# On the Equivalence of the Constrained RLS and the GSC-RLS Beamformers

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***Abstract*—This paper compares the transients of the constrained RLS (CRLS) algorithm with the generalized sidelobe canceller (GSC) employing an RLS algorithm. It is shown that the requirement for transient-equivalence is satisfied by the GSC structure using proper initialization, and any restriction concerning orthogonality of the matrices involved may be relaxed. This result differs from the more restrictive case for transient-equivalence of the constrained LMS (CLMS) algorithm and the GSC employing LMS algorithm. Simulations in beamforming and system-identification applications confirms the theoretical results.**

## I. Introduction

ADAPTATION algorithms which satisfy linear constraints finds applications in several areas of signal processing, such as beamforming and blind multiuser detection in code-division multiple-access (CDMA) systems. By imposing a set of linear constraints on the adaptive filter, the necessity of a desired signal can often be relaxed, resulting in what is commonly referred to as blind algorithms. Linear constraints usually reflects the prior knowledge of the system, like direction of arrival (DOA) of user signals in antenna array processing [1], [2], user spreading code in blind adaptive multiuser detection [3], [4], or linear phase [5].

The constrained LMS (CLMS) algorithm, which does not require re-initialization and incorporates the constraints into the solution was first introduced by Frost [1]. More recently, other constrained adaptation algorithms were introduced which are tailored to specific applications or present advantageous performance regarding convergence and robustness see, e.g., [6], [7]–[12]. The constrained RLS (CRLS) algorithm introduced in [6] is one solution which tries to overcome the problem of slow convergence experienced with the CLMS algorithm in situations when the input signal is strongly correlated.

An alternative approach to implement linearly constrained (LC) adaptive filters was introduced in [2], which became known as the generalized sidelobe canceller (GSC). By transforming the constrained minimization problem into an unconstrained minimization problem, the GSC structure allows that any adaptation algorithm be directly applied. The GSC structure makes use of a blocking matrix that must be orthogonal to the corresponding constraint matrix.

It has been shown in [2] that in order for the transients of the CLMS algorithm and the GSC employing an LMS algorithm to be the same, the blocking matrix needs to be unitary. The requirement of a unitary blocking matrix can lead to a computationally complex implementation of the GSC structure. This is because the computations required for the multiplication of the input-signal vector with the blocking matrix may exceed the filtering operation by an order of magnitude. In these situations employing other approaches may be more efficient [9], [11]. For the case of non-unitary blocking matrices, the transient, or equivalently, the convergence speed, depends on the step size and the particular blocking matrix chosen. In other words, if the blocking matrix changes the step size changes, including the limits for stability. An equivalence-comparison of the transients has not yet been performed for the CRLS algorithm and the GSC structure employing an RLS algorithm, herein referred to as the GSC-RLS algorithm.

The goal of this paper is to investigate what are the requirements for transient-equivalence when considering the implementations of the CRLS and the GSC-RLS algorithms. The following question may be asked: Is the requirement of unitary blocking matrix related to the implementation of the LMS algorithm carried over to the case of the RLS algorithm? It is shown that this is not the case, and that the transients of the CRLS and the GSC-RLS algorithm are equal with probability one. As a consequence, any valid non-unitary blocking matrix used in the GSC-RLS structure will always produce the same curves as the CRLS algorithm.

## II. The Constrained Least-Squares Filter

In linearly constrained adaptive filtering, the constraints are given by a set of $p$ equations

$$\mathbf{C}^{\mathrm{H}}\mathbf{w} = \mathbf{f} \tag{1}$$

where $\mathbf{C}$ is a $N \times p$ constraint matrix and $\mathbf{f}$ is a vector containing the $p$ constraint values. The constrained recursive least-squares (CRLS) algorithm to be discussed in Section III solves the following optimization problem

$$\mathbf{w}(k) = \arg\min_{\mathbf{w}} \mathbf{e}^{\mathrm{H}}(k)\mathbf{e}(k) \quad \text{subject to} \quad \mathbf{C}^{\mathrm{H}}\mathbf{w} = \mathbf{f} \tag{2}$$

where the error vector $\mathbf{e}(k)$ is defined as

$$\mathbf{e}(k) = \mathbf{d}(k) - \mathbf{X}^{\mathrm{H}}(k)\mathbf{w} \tag{3}$$

and

$$\mathbf{d}(k) = \left[d(k)\ \lambda^{1/2}d(k-1)\ \cdots\ \lambda^{k/2}d(0)\right]^{\mathrm{T}} \tag{4}$$

$$\mathbf{X}(k) = \left[\mathbf{x}(k)\ \lambda^{1/2}\mathbf{x}(k-1)\ \cdots\ \lambda^{k/2}\mathbf{x}(0)\right] \tag{5}$$

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are the $(k+1)\times 1$ reference vector and the $N\times(k+1)$ input matrix, respectively, and $\lambda$ is the forgetting factor ($0<\lambda\leq 1$). Applying the method of Lagrange multipliers, the constrained LS solution at time instant $k$ is given by [13]

$$\mathbf{w}(k)=\mathbf{R}^{-1}(k)\mathbf{p}(k)+\mathbf{R}^{-1}(k)\mathbf{C}\times [\mathbf{C}^{\mathrm{H}}\mathbf{R}^{-1}(k)\mathbf{C}]^{-1}[\mathbf{f}-\mathbf{C}^{\mathrm{H}}\mathbf{R}^{-1}(k)\mathbf{p}(k)] \tag{6}$$

where $\mathbf{R}(k)$ is the $N\times N$ *deterministic* correlation matrix and $\mathbf{p}(k)$ is the $N\times 1$ *deterministic* cross-correlation vector, defined as

$$\mathbf{R}(k)=\mathbf{X}(k)\mathbf{X}^{\mathrm{H}}(k)=\sum_{i=0}^{k}\lambda^{k-i}\mathbf{x}(i)\mathbf{x}^{\mathrm{H}}(i) \tag{7}$$

$$\mathbf{p}(k)=\mathbf{X}(k)\mathbf{d}^{*}(k)=\sum_{i=0}^{k}\lambda^{k-i}\mathbf{x}(i)d^{*}(i). \tag{8}$$

