A FAMILY OF WAVELETS AND A NEW ORTHOGONAL MULTIRESOLUTION ANALYSIS BASED ON THE NYQUIST CRITERION

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ABSTRACT- *A generalisation of the Shannon complex wavelet is introduced, which is related to raised cosine filters. This approach is used to derive a new family of orthogonal complex wavelets based on the Nyquist criterion for Intersymbolic Interference (ISI) elimination. An orthogonal Multiresolution Analysis (MRA) is presented, showing that the roll-off parameter should be kept below 1/3. The pass-band behaviour of the Wavelet Fourier spectrum is examined. The left and right roll-off regions are asymmetric; nevertheless the Q-constant analysis philosophy is maintained. Finally, a generalisation of the (square root) raised cosine wavelets is proposed.*

Key-words- Multiresolution Analysis, Wavelets, Nyquist Criterion, Intersymbolic Interference (ISI).

1. INTRODUCTION

Wavelet analysis has matured rapidly over the past years and has been proved to be valuable for both scientists and engineers [1,2]. Wavelet transforms have recently gained numerous applications throughout an amazing number of areas [3, 4]. Another strongly related tool is the Multiresolution analysis (MRA). Since its introduction in 1989 [5], MRA representation has emerged as a very attractive approach in signal processing, providing a local emphasis of features of importance to a signal [2, 6, 7]. The purpose of this paper is twofold: first, to introduce a new family of wavelets and then to provide a new and complete orthogonal multiresolution analysis. We adopt the symbol := to denote equals by definition. As usual, $Sinc(t)$:=sin(πt)/πt and $Sa(t)$:=sin(t)/t. The gate function of length T is denoted by $\prod\left(\frac{t}{T}\right)$. Wavelets are denoted by $y(t)$ and scaling functions by $f(t)$. The paper is organised as follows. Section 2 generalises the Sinc MRA. A new orthogonal MRA based on the raised cosine is introduced in section 3. A new family of orthogonal wavelet is also given. Further generalisations are carried out in section 4. Finally, Section 5 presents the conclusions.

2. A GENERALISED SHANNON WAVELET (RAISED-COSINE WAVELET)

The scaling function for the Shannon MRA (or Sinc MRA) is given by the sample function: $f^{(Sha)}(t) = Sinc(t)$. A naive and interesting generalisation of the complex Shannon wavelet can be done by using spectral properties of the raised-cosine filter [8]. The most used filter in Digital Communication Systems, the raised cosine spectrum $P(w)$ with a roll-off factor α, was conceived to eliminate the ***Intersymbolic Interference*** (ISI). Its transfer function is given by

$$P(w) = \begin{cases} \frac{1}{2p} & 0 \le |w| < (1-a)p \\ \frac{1}{4p}\left\{1+\cos\frac{1}{2a}\left(|w| - p(1-a)\right)\right\} & (1-a)p \le |w| < (1+a)p \\ 0 & |w| \ge (1+a)p. \end{cases} \quad (1)$$

The "raised cosine" frequency characteristic therefore consists of a flat spectrum portion followed by a roll-off portion with a sinusoidal format. Such spectral shape is very often used in the design of base-band digital systems. It is derived from the pulse shaping design criterion that would yield zero ISI, the so-called Nyquist Criterion. Note that $P(w)$ is a real and non-negative function [8], and in addition

$$\sum_{l \in Z} P(w + l2p) = \frac{1}{2p}. \quad (2)$$

Furthermore, the following normalisation condition holds: $\frac{1}{2p}\int_{-\infty}^{+\infty} P(w)dw = 1$.

We propose here the replacement of the Shannon scaling function on the frequency domain by a raised cosine, with parameter α (Fig. 1). We assume then that Φ(w)=$P(w)$. In the time domain this corresponds exactly to the impulse response of a Nyquist raised-cosine filter.

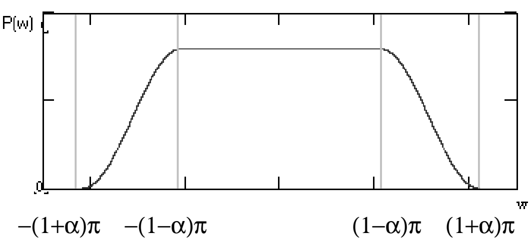

Figure 1. Fourier Spectrum of the raised cosine scaling function.

