactly the same way as for $F(z)$, by solving a classical input-output identification problem (with a unit norm constraint to avoid a possible bias).

It is straightforward to verify that if $\Phi(z)$ and $\tilde{\Phi}(z)$ are two $\mathbb{C}^m$-polynomials, then

$$[I_m - T(z)]\Phi(z) = \mathbf{0}_m \iff \Phi(z) = \gamma(z)P(z) \qquad (16)$$

and

$$[I_m - \tilde{T}(z)]\tilde{\Phi}(z) = \mathbf{0}_m \iff \tilde{\Phi}(z) = \tilde{\gamma}(z)Q(z) \qquad (17)$$

where $\gamma(z)$ and $\tilde{\gamma}(z)$ are some scalar polynomials, depending on the degrees of $\Phi(z)$ and $\tilde{\Phi}(z)$, respectively. The scalar polynomial $\gamma(z)$ reduces to a constant if, and only if, the degree of $\Phi(z)$ is equal to that of $P(z)$, in which case, we get $\Phi(z) = \alpha_0 P(z)$ for some constant α_0. The same holds true for $\tilde{\gamma}(z)$.

We may quote that the matrix $[I_m - T(z)]$, respectively $[I_m - \tilde{T}(z)]$, is the orthogonal projection operator on the rational subspace spanned by the polynomial $P(z)$, respectively $Q(z)$.

Observe that, if we rather consider the left kernel of $[I_m - T(z)]$, respectively $[I_m - \tilde{T}(z)]$, we obtain the following relations

$$\Phi^\sharp(z)[I_m - T(z)] = \mathbf{0}_m \iff \Phi^\sharp(z) = \gamma(z)P^\sharp(z) \qquad (18)$$

and

$$\tilde{\Phi}^\sharp(z)[I_m - \tilde{T}(z)] = \mathbf{0}_m \iff \tilde{\Phi}^\sharp(z) = \tilde{\gamma}(z)Q^\sharp(z) \qquad (19)$$

where $\gamma(z)$ and $\tilde{\gamma}(z)$ are some scalar polynomials. This shows that one may obtain directly the equalizers using the same principle of the classical (subspace, least squares) second order methods.

As concerning the implementation, the polynomials $\Phi(z)$, $\Phi^\sharp(z)$, $\tilde{\Phi}(z)$ and $\tilde{\Phi}^\sharp(z)$ can be obtained in a close form from optimizing a quadratic cost function

$$\hat{\vec{\mathbf{h}}} = \arg\min_{\tilde{\mathbf{h}}\in\mathcal{S}} \tilde{\mathbf{h}}^{\mathbf{H}}\, \mathcal{K}\, \tilde{\mathbf{h}},$$

where $\mathcal{S}$ is a set that specifies the domain of $\vec{\mathbf{h}}$ and $\mathcal{K}$ is characteristic to the kernel under consideration. Usually $\vec{\mathbf{h}}$ is subject to some constraint as $||\vec{\mathbf{h}}|| = 1$ to rule out the trivial zero solution. However these methods may not be robust against modelling errors especially when the channel matrix is close to being singular or when there is a poor SNR.

V. The Algorithm and Simulation Examples

In this section we present some computer simulations to verify the properties of the proposed approach presented hitherto.

The different steps to compute a solution of Problem 1 are outlined below:

1) Estimation of $F(z)$ with the adaptive FIR system identification where $Y_2(n)$ is the input and $Y_1(n)$ is the output.

2) Estimation of the order κ *e.g.* using SVD of the block Hankel matrix Γ_{M+1} (*generic to all second order methods*).

3) Find the polynomials $A(z)$ and $B(z)$ with (10a) followed by (10b).

4) Equalization: $\tilde{s}(n) = A(z)Y_1(n) + B(z)Y_2(n)$.

Such an algorithm has been used to perform some simulations. We considered the input s to the channel as an i.i.d. binary source sequence with variance $\sigma_s^2 = 1$. The number of (virtual) subchannels has been set to $L = 4$. The channel coefficients considered here are listed in Table I.