## III. The constrained RLS algorithm

A recursive update of the optimal LS solution in (6) will be addressed in this section. Note that the solution in (6) can be divided into two terms

$$\mathbf{w}(k)=\mathbf{w}_{uc}(k)+\mathbf{w}_{c}(k) \tag{9}$$

where

$$\mathbf{w}_{uc}(k)=\mathbf{R}^{-1}(k)\mathbf{p}(k) \tag{10}$$

and

$$\mathbf{w}_{c}(k)=\mathbf{R}^{-1}(k)\mathbf{C}\times [\mathbf{C}^{\mathrm{H}}\mathbf{R}^{-1}(k)\mathbf{C}]^{-1}[\mathbf{f}-\mathbf{C}^{\mathrm{H}}\mathbf{R}^{-1}(k)\mathbf{p}(k)] \tag{11}$$

The coefficient-vector $\mathbf{w}_{uc}(k)$ is an unconstrained solution (the deterministic Wiener solution) and is independent of the constraints, whereas $\mathbf{w}_{c}(k)$ depends on the constraints imposed by $\mathbf{C}^{\mathrm{H}}\mathbf{w}(k)=\mathbf{f}$. The coefficient-vector $\mathbf{w}_{uc}(k)$ already has a recursive expression given by the unconstrained RLS algorithm [14]

$$\mathbf{w}_{uc}(k)=\mathbf{w}_{uc}(k-1)+e_{uc}^{*}(k)\boldsymbol{\kappa}(k) \tag{12}$$

where $\boldsymbol{\kappa}(k)=\mathbf{R}^{-1}(k)\mathbf{x}(k)$ is the gain vector, and $e_{uc}(k)=d(k)-\mathbf{w}_{uc}^{\mathrm{H}}(k-1)\mathbf{x}(k)$ is the *a priori* unconstrained error.

In order to derive a recursive update for $\mathbf{w}_{c}(k)$, let us define the auxiliary matrices $\boldsymbol{\Gamma}(k)$ and $\boldsymbol{\Psi}(k)$, which have dimensions $(N\times p)$ and $(p\times p)$, respectively.

$$\boldsymbol{\Gamma}(k)=\mathbf{R}^{-1}(k)\mathbf{C} \tag{13}$$

$$\boldsymbol{\Psi}(k)=\mathbf{C}^{\mathrm{H}}\boldsymbol{\Gamma}(k)=\mathbf{C}^{\mathrm{H}}\mathbf{R}^{-1}(k)\mathbf{C} \tag{14}$$

such that

$$\mathbf{w}_{c}(k)=\boldsymbol{\Gamma}(k)\boldsymbol{\Psi}^{-1}(k)[\mathbf{f}-\mathbf{C}^{\mathrm{H}}\mathbf{w}_{uc}(k)] \tag{15}$$

For the case of a single constraint, $\boldsymbol{\Psi}(k)$ is a scalar and computation of $\boldsymbol{\Psi}^{-1}(k)$ is trivial. In case of multiple constraints, a recursive formula will be required to reduce the computational complexity. Since the objective of this paper is to perform an equivalence study and not to derive efficient updating schemes, Table I shows only the basic recursions of the CRLS algorithm as stated by Equations (12)–(15). For a more efficient implementation of the CRLS algorithm, see [6] and [5]. For the equivalence study to be carried out in Section V we show in the Appendix that the CRLS recursions can be written as (see Appendix)

$$\mathbf{w}(k)=\mathbf{w}(k-1)+e^{*}(k)\mathbf{R}^{-1}(k)\mathbf{x}(k)- e^{*}(k)\mathbf{R}^{-1}(k)\mathbf{C}[\mathbf{C}^{\mathrm{H}}\mathbf{R}^{-1}(k)\mathbf{C}]^{-1}\mathbf{C}^{\mathrm{H}}\mathbf{R}^{-1}(k)\mathbf{x}(k) \tag{16}$$

Equation (16) is of pure theoretical interest, since it will not render an efficient implementation.

TABLE I
THE CONSTRAINED RLS ALGORITHM.

| |
|---|
| Initialization: |
| $\mathbf{w}_{uc}(0)$ and $\mathbf{R}^{-1}(0)$ |
| for $k=1,2,\ldots$ |
| { |
| $\mathbf{k}(k)=\mathbf{R}^{-1}(k-1)\mathbf{x}(k)$ |
| $\boldsymbol{\kappa}(k)=\frac{\mathbf{k}(k)}{\lambda+\mathbf{x}^{\mathrm{H}}(k)\mathbf{k}(k)}$ |
| $\mathbf{R}^{-1}(k)=\frac{1}{\lambda}\left[\mathbf{R}^{-1}(k-1)-\frac{\mathbf{k}(k)\mathbf{k}^{\mathrm{H}}(k)}{\lambda+\mathbf{x}^{\mathrm{H}}(k)\mathbf{k}(k)}\right]$ |
| $e_{uc}(k)=d(k)-\mathbf{w}_{uc}^{\mathrm{H}}(k-1)\mathbf{x}(k)$ |
| $\mathbf{w}_{uc}(k)=\mathbf{w}_{uc}(k-1)+e_{uc}^{*}(k)\boldsymbol{\kappa}(k)$ |
| $\boldsymbol{\Gamma}(k)=\mathbf{R}^{-1}(k)\mathbf{C}$ |
| $\boldsymbol{\Psi}(k)=\mathbf{C}^{\mathrm{H}}\boldsymbol{\Gamma}(k)$ |
| $\mathbf{w}(k)=\mathbf{w}_{uc}(k)+\boldsymbol{\Gamma}(k)\boldsymbol{\Psi}^{-1}(k)\left[\mathbf{f}-\mathbf{C}^{\mathrm{H}}\mathbf{w}_{uc}(k)\right]$ |
| } |

## IV. Generalized Sidelobe Canceller Structure

This section reviews the generalized sidelobe canceller (GSC) structure, and provides the necessary definitions for the equivalence study in the next section. For a more detailed treatment of the GSC structure, see, e.g, [2] and [14]. Many implementations of LC adaptive filters utilize the advantages of the GSC model [14], mainly because this model employs unconstrained adaptation algorithms that have been extensively studied in the literature. Figure 1 shows the schematic of the GSC model.