The generalised Shannon scaling function is therefore:

$$f^{(GSha)}(t) = \frac{\cos apt}{1-(2at)^2} Sinc(t). \quad (3)$$

In the particular case α=0, the scaling function simplifies to the classical Shannon scaling function. As a consequence of the Nyquist criterion, the scaling function presents zero crossing points on the unidimensional grid of integers, n=±1,±2,±3,… .This scaling function f defines a non-orthogonal MRA. Figure 2 shows the scaling function corresponding to a generalised Shannon MRA for a few values of α.

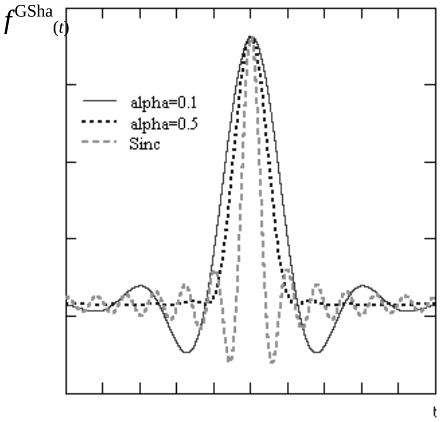

Figure 2. Scaling function for the raised cosine wavelet (generalised Shannon scaling function).

3. MULTIRESOLUTION ANALYSIS BASED ON NYQUIST FILTERS

A very simple way to build an orthogonal MRA via the raised cosine spectrum [8] can be accomplished by invoking Meyer's central condition [6]:

$$\sum_{n\in Z} |\,\Phi(w+2pn)\,|^2 = \frac{1}{2p}\,. \qquad (4)$$

Comparing eqn(2) to eqn(4), we choose $\Phi(w)=\sqrt{P(w)}$ (i.e. a square root of the raised cosine spectrum). Let then

$$\Phi(w)=\begin{cases}\frac{1}{\sqrt{2p}} & 0\le |w|<(1-a)p\\ \frac{1}{\sqrt{2p}}\cos\frac{1}{4a}\left(|w|-(1-a)p\right) & (1-a)p\le |w|<(1+a)p\\ 0 & |w|\ge(1+a)p\,.\end{cases} \qquad (5)$$

Clearly, $\sum_n |\,\Phi(w+2pn)\,|^2=\frac{1}{2p}$, so the square root of the raised cosine shape allows an orthogonal MRA. The scaling function $f(t)$ is plotted in the spectral domain (Fig. 3).

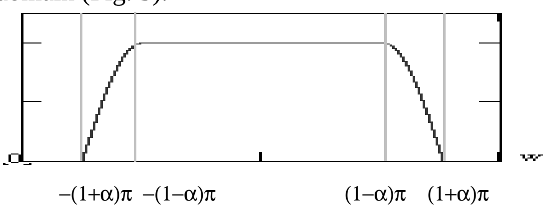

Figure 3. Spectral Characteristic of the "*de Oliveira*" orthogonal MRA.

The cosine pulse function PCOS defined below plays an important role on the raised cosine MRA.

Definition 1. The cosine pulse function of parameters t_0, q_0, w_0 and B is defined by

$$PCOS(w;t_0,q_0,w_0,B):=\cos(wt_0+q_0)\prod\left(\frac{w-w_0}{2B}\right),$$

$t_0,q_0,w_0,B\in\Re$, $0<B<w_0$. ❑

It corresponds to a cosine pulse (in the frequency domain), with frequency t_0 and phase q_0, with duration $2B$ rad/s, centred at w_0 rad/s.

Some interesting particular cases include:

1) The Gate function: $\prod\left(\frac{w}{2B}\right)=PCOS(w;0,0,0,B)$,

2) A Gate shifted by w_0

$$\prod\left(\frac{w-w_0}{2B}\right)=PCOS(w;0,0,w_0,B),$$

3) An infinite cosine pulse:

$$\cos(wt_0+q_0)=PCOS(w;t_0,q_0,0,B\to+\infty).$$

Denoting the inverse Fourier transform of $PCOS$ by $pcos(t;t_0,q_0,w_0,B):=\Im^{-1}PCOS(w;t_0,q_0,w_0,B)$, the following result can be proved.