Table I - Channel coefficients

$\mathbf{h}_0$	$\mathbf{h}_1$	$\mathbf{h}_2$	$\mathbf{h}_3$	$\mathbf{h}_4$	$\mathbf{h}_5$
-0.692	-1.441	0.816	1.191	-1.604	-0.805
0.858	0.571	0.712	-1.202	0.257	0.529
1.254	-0.400	1.290	-0.020	-1.056	0.219
-1.594	0.690	0.669	-0.157	1.415	-0.922

In the first simulation we have considered noisy observations, with a signal to noise ratio of $SNR = 20dB$ and have supposed that the channel order was correctly estimated ($M = 4$). The classical recursive least squares algorithm was used in the identification of the polynomial matrix $F(z)$, without any constraint in the norm of this matrix. Figure 2 shows the combined response obtained by solving Problem 1 for a maximum delay, following steps 1 to 3 above. In Figure 3, we display the combined response corresponding to Bezout identity (4). It is worth noting that we were able to solve the Bezout identity in a blind setting even in a noisy environment and without any constraint to monitor the bias in the estimation of $F(z)$.

Finally, we verify in Figure 4, the undermodeling properties of the proposed approach as quoted in Section 3. In this experiment, the equalizer order was set to $M = 2$. As expected, the number of nonzero terms in the combined response is exactly $\kappa + 1 - M = 3$.

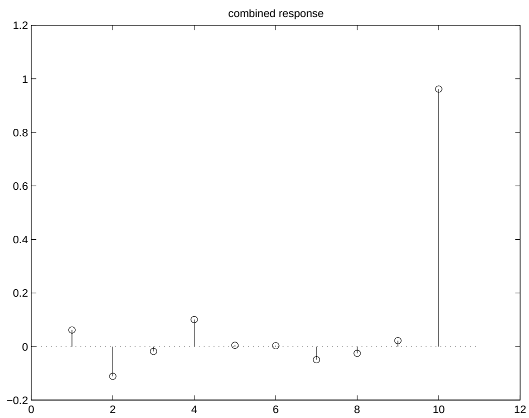

Fig. 2. Combined response with $SNR = 20dB$ and with maximal delay.

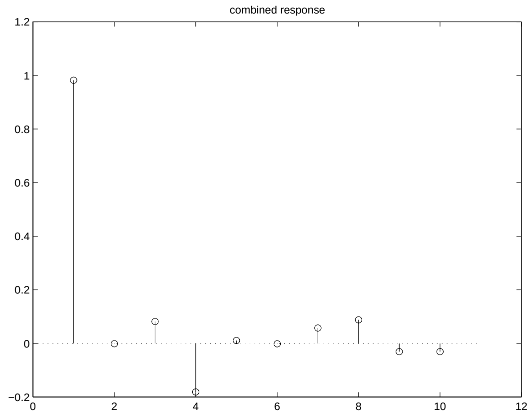

Fig. 3. Combined response with $SNR = 20dB$ and with null delay.

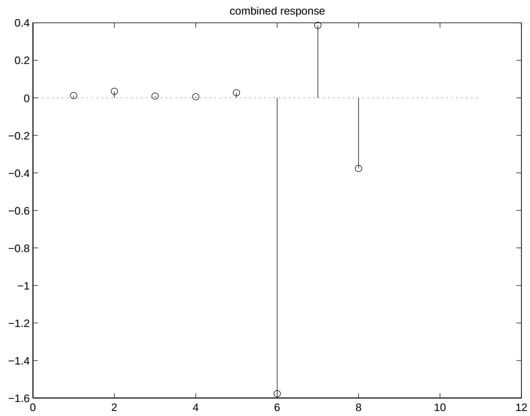

Fig. 4. Combined response without noise and $M = 2$.

VI. Concluding remarks

We have proposed a polynomial approach to solve the Bezout identity underlying the equalization of a *single-input multiple-output channel*, in a blind environment. The approach allows one to solve this identity with a maximum and/or a minimum delay.

It must be mentioned that the equalizer polynomials that satisfy the Bezout equation correspond to a Zero Forcing (ZF) equalizer. As it is well known, the ZF equalizer tends to perform poorly when the channel noise is significant and when the channel $H(z)$ approaches the boundary of the minimum phase domain. Therefore, the use of unit-norm constraints in the estimation of $F(z)$ to reduce the effect of noise and a robust resolution of equation (10a) by exploiting the displacement structure of the block Hankel matrix are under study.

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