Matrix $\mathbf{B}$ in Figure 1 is a full-rank $N\times(N-p)$ blocking matrix designed to filter out completely the components of the input signal that are in the same directions as the constraints. Matrix $\mathbf{B}$ must span the null space of the constraint matrix $\mathbf{C}$, i.e., $\mathbf{B}^{\mathrm{H}}\mathbf{C}=\mathbf{0}$. The upper part of the GSC structure implements the constraints through the $N\times 1$ vector $\mathbf{F}$ defined as

$$\mathbf{F}=\mathbf{C}\left(\mathbf{C}^{\mathrm{H}}\mathbf{C}\right)^{-1}\mathbf{f} \tag{17}$$

The $(N-p)\times 1$ vector $\mathbf{w}_{\text{GSC}}$ can be adapted freely using any unconstrained adaptation algorithm. The desired signal

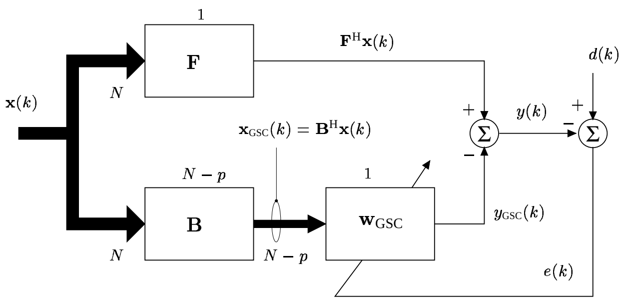


Fig. 1. Generalized sidelobe canceling (GSC) model.

as defined in Figure 1, incorporates an external reference signal $d(k)$ such that the resulting desired signal fed back to the adaptation algorithm in the GSC structure becomes $d_{\mathrm{GSC}}(k) = \mathbf{F}^{\mathrm{H}}\mathbf{x}(k) - d(k)$. This more general case includes common applications where $d(k) = 0$, e.g., blind beamforming [1] and blind multiuser detection [4].

The optimal least-squares solution of $\mathbf{w}_{\mathrm{GSC}}(k)$ is given by

$$\mathbf{w}_{\mathrm{GSC}}(k) = \mathbf{R}_{\mathrm{GSC}}^{-1}(k)\mathbf{p}_{\mathrm{GSC}}(k) \tag{18}$$

where $\mathbf{R}_{\mathrm{GSC}}(k)$ and $\mathbf{p}_{\mathrm{GSC}}(k)$ are the deterministic autocorrelation matrix the and cross-correlation vector, respectively. From Figure 1 it follows that

$$\begin{aligned}\mathbf{R}_{\mathrm{GSC}}(k) &= \sum_{i=0}^{k} \lambda^{k-i}\mathbf{x}_{\mathrm{GSC}}(k)\mathbf{x}_{\mathrm{GSC}}^{\mathrm{H}}(k) \\ &= \mathbf{B}^{\mathrm{H}}\mathbf{R}(k)\mathbf{B}\end{aligned} \tag{19}$$

$$\begin{aligned}\mathbf{p}_{\mathrm{GSC}}(k) &= \sum_{i=0}^{k} \lambda^{k-i}[\mathbf{F}^{\mathrm{H}}\mathbf{x}(k) - d(k)]^*[\mathbf{B}^{\mathrm{H}}\mathbf{x}(k)] \\ &= -\mathbf{B}^{\mathrm{H}}\mathbf{p}(k) + \mathbf{B}^{\mathrm{H}}\mathbf{R}(k)\mathbf{F}\end{aligned} \tag{20}$$

The RLS recursions for the GSC structure becomes

$$\mathbf{w}_{\mathrm{GSC}}(k) = \mathbf{w}_{\mathrm{GSC}}(k-1) + e_{\mathrm{GSC}}^*(k)\boldsymbol{\kappa}_{\mathrm{GSC}}(k) \tag{21}$$

where $e_{\mathrm{GSC}}(k) = d_{\mathrm{GSC}}(k) - \mathbf{w}_{\mathrm{GSC}}^{\mathrm{H}}(k-1)\mathbf{x}_{\mathrm{GSC}}(k) = -e(k)$ is the *a priori error* and $\boldsymbol{\kappa}_{\mathrm{GSC}}(k) = \mathbf{R}_{\mathrm{GSC}}^{-1}(k)\mathbf{x}_{\mathrm{GSC}}(k)$ is the gain vector. In next section we will compare the recursions in Equation (21) with those of the CRLS algorithm in Equation (16).

## V. Equivalence of the CRLS and GSC-RLS implementations

This section addresses the relationship between the CRLS algorithm in Section III and the GSC-RLS algorithm in Section IV. It is well known that the CRLS and the GSC-RLS formulations have the same optimal solution [13]. However, no results comparing their transient behavior have yet been provided. Analysis of the CLMS and GSC-LMS algorithms reveals that the transients of both algorithms only become equal if $\mathbf{B}$ is unitary, i.e., $\mathbf{B}^{\mathrm{H}}\mathbf{B} = \mathbf{I}$ [2]. Our goal in this section is to investigate under what circumstances the transients of the CRLS and the GSC-RLS algorithms are identical with probability one.