Proposition 1. Given t_0, q_0,w_0 and B parameters of a $PCOS$, the inverse spectrum $pcos$ is given by:

$$pcos(t;t_0,q_0,w_0,B)=$$
$$\frac{B}{2p}\left\{e^{j(w_0t+w_0t_0+q_0)}.Sa[B(t+t_0)]+e^{j(w_0t-w_0t_0-q_0)}.Sa[B(t-t_0)]\right\} \qquad (6)$$

Proof. Follows applying the convolution property for the following couple of transform pairs:

$\frac{1}{2}\left[d(t+t_0)e^{jq_0}+d(t-t_0)e^{-jq_0}\right]\leftrightarrow\cos(wt_0+q_0)$ and $\frac{B}{p}e^{jw_0t}.Sa(Bt)\leftrightarrow\prod\left(\frac{w-w_0}{2B}\right)$. ❑

It is interesting to check some particular cases:

$pcos(w;0,0,0,B)\leftrightarrow PCOS(w;0,0,0,B)$ Û

$\frac{B}{p}.Sa(Bt)\leftrightarrow\prod\left(\frac{w}{2B}\right)$

$pcos(t;t_0,0,0,B\to+\infty)\leftrightarrow PCOS(w;t_0,0,0,B\to+\infty)$

Û $\frac{1}{2}\left[d(t+t_0)+d(t-t_0)\right]\leftrightarrow\cos(wt_0)$, which follows from the property of the sequence

$$\lim_{e\to 0}\frac{1}{pe}.Sa\left(\frac{t}{e}\right)=d(t)\,. \qquad (7)$$

Property 1. (Time shift): A shift T in time is equivalent to the following change of parameters: $pcos(t-T;t_0,q_0,w_0,B)=pcos(t;t_0-T,q_0,w_0,B)$. ❑

In order to find out the scaling function of the new orthogonal MRA introduced in this section, let us take the inverse Fourier transform of $\Phi(w)$.

The spectrum $\Phi(w)$ can be rewritten as a sum of contributions from three different sections (a central flat section and two cosine-shaped ends):

$$\sqrt{2p}\,\Phi(w) = \prod\left(\frac{w}{2p-2B}\right) + \cos(wt_0+q_0)\prod\left(\frac{w-p}{2B}\right) + \cos(-wt_0+q_0)\prod\left(\frac{-w-p}{2B}\right) \qquad (8)$$

with parameters $B := pa$, $t_0 := \frac{1}{4a}$ and $q_0 := -\frac{(1-a)p}{4a}$.

It follows from Definition 1 that

$$\sqrt{2p}\,\Phi(w) = PCOS(w;0,0,0,2p-2B) + PCOS\left(w;\frac{1}{4a},-\frac{(1-a)p}{4a},p,pa\right) + PCOS\left(-w;\frac{1}{4a},-\frac{(1-a)p}{4a},p,pa\right)$$

and therefore

$$\sqrt{2p}\,f^{(deO)}(t) = pcos(t;0,0,0,2p-2B) + pcos\left(t;\frac{1}{4a},-\frac{(1-a)p}{4a},p,pa\right) + pcos\left(-t;\frac{1}{4a},-\frac{(1-a)p}{4a},p,pa\right)$$

After a somewhat tedious algebraic manipulation, we derive

$$f^{(deO)}(t) = \frac{1}{\sqrt{2p}}.(1-a).Sinc[(1-a)t] + \frac{1}{\sqrt{2p}}.\frac{4a}{p}.\frac{1}{1-(4at)^2}\{\cos p(1+a)t + 4at.\sin p(1-a)t\}. \qquad (9)$$

A sketch of the above MRA scaling function is shown in figure 4, assuming a few roll-off values.

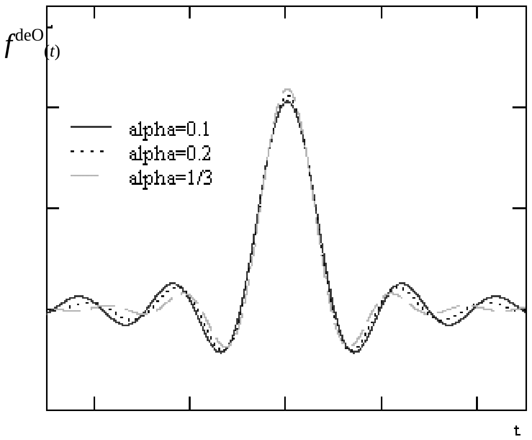

Figure 4. "*de Oliveira*" Scaling Function for an Orthogonal MRA (α=0.1, 0.2 and 1/3).