We will study the coefficient-vector evolution defined as

$$\Delta\mathbf{w}(k) = \mathbf{w}(k) - \mathbf{w}(k-1) \tag{22}$$

Equation (16) gives us the coefficient-vector evolution for the CRLS algorithm as

$$\begin{aligned}\Delta\mathbf{w}(k) = e^*(k)&\left\{\mathbf{I} - \mathbf{R}^{-1}(k)\mathbf{C}\left[\mathbf{C}^{\mathrm{H}}\mathbf{R}^{-1}(k)\mathbf{C}\right]^{-1}\mathbf{C}^{\mathrm{H}}\right\} \\ &\times \mathbf{R}^{-1}(k)\mathbf{x}(k)\end{aligned} \tag{23}$$

For the GSC-RLS algorithm, considering that $\mathbf{w}(k) = \mathbf{F} - \mathbf{B}\mathbf{w}_{GSC}(k)$, Equation (21) gives us

$$\begin{aligned}\Delta\mathbf{w}(k) &= \mathbf{F} - \mathbf{B}\mathbf{w}_{\mathrm{GSC}}(k) - \mathbf{w}(k-1) \\ &= \mathbf{F} - \mathbf{B}\left[\mathbf{w}_{\mathrm{GSC}}(k-1) + \right. \\ &\quad \left. e_{\mathrm{GSC}}^*(k)\mathbf{R}_{\mathrm{GSC}}^{-1}(k)\mathbf{x}_{\mathrm{GSC}}(k)\right] - \mathbf{w}(k-1) \\ &= e^*(k)\mathbf{B}[\mathbf{B}^{\mathrm{H}}\mathbf{R}(k)\mathbf{B}]^{-1}\mathbf{B}^{\mathrm{H}}\mathbf{x}(k) \\ &= e^*(k)\left\{\mathbf{B}\left[\mathbf{B}^{\mathrm{H}}\mathbf{R}(k)\mathbf{B}\right]^{-1}\mathbf{B}^{\mathrm{H}}\mathbf{R}(k)\right\}\mathbf{R}^{-1}(k)\mathbf{x}(k)\end{aligned} \tag{24}$$

where $\mathbf{w}(k-1) = \mathbf{F} - \mathbf{B}\mathbf{w}_{\mathrm{GSC}}(k-1)$ was used together with Equation (19). In order for Equations (23) and (24) to be identical it is required that the following matrix equality holds:

$$\begin{aligned}&\mathbf{B}(\mathbf{B}^{\mathrm{H}}\mathbf{R}(k)\mathbf{B})^{-1}\mathbf{B}^{\mathrm{H}}\mathbf{R}(k) \\ &\quad + \mathbf{R}^{-1}(k)\mathbf{C}(\mathbf{C}^{\mathrm{H}}\mathbf{R}^{-1}(k)\mathbf{C})^{-1}\mathbf{C}^{\mathrm{H}} = \mathbf{I}\end{aligned} \tag{25}$$

The initialization of both schemes (CRLS and GSC-RLS) should be equivalent, which means $\mathbf{R}_{\mathrm{GSC}}^{-1}(0) = \left[\mathbf{B}^{\mathrm{H}}\mathbf{R}(0)\mathbf{B}\right]^{-1}$ and $\mathbf{w}(0) = \mathbf{F}$ for the particular case where $\mathbf{w}_{\mathrm{GSC}}(0) = \mathbf{0}$. The equivalence of the CRLS and the GSC-RLS using the correct initialization is ensured by the following lemma:

**Lemma 1.** *For $\mathbf{B}^H\mathbf{C} = \mathbf{0}$, if $\mathbf{R}^{-1}(k)$ exists and is symmetric, if rank$(\mathbf{B}) = N - p$, and if rank$(\mathbf{C}) = p$, Equation (25) holds true.*

*Proof:* Define the matrices $\bar{\mathbf{B}} = \mathbf{R}^{\mathrm{H}/2}(k)\mathbf{B}$ and $\bar{\mathbf{C}} = \mathbf{R}^{-1/2}(k)\mathbf{C}$ where $\mathbf{R}(k) = \mathbf{R}^{1/2}(k)\mathbf{R}^{\mathrm{H}/2}(k)$. With these notations, the left hand side of Equation (25) becomes $\bar{\mathbf{B}}(\bar{\mathbf{B}}^{\mathrm{H}}\bar{\mathbf{B}})^{-1}\bar{\mathbf{B}}^{\mathrm{H}} + \bar{\mathbf{C}}(\bar{\mathbf{C}}^{\mathrm{H}}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}^{\mathrm{H}}$, and it remains to show that this addition of matrices equals identity. For this purpose, let us introduce the matrix $\mathbf{T} = [\bar{\mathbf{C}}\ \bar{\mathbf{B}}]$. $\mathbf{T}$ is a full-rank $(N \times N)$ matrix, and, consequently, $\mathbf{T}^{-1}$ exists. We have,

$$\mathbf{T}^{\mathrm{H}}\mathbf{T} = \begin{bmatrix}\bar{\mathbf{C}}^{\mathrm{H}}\bar{\mathbf{C}} & \bar{\mathbf{C}}^{\mathrm{H}}\bar{\mathbf{B}} \\ \bar{\mathbf{B}}^{\mathrm{H}}\bar{\mathbf{C}} & \bar{\mathbf{B}}^{\mathrm{H}}\bar{\mathbf{B}}\end{bmatrix} = \begin{bmatrix}\bar{\mathbf{C}}^{\mathrm{H}}\bar{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{B}}^{\mathrm{H}}\bar{\mathbf{B}}\end{bmatrix} \tag{26}$$

where the relation $\bar{\mathbf{B}}^{\mathrm{H}}\bar{\mathbf{C}} = \mathbf{0}$ was used. We have

$$(\mathbf{T}^{\mathrm{H}}\mathbf{T})^{-1} = \begin{bmatrix}(\bar{\mathbf{C}}^{\mathrm{H}}\bar{\mathbf{C}})^{-1} & \mathbf{0} \\ \mathbf{0} & (\bar{\mathbf{B}}^{\mathrm{H}}\bar{\mathbf{B}})^{-1}\end{bmatrix} \tag{27}$$

Therefore,

$$\begin{aligned}\mathbf{T}(\mathbf{T}^{\mathrm{H}}\mathbf{T})^{-1}\mathbf{T}^{\mathrm{H}} &= \mathbf{T}\mathbf{T}^{-1}\mathbf{T}^{-H}\mathbf{T}^{\mathrm{H}} = \mathbf{I} \\ &= [\bar{\mathbf{C}}\ \bar{\mathbf{B}}]\begin{bmatrix}(\bar{\mathbf{C}}^{\mathrm{H}}\bar{\mathbf{C}})^{-1} & \mathbf{0} \\ \mathbf{0} & (\bar{\mathbf{B}}^{\mathrm{H}}\bar{\mathbf{B}})^{-1}\end{bmatrix}\begin{bmatrix}\bar{\mathbf{C}}^{\mathrm{H}} \\ \bar{\mathbf{B}}^{\mathrm{H}}\end{bmatrix} \\ &= \bar{\mathbf{C}}(\bar{\mathbf{C}}^{\mathrm{H}}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}^{\mathrm{H}} + \bar{\mathbf{B}}(\bar{\mathbf{B}}^{\mathrm{H}}\bar{\mathbf{B}})^{-1}\bar{\mathbf{B}}^{\mathrm{H}} = \mathbf{I}\end{aligned}$$

□

As a consequence of Lemma 1, and Equations (23) and (24), we can conclude that the necessary requirement for equivalent transients of the CRLS and the GSC-RLS algorithms is that $\mathbf{B}^{\mathrm{H}}\mathbf{C} = \mathbf{0}$, which holds true in any GSC structure. This is a looser requirement than the transient-equivalence of the CLMS and GSC-LMS algorithms, which in addition to $\mathbf{B}^{\mathrm{H}}\mathbf{C} = \mathbf{0}$, requires $\mathbf{B}$ to be unitary.

## VI. Simulations

In this section the equivalence of the CRLS and GSC-RLS algorithms are investigated in two applications. The first application is a beamforming application where the desired signal is set to zero, i.e., $d(k) = 0$. The second application using the more general desired signal with $d(k) \neq 0$ is a system-identification application where the adaptive filter is constrained to have linear phase.

### A. Beamforming with derivative constraints

A uniform linear array with $M = 12$ antennas with element spacing equal to half wave-length was used in a system with $K = 5$ users, where the signal of one user is of interest and the other 4 are treated as interferers. The direction of arrival (DOA) and the signal-to-noise ratio (SNR) for the different signals can be found in Table II.

A second-order derivative constraint matrix [15] was used giving a total of three constraints (see [15] for further details). The GSC implementation used a non-unitary blocking matrix constructed through a sequence of sparse matrices as presented in [16] rendering an implementation of the multiplication $\mathbf{Bx}(k)$ of low computational complexity.

The simulations were averaged over 50 trials and both the CRLS and the GSC-RLS algorithms used $\lambda = 0.99$. Figure 2 shows the evolution of coefficient-error norm for the CRLS and the GSC-RLS algorithms. Figure 2 also plots the results for the CLMS and the GSC-LMS algorithms. As can be seen from the figure, the CLMS and the GSC-LMS algorithms only become identical when using the unitary blocking matrix, whereas the CRLS and the GSC-RLS algorithms are identical for the non-unitary blocking matrix. This fact is further illustrated in Figure 3, where the norm of the difference between the CRLS and the GSC-RLS solutions is plotted.

TABLE II

SIGNAL PARAMETERS

| SIGNAL | DOA | SNR |
|---|---|---|
| desired | $0^{\circ}$ | 15 dB |
| interferer 1 | $22^{\circ}$ | 20 dB |
| interferer 2 | $-15^{\circ}$ | 25 dB |
| interferer 3 | $-20^{\circ}$ | 25 dB |
| interferer 4 | $-50^{\circ}$ | 20 dB |

### B. Identification of plant with linear phase

An experiment was carried out in a system-identification problem where the filter coefficients were constrained to preserve linear phase at every iteration. For this example we used

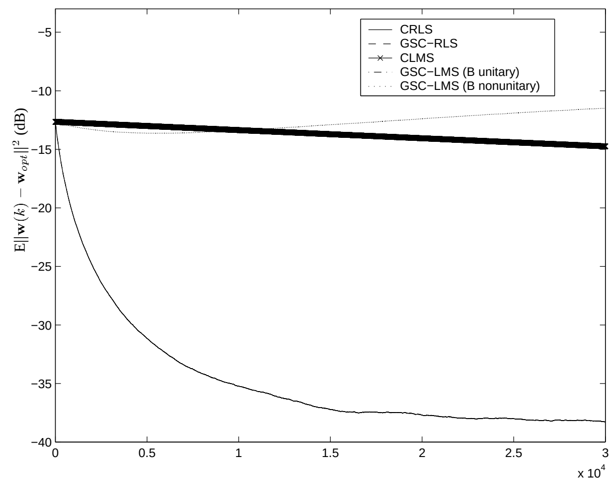


Fig. 2. Coefficient-error vector as a function of the iteration $k$ for a beamforming application using derivative constraints.

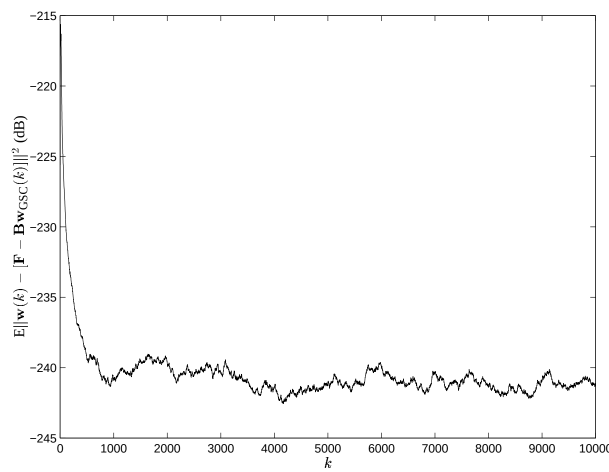


Fig. 3. Norm of the difference of the CRLS and the GSC-RLS coefficient vectors as a function of the iteration $k$ for a beamforming application using derivative constraints.