The scaling function $f^{(deO)}(t)$ can be expressed in a more elegant and compact representation with the help of the following special functions:

<u>Definition 2</u>. (Special functions); ν is a real number, $H_n(t) := nSinc(nt)$, 0≤ν≤1, and

$$M_{n_2}^{n_1}(t) := \frac{1}{p}\frac{2|n_1-n_2|}{1-[2t(n_1-n_2)]^2}\{\cos pn_1t + 2(n_1-n_2)t.\sin pn_2t\}$$

❑

It follows that

$\sqrt{2p}\,f^{(Sha)}(t) = H_1(t)$,

$\sqrt{2p}\,f^{(deO)}(t) = H_{1-a}(t) + M_{1-a}^{1+a}(t)$.

Clearly, $\lim_{a\to 0} f^{(deO)}(t) = f^{(Sha)}(t)$.

The low-pass $H(.)$ filter of the MRA can be found by using the so-called two-scale relationship for the scaling function [7]:

$$\Phi(w) = \frac{1}{\sqrt{2}}H\left(\frac{w}{2}\right)\Phi\left(\frac{w}{2}\right). \qquad (10)$$

How should H be chosen to make eqn(10) hold? Initially, let us sketch the spectrum of Φ(*w*) and Φ(*w*/2) as shown in figure 5.

The main idea is to not allow overlapping between the roll-off portions of these spectra. Imposing that 2π(1−α)>(1+α)π, it follows that α<1/3 (remember that 0<α<1). This is a simplifying hypothesis. It is quite usual the use of small roll-off factors in Digital Communication Systems.

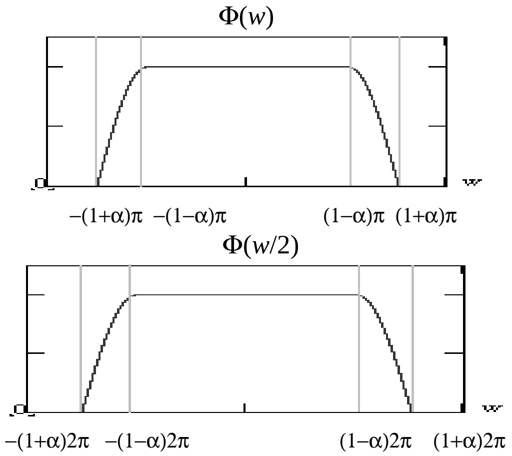

Figure 5. Draft of Φ(*w*) and Φ(*w*/2).

It is suggested to assume that $H\left(\frac{w}{2}\right) = \frac{1}{\sqrt{p}}\Phi(w)$.

Substituting this transfer function into the refinement equation (eqn(10)), results in

$$\Phi(w) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{p}}\Phi(w).\Phi\left(\frac{w}{2}\right). \qquad (11)$$

The above equation is actually an identity for |w| > (1+α)π. Into the region |*w*| < (1+α)π, it can be seen that $\Phi\left(\frac{w}{2}\right) = \sqrt{2p}$, under the constraint α<1/3.

The orthogonal "*de Oliveira*" wavelet can be found by the following procedure [7]:

$$\Psi(w) = e^{-jw/2}\frac{1}{\sqrt{2}}H^*\left(\frac{w}{2}-p\right)\Phi\left(\frac{w}{2}\right). \qquad (12)$$

Inserting the shape of the filter *H* in the above equation, it follows that:

$$\Psi(w) = e^{-jw/2}\frac{1}{\sqrt{2p}}\Phi(w-2p).\Phi\left(\frac{w}{2}\right). \qquad (13)$$

In order to evaluate the spectrum of the mother wavelet, we plot both $\Phi(w-2p)$ and $\Phi\left(\frac{w}{2}\right)$, again under the constraint α<1/3 (Fig. 6). In this case, (1+α)π<(1−α)2π and (1+α)2π<(1−α/3)3π.

Defining a shaping pulse

$$S^{(deO)}(w) := \frac{1}{\sqrt{2p}}\Phi(w-2p).\Phi\left(\frac{w}{2}\right), \qquad (14)$$

the wavelet specified by eqn(12) can be rewritten as $\Psi^{(deO)}(w) = e^{-jw/2}S^{(deO)}(w)$. The term $e^{-jw/2}$ accounts for the *wave,* while the term $S(w)$ accounts for *let*.