$N = 10$ and, in order to fulfill the linear phase requirement, we made

$$\mathbf{C} = \begin{bmatrix} \mathbf{I}_{N/2} \\ \mathbf{0}^{\mathrm{T}} \\ -\mathbf{J}_{N/2} \end{bmatrix} \tag{28}$$

with $\mathbf{J}$ being a reversal matrix (an identity matrix with all lines in reversed order), and

$$\mathbf{f} = [0 \cdots 0]^{\mathrm{T}} \tag{29}$$

Due to the symmetry of $\mathbf{C}$ and the fact that $\mathbf{f}$ is a null vector, efficient structures can be employed to construct the blocking matrix [5]. The GSC implementation used a non-unitary blocking matrix given by

$$\mathbf{B} = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 1 \\ \mathbf{J}_{N/2} & \mathbf{0} \end{bmatrix} \tag{30}$$

The input signal consists of zero-mean unity-variance colored noise with eigenvalue spread around $1350$ and the reference signal was obtained after filtering the input by a linear-phase FIR filter and adding measurement noise with variance equal to $10^{-6}$.

The simulations were averaged over 50 trials and both the CRLS and the GSC-RLS algorithms used $\lambda = 0.95$. Figure 4 shows the evolution of coefficient-error norm for the CRLS and the GSC-RLS algorithms. Similarly as in the beamforming example, the curves for the CRLS and the GSC-RLS algorithms are identical, and the CLMS and the GSC-LMS become identical only when the blocking matrix is unitary. Figure 5, plots the norm of the difference between the CRLS and the GSC-RLS solutions.

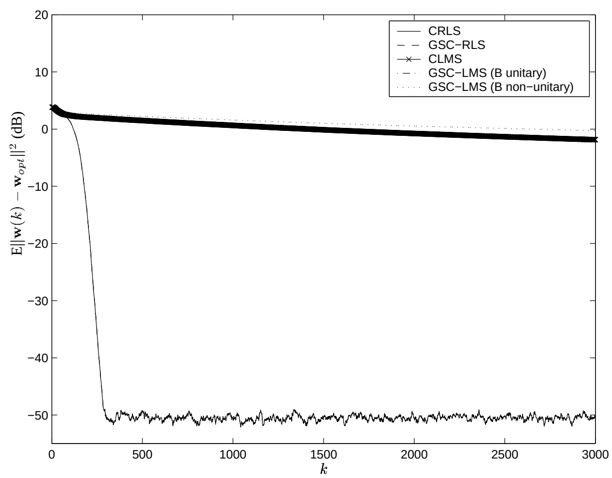


Fig. 4. Coefficient-error vector as a function of the iteration $k$ for a system-identification application.

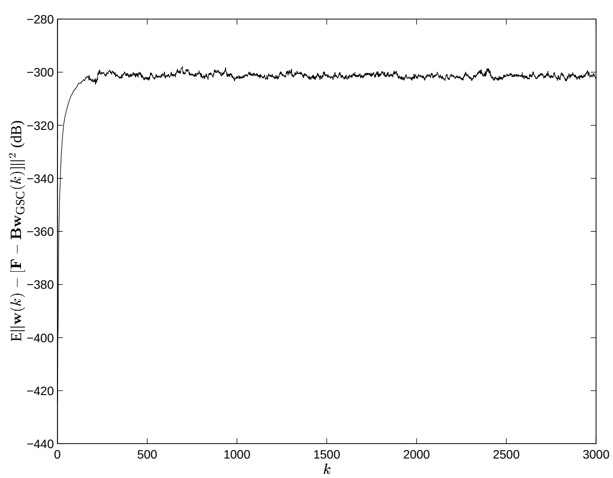


Fig. 5. Norm of the difference of the CRLS and the GSC-RLS coefficient vectors as a function of the iteration $k$ for a system-identification application.

## VII. **Conclusions**

This paper investigated the condition for which the transients of the constrained RLS algorithm and the GSC structure using an RLS algorithm become equal. It was shown that for a proper initialization, the only requirement is that the blocking matrix of the GSC structure is orthogonal to the space spanned by the constraints, which is always true for a properly designed GSC.

## **Appendix**

In this Appendix it is shown that the CRLS algorithm [6] can be written on the form given by Equation (16). Let us start by finding a recursive expression of $\mathbf{\Gamma}(k)$. We first note that $\mathbf{R}^{-1}(k)$ can be written as [14]

$$\mathbf{R}^{-1}(k) = \frac{1}{\lambda}[\mathbf{R}^{-1}(k-1) - \mathbf{R}^{-1}(k)\mathbf{x}(k)\mathbf{x}^{\mathrm{H}}(k)\mathbf{R}^{-1}(k-1)] \tag{31}$$

Using (31) in (13) results

$$\begin{aligned}\mathbf{\Gamma}(k) &= \frac{1}{\lambda}\left[\mathbf{R}^{-1}(k-1)\mathbf{C}\right.\\ &\quad \left.-\mathbf{R}^{-1}(k)\mathbf{x}(k)\mathbf{x}^{\mathrm{H}}(k)\mathbf{R}^{-1}(k-1)\mathbf{C}\right]\\ &= \frac{1}{\lambda}\left[\mathbf{\Gamma}(k-1) - \boldsymbol{\kappa}(k)\mathbf{x}^{\mathrm{H}}(k)\mathbf{\Gamma}(k-1)\right]\end{aligned} \tag{32}$$