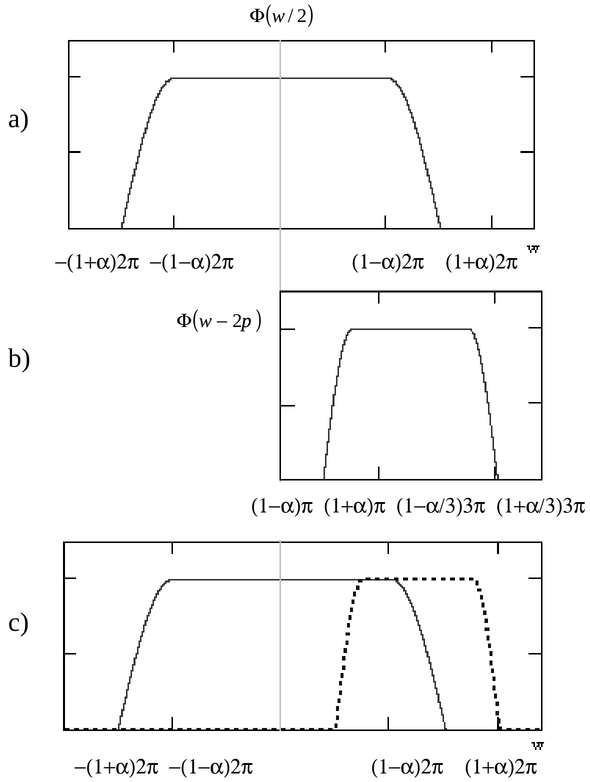

Figure 6. Sketch of the scaling function spectrum: (a) scaled version $\Phi\left(\frac{w}{2}\right)$ (b) translated version $\Phi(w-2p)$ (c) simultaneous plot of (a) and (b).

From the figure 6, it follows by inspection that:

$$S^{(deO)}(w) = \begin{cases} 0 & \text{if } w < p(1-a) \\ \Phi(w-2p) & \text{if } p(1-a) \le w < p(1+a) \\ \frac{1}{\sqrt{2p}} & \text{if } p(1+a) \le w < 2p(1-a) \\ \Phi\left(\frac{w}{2}\right) & \text{if } 2p(1-a) \le w < 2p(1+a) \\ 0 & \text{if } w \ge 2p(1+a). \end{cases} \qquad (15)$$

Inserting the (square root) raised cosine format of Φ(.), results in:

$$S^{(deO)}(w) = \begin{cases} 0 & \text{if } w < p(1-a) \\ \frac{1}{\sqrt{2p}}\cos\frac{1}{4a}\left(w - p(1+a)\right) & \text{if } p(1-a) \le w < p(1+a) \\ \frac{1}{\sqrt{2p}} & \text{if } p(1+a) \le w < 2p(1-a) \\ \frac{1}{\sqrt{2p}}\cos\frac{1}{8a}\left(w - 2p(1-a)\right) & \text{if } 2p(1-a) \le w < 2p(1+a) \\ 0 & \text{if } w \ge 2p(1+a). \end{cases} \qquad (16)$$

The complex "*de Oliveira*" wavelet is given by $\Psi^{(deO)}(w) = e^{-jw/2}S^{(deO)}(w)$. and its modulo $|\Psi^{(deO)}(w)| = S^{(deO)}(w)$ is depicted in figure 7. Observe furthermore that making α→0, the wavelet reduces to the complex Shannon wavelet.

It is quite apparent from figure 7 the band-pass behaviour of the wavelet $\Psi^{(deO)}(w)$. Observe that the left and right roll-off is not exactly symmetrical. Instead, despite their similar shape, they occur at different scales, a typical behaviour of wavelets.