In order to obtain a recursive expression for $\mathbf{\Psi}^{-1}(k)$, pre-multiply (32) by $\mathbf{C}^{\mathrm{H}}$ and apply the *Matrix Inversion Lemma*

$$\begin{aligned}\mathbf{\Psi}^{-1}(k) &= \lambda\left\{\mathbf{C}^{\mathrm{H}}\mathbf{\Gamma}(k-1) - \mathbf{C}^{\mathrm{H}}\boldsymbol{\kappa}(k)\mathbf{x}^{\mathrm{H}}(k)\mathbf{\Gamma}(k-1)\right\}^{-1}\\ &= \lambda\left[\mathbf{\Psi}^{-1}(k-1)\right.\\ &\quad \left.+\frac{\mathbf{\Psi}^{-1}(k-1)\mathbf{C}^{\mathrm{H}}\boldsymbol{\kappa}(k)\mathbf{x}^{\mathrm{H}}(k)\mathbf{\Gamma}(k-1)\mathbf{\Psi}^{-1}(k-1)}{1-\mathbf{x}^{\mathrm{H}}(k)\mathbf{\Gamma}(k-1)\mathbf{\Psi}^{-1}(k-1\mathbf{C}^{\mathrm{H}}\boldsymbol{\kappa}(k)}\right]\end{aligned} \tag{33}$$

In order to simplify the notation, define $\boldsymbol{\ell}(k)$ as

$$\boldsymbol{\ell}(k) = \frac{\mathbf{\Psi}^{-1}(k-1)\mathbf{C}^{\mathrm{H}}\boldsymbol{\kappa}(k)}{1-\mathbf{x}^{\mathrm{H}}(k)\mathbf{\Gamma}(k-1)\mathbf{\Psi}^{-1}(k-1)\mathbf{C}^{\mathrm{H}}\boldsymbol{\kappa}(k)} \tag{34}$$

which gives

$$\begin{aligned}\mathbf{\Psi}^{-1}(k) &= \lambda\left[\mathbf{\Psi}^{-1}(k-1)\right.\\ &\quad \left.+\boldsymbol{\ell}(k)\mathbf{x}^{\mathrm{H}}(k)\mathbf{\Gamma}(k-1)\mathbf{\Psi}^{-1}(k-1)\right]\end{aligned} \tag{35}$$

From (34), we know that

$$\begin{aligned}\boldsymbol{\ell}(k) &= \boldsymbol{\ell}(k)\mathbf{x}^{\mathrm{H}}(k)\mathbf{\Gamma}(k-1)\mathbf{\Psi}^{-1}(k-1)\mathbf{C}^{\mathrm{H}}\boldsymbol{\kappa}(k)\\ &\quad +\mathbf{\Psi}^{-1}(k-1)\mathbf{C}^{\mathrm{H}}\boldsymbol{\kappa}(k)\end{aligned} \tag{36}$$

Post-multiplying (33) by $\mathbf{C}^{\mathrm{H}}\boldsymbol{\kappa}(k)$ and divide by $\lambda$ gives the same expression as in (36), therefore

$$\boldsymbol{\ell}(k) = \frac{1}{\lambda}\mathbf{\Psi}^{-1}(k)\mathbf{C}^{\mathrm{H}}\boldsymbol{\kappa}(k) \tag{37}$$

To show the formulation of the CRLS algorithm given by (16), substitute $e_{uc}(k)$ by $d(k) - \mathbf{w}_{uc}^{\mathrm{H}}(k-1)\mathbf{x}(k)$, and $\mathbf{\Psi}(k)$ and $\mathbf{\Gamma}(k)$ by their recursive expressions. Using the recursions given by Equations (9) and (11), the coefficient update for the CRLS algorithm is given by

$$
\begin{aligned}
\mathbf{w}(k) &= \mathbf{w}_{uc}(k) + \mathbf{w}_c(k) \\
&= \mathbf{w}_{uc}(k-1) + \boldsymbol{\Gamma}(k)\boldsymbol{\Psi}^{-1}(k)[\mathbf{f} - \mathbf{C}^{\mathrm{H}}\mathbf{w}_{uc}(k-1)] \\
&\quad + e_{uc}^*(k)\boldsymbol{\kappa}(k) - \boldsymbol{\Gamma}(k)\boldsymbol{\Psi}^{-1}(k)\mathbf{C}^{\mathrm{H}}\boldsymbol{\kappa}(k)e_{uc}^*(k) \\
&= \mathbf{w}_{uc}(k-1) + \boldsymbol{\Gamma}(k-1)\boldsymbol{\Psi}^{-1}(k-1)\left[\mathbf{f} - \mathbf{C}^{\mathrm{H}}\mathbf{w}_{uc}(k-1)\right] \\
&\quad - \boldsymbol{\kappa}(k)\mathbf{x}^{\mathrm{H}}(k)\boldsymbol{\Gamma}(k-1)\boldsymbol{\Psi}^{-1}(k-1)\left[\mathbf{f} - \mathbf{C}^{\mathrm{H}}\mathbf{w}_{uc}(k-1)\right] \\
&\quad + e_{uc}^*(k)\boldsymbol{\kappa}(k) \\
&\quad + \boldsymbol{\Gamma}(k-1)\boldsymbol{\ell}(k)\mathbf{x}^{\mathrm{H}}(k)\boldsymbol{\Gamma}(k-1)\boldsymbol{\Psi}^{-1}(k-1) \times \\
&\qquad \left[\mathbf{f} - \mathbf{C}^{\mathrm{H}}\mathbf{w}_{uc}(k-1)\right] \\
&\quad - \boldsymbol{\kappa}(k)\mathbf{x}^{\mathrm{H}}(k)\boldsymbol{\Gamma}(k-1)\boldsymbol{\ell}(k)\mathbf{x}^{\mathrm{H}}(k)\boldsymbol{\Gamma}(k-1)\boldsymbol{\Psi}^{-1}(k-1) \times \\
&\qquad \left[\mathbf{f} - \mathbf{C}^{\mathrm{H}}\mathbf{w}_{uc}(k-1)\right] \\
&\quad - \boldsymbol{\Gamma}(k)\boldsymbol{\Psi}^{-1}(k)\mathbf{C}^{\mathrm{H}}\boldsymbol{\kappa}(k)e_{uc}^*(k)
\end{aligned} \tag{38}
$$