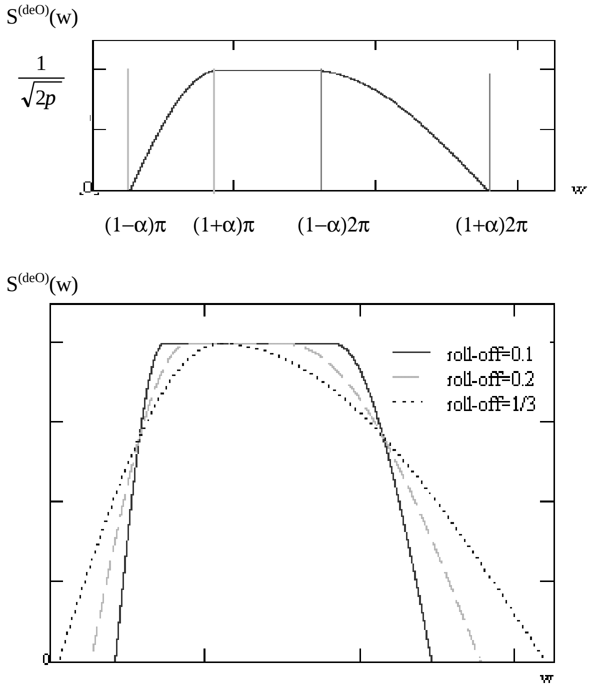

Figure 7. Modulo of the "*de Oliveira*" Wavelet (frequency domain).

The time domain representation of the wavelet can be derived by taking the inverse Fourier transform: $y^{(deO)}(t) = \Im^{-1}\Psi^{(deO)}(w)$.

Denoting by $s^{(deO)}(t) \leftrightarrow S^{(deO)}(w)$ the corresponding transform pair, it follows that $y^{(deO)}(t) = s^{(deO)}(t - \frac{1}{2})$.

The shaping pulse can be rewritten as:

$$\sqrt{2p}\,S^{(deO)}(w) = PCOS\left(w; \frac{1}{4a}, -\frac{p(1+a)}{4a}, p, pa\right) + PCOS\left(w; 0, 0, \frac{3p}{2}(1-\frac{a}{3}), \frac{p}{2}(1-3a)\right) + PCOS\left(w; \frac{1}{8a}, -\frac{2p(1-a)}{8a}, 2p, 2pa\right)$$

Finally, applying the inverse transform, we have

$$\sqrt{2p}\,s^{(deO)}(t) = pcos\left(t; \frac{1}{4a}, -\frac{p(1+a)}{4a}, p, pa\right) + pcos\left(t; 0, 0, \frac{3p}{2}(1-\frac{a}{3}), \frac{p}{2}(1-3a)\right) + pcos\left(t; \frac{1}{8a}, -\frac{2p(1-a)}{8a}, 2p, 2pa\right) \qquad (17)$$

The *pcos*(.) signal is a complex signal when there are no symmetries in *PCOS*(.). The real and imaginary parts of the *pcos* function can be handled separately, according to

$pcos(t; t_0, q_0, w_0, B) = rpcos(t) + j.ipcos(t)$, where $rpcos(t) := \Re e(pcos(t; t_0, q_0, w_0, B))$ and $ipcos(t) := \Im m(pcos(t; t_0, q_0, w_0, B))$.

Aiming to investigate the wavelet behaviour, we propose to separate the Real and Imaginary parts of $s^{(deO)}(t)$, introducing new functions *rpc*(.) and *ipc*(.)

$$\Re e\,\psi^{(deO)}(t) = \Re e\left\{s^{(deO)}(t-\frac{1}{2})\right\} \quad \Im m\left\{\psi^{(deO)}(t)\right\} = \Im m\left\{s^{(deO)}(t-\frac{1}{2})\right\} \tag{18}$$

Proposition 2. Let $\Delta\omega^{(+1)} := \omega_0 + B$ and $\Delta\omega^{(-1)} := \omega_0 - B$; $\Delta\theta^{(+1)} := Bt_0 + \omega_0 t_0 + \theta_0$ and $\Delta\theta^{(-1)} := Bt_0 - \omega_0 t_0 - \theta_0$ be auxiliary parameters. Then

$$rpc(t) = \frac{1}{2\pi}\,\frac{-t_0.\sum_{i\in\{-1,+1\}} \operatorname{sen}\Delta\theta^{(i)}\cos\Delta\omega^{(i)}t + t.\sum_{i\in\{-1,+1\}}(i)\cos\Delta\theta^{(i)}\operatorname{sen}\Delta\omega^{(i)}t}{t^2 - t_0^2}$$

$$ipc(t) = \frac{1}{2\pi}\,\frac{-t_0.\sum_{i\in\{-1,+1\}} \operatorname{sen}\Delta\theta^{(i)}\operatorname{sen}\Delta\omega^{(i)}t + t.\sum_{i\in\{-1,+1\}}(-i)\cos\Delta\theta^{(i)}\cos\Delta\omega^{(i)}t}{t^2 - t_0^2}$$

❑

Proof. Follows from trigonometry identities.