In Equation (38) the term $\mathbf{w}_{uc}(k-1) + \boldsymbol{\Gamma}(k-1)\boldsymbol{\Psi}^{-1}(k-1)\left[\mathbf{f} - \mathbf{C}^{\mathrm{H}}\mathbf{w}_{uc}(k-1)\right]$ corresponds to $\mathbf{w}(k-1)$. Furthermore with $e_{uc}^*(k) = d^*(k) - \mathbf{x}^{\mathrm{H}}(k)\mathbf{w}_{uc}(k-1)$, the second and third expressions correspond to $\boldsymbol{\kappa}(k)\left(d^*(k) - \mathbf{x}^{\mathrm{H}}(k)\{\mathbf{w}_{uc}(k-1) + \boldsymbol{\Gamma}(k-1)\boldsymbol{\Psi}^{-1}(k-1)\times [\mathbf{f} - \mathbf{C}^{\mathrm{H}}\mathbf{w}_{uc}(k-1)]\}\right)$ which simplifies to $\boldsymbol{\kappa}(k)[d^*(k) - \mathbf{x}^{\mathrm{H}}(k)\mathbf{w}(k-1)] = \boldsymbol{\kappa}(k)e^*(k)$. As a consequence Equation (38) simplifies to

$$
\begin{aligned}
\mathbf{w}(k) &= \mathbf{w}(k-1) + \boldsymbol{\kappa}(k)e^*(k) \\
&\quad + \boldsymbol{\Gamma}(k-1)\boldsymbol{\ell}(k)\mathbf{x}^{\mathrm{H}}(k)\boldsymbol{\Gamma}(k-1)\boldsymbol{\Psi}^{-1}(k-1) \times \\
&\quad \left[\mathbf{f} - \mathbf{C}^{\mathrm{H}}\mathbf{w}_{uc}(k-1)\right] \\
&\quad - \boldsymbol{\kappa}(k)\mathbf{x}^{\mathrm{H}}(k)\boldsymbol{\Gamma}(k-1)\boldsymbol{\ell}(k)\mathbf{x}^{\mathrm{H}}(k)\boldsymbol{\Gamma}(k-1)\boldsymbol{\Psi}^{-1}(k-1) \times \\
&\quad \left[\mathbf{f} - \mathbf{C}^{\mathrm{H}}\mathbf{w}_{uc}(k-1)\right] \\
&\quad - \boldsymbol{\Gamma}(k)\underbrace{\boldsymbol{\Psi}^{-1}(k)\mathbf{C}^{\mathrm{H}}\boldsymbol{\kappa}(k)}_{\lambda\boldsymbol{\ell}(k)}[d^*(k) - \mathbf{x}^{\mathrm{H}}(k)\mathbf{w}_{uc}(k-1)] \\
&= \mathbf{w}(k-1) + \boldsymbol{\kappa}(k)e^*(k) \\
&\quad + \underbrace{[\boldsymbol{\Gamma}(k-1) - \boldsymbol{\kappa}(k)\mathbf{x}^{\mathrm{H}}(k)\boldsymbol{\Gamma}(k-1)]}_{\lambda\boldsymbol{\Gamma}(k)} \\
&\quad \times \boldsymbol{\ell}(k)\mathbf{x}^{\mathrm{H}}(k)\boldsymbol{\Gamma}(k-1)\boldsymbol{\Psi}^{-1}(k-1)\left[\mathbf{f} - \mathbf{C}^{\mathrm{H}}\mathbf{w}_{uc}(k-1)\right] \\
&\quad - \boldsymbol{\Gamma}(k)\lambda\boldsymbol{\ell}(k)[d^*(k) - \mathbf{x}^{\mathrm{H}}(k)\mathbf{w}_{uc}(k-1)] \\
&= \mathbf{w}(k-1) + \boldsymbol{\kappa}(k)e^*(k) \\
&\quad + \lambda\boldsymbol{\Gamma}(k)\boldsymbol{\ell}(k)\mathbf{x}^{\mathrm{H}}(k)\boldsymbol{\Gamma}(k-1)\boldsymbol{\Psi}^{-1}(k-1) \times \\
&\quad \left[\mathbf{f} - \mathbf{C}^{\mathrm{H}}\mathbf{w}_{uc}(k-1)\right] \\
&\quad - \lambda\boldsymbol{\Gamma}(k)\boldsymbol{\ell}(k)[d^*(k) - \mathbf{x}^{\mathrm{H}}(k)\mathbf{w}_{uc}(k-1)] \\
&= \mathbf{w}(k-1) + \boldsymbol{\kappa}(k)e^*(k) - \lambda\boldsymbol{\Gamma}(k)\boldsymbol{\ell}(k) \\
&\quad \times \left(d^*(k) - \mathbf{x}^{\mathrm{H}}(k) \times \right. \\
&\left. \underbrace{\{\mathbf{w}_{uc}(k-1) + \boldsymbol{\Gamma}(k-1)\boldsymbol{\Psi}^{-1}(k-1)\left[\mathbf{f} - \mathbf{C}^{\mathrm{H}}\mathbf{w}_{uc}(k-1)\right]\}}_{\mathbf{w}(k-1)}\right) \\
&= \mathbf{w}(k-1) + \boldsymbol{\kappa}(k)e^*(k) - \lambda\boldsymbol{\Gamma}(k)\boldsymbol{\ell}(k) \times \\
&\quad \left[d(k) - \mathbf{w}^{\mathrm{H}}(k-1)\mathbf{x}(k)\right]^* \\
&= \mathbf{w}(k-1) + e^*(k)\boldsymbol{\kappa}(k) - \lambda e^*(k)\boldsymbol{\Gamma}(k)\boldsymbol{\ell}(k)
\end{aligned} \tag{39}
$$

which is equal to the CRLS update given by (16).

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