At this point, an alternative notation $rpc(t) = rpc\left(t; \Delta\omega^{(+1)}, \Delta\omega^{(-1)}, \Delta\theta^{(+1)}, \Delta\theta^{(-1)}\right)$ and $ipc(t) = ipc\left(t; \Delta\omega^{(+1)}, \Delta\omega^{(-1)}, \Delta\theta^{(+1)}, \Delta\theta^{(-1)}\right)$ can be introduced to explicit the dependence on these new parameters. Handling apart the real and imaginary parts of $s^{(deO)}(t)$, we arrive at

$$\sqrt{2\pi}\,\Re e\left(s^{(deO)}(t)\right) = rpc\left(t;\pi(1+\alpha),\pi(1-\alpha),0,\frac{\pi}{2}\right) + rpc\left(t;2\pi(1-\alpha),\pi(1+\alpha),0,0\right) + \mathrm{rpc}\left(t;2\pi(1+\alpha),2\pi(1-\alpha),\frac{\pi}{2},0\right) \tag{19}$$

Applying now proposition 2, after many algebraic manipulations:

$$\sqrt{2\pi}\,\Re e\left(s^{(deO)}(t)\right) = \frac{1}{2}\left\{\mathrm{H}_{2(1-\alpha)}(t) - \mathrm{H}_{(1+\alpha)}(t) + \mathrm{M}_{1+\alpha}^{1-\alpha}(t) + \mathrm{M}_{2(1-\alpha)}^{2(1+\alpha)}(t)\right\}, \tag{20}$$

and $\Re e\left(\psi^{(deO)}(t)\right) = \Re e\left(s^{(deO)}(t-1/2)\right)$.

The analysis of the imaginary part can be done in a similar way.

Definition 3. (special functions); ν is a real number, $\overline{H}_\nu(t) := \nu\,\dfrac{\cos(\nu\pi t)}{\nu\pi t}$, 0≤ν≤1, and

$$\overline{\mathrm{M}}_{\nu_2}^{\nu_1}(t) := \frac{1}{\pi}\,\frac{2\,|\nu_1-\nu_2|}{1-\left[2t(\nu_1-\nu_2)\right]^2}\left\{\sin\pi\nu_1 t - 2(\nu_1-\nu_2)t.\cos\pi\nu_2 t\right\}$$

❑

The imaginary part of the wavelet can be found by

$$\sqrt{2\pi}\,\Im m\left(s^{(deO)}(t)\right) = \frac{1}{2}\left\{\overline{\mathrm{H}}_{2(1-\alpha)}(t) - \overline{\mathrm{H}}_{(1+\alpha)}(t) + \overline{\mathrm{M}}_{1+\alpha}^{1-\alpha}(t) + \overline{\mathrm{M}}_{2(1-\alpha)}^{2(1+\alpha)}(t)\right\} \tag{21}$$

and $\Im m\left(\psi^{(deO)}(t)\right) = \Im m\left(s^{(deO)}(t-1/2)\right)$.

The real part (as well as the imaginary part) of the complex wavelet $\psi^{(deO)}(t)$ are plotted in figure 8, for α=0.1, 0.2 and 1/3.

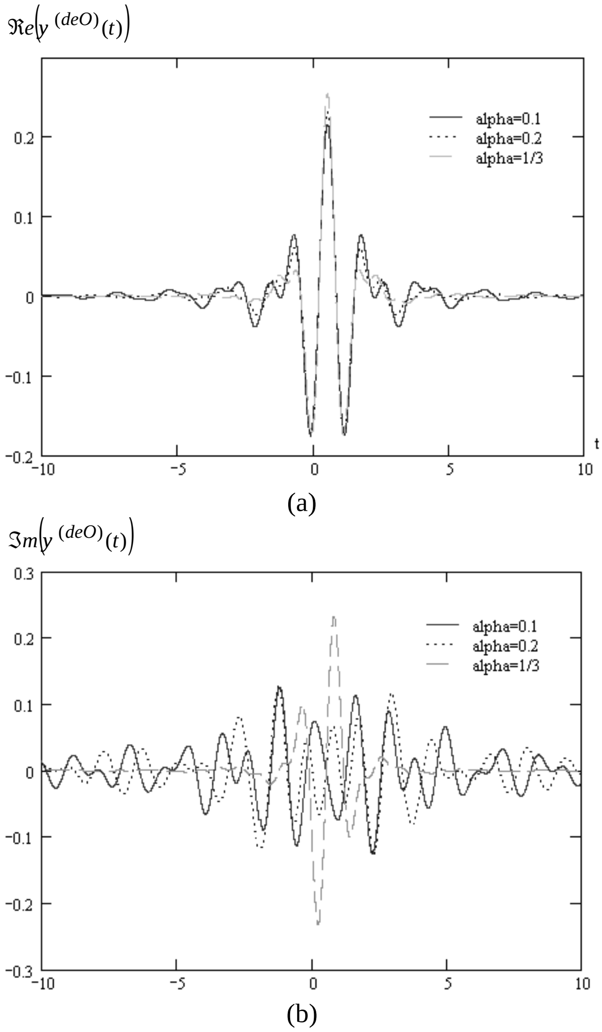

Figure 8. Wavelet $\psi^{(deO)}(t)$:
(a) real part and (b) imaginary part.

4. FURTHER GENERALISATIONS

Generally, the approach presented in the last section is not restricted to raised cosine filters.

Algorithm of MRA Construction. Let *P*(ω) be a real band-limited function, *P*(ω)=0 ω>2π, which satisfies the vestigial side band symmetry condition, i.e.,

$$\left\{P(\omega) + P(\omega - 2\pi)\right\} = \frac{1}{2\pi} \quad \text{for } |\,\omega\,| < \pi$$

Then the scaling function $\Phi(\omega) = \sqrt{P(\omega)}$ defines an orthogonal MRA. ❑

Proposition 3. If *P*(ω;α) is a Nyquist filter of roll-off α, and λ(α) is an arbitrary probability density function, 0<α<1, then the scaling function $\Phi(\omega) = \sqrt{\int_0^1 \lambda(\alpha)P(\omega;\alpha)d\alpha}$ defines an orthogonal MRA.

Proof. It is enough to show that $\sum_{n\in Z}|\Phi(\omega + 2\pi n)|^2 = \frac{1}{2\pi}$.

Given an integer *n*, then $\Phi(\omega + 2\pi n) = \sqrt{\int_0^1 \lambda(\alpha)P(\omega + 2\pi n;\alpha)d\alpha}$. Taking the square of both members and adding equations for each integer *n*,

$$\sum_{n\in Z}|\Phi(w+2\pi n)|^2=\int_0^1 \lambda(\alpha)\sum_{n\in Z}P(w+2\pi n;\alpha)\,d\alpha \cdot$$

Since $P(w;\alpha)$ is a Nyquist filter,

$$\sum_{n\in Z}P(w+2\pi n;\alpha)=\frac{1}{2\pi}$$ and the proof follows. ❑

The most interesting case of such generalisation corresponds to a "weighting" of square-root-raise-cosine filter.

<u>Corollary</u>. (Generalised raised-cosine MRA). If $P(w;\alpha)$ is the raised cosine spectrum with a (continuous) roll-off parameter α, i.e.,

$$P(w;\alpha):=\begin{cases}\frac{1}{2\pi} & 0\leq |w|<(1-\alpha)\pi\\ \frac{1}{4\pi}\left\{1+\cos\frac{1}{2\alpha}\left(|w|-\pi(1-\alpha)\right)\right\} & (1-\alpha)\pi\leq |w|<(1+\alpha)\pi\\ 0 & |w|\geq(1+\alpha)\pi\end{cases}.$$

and $\lambda(\alpha)$ is an arbitrary probability density function defined over the interval $0<\alpha<1$, the scaling function $\Phi(w)=\sqrt{\int_0^1 \lambda(\alpha)P(w;\alpha)\,d\alpha}$ defines an orthogonal MRA. ❑

5. CONCLUSIONS

This paper introduced a new family of complex orthogonal wavelets, which was derived from the classical Nyquist criterion for ISI elimination in Digital Communication Systems. Properties of both the scaling function and the mother wavelet were investigated. This wavelet family can be used to perform an orthogonal Multiresolution Analysis. A new function termed *PCOS* was introduced, which is offered as a powerful tool in matters that concern raised cosines. An algorithm for the construction of MRA based on vestigial side band filters was presented. A generalisation of the (square root) raised cosine wavelet was also proposed yielding a broad class of orthogonal wavelets and MRA.

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