Topological Structures Associated with Discrete Memoryless Channels

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Abstract—**The aim of this paper is to identify the algebraic and geometric structures associated to discrete memoryless channels. The procedure employed to achieve this goal is based on the following steps: knowing the graph associated to discrete memoryless channel, 1) to determine the set of surfaces in which the graph is embedded; 2) to establish the set of algebraic structures inherited by the surfaces through the first homology group; and 3) to identify the regular tessellations which may be used in the design of modulators and quantizers.**

I. Introduction

TWO were the facts that led us to consider in this paper the geometric and algebraic aspects associated to the surfaces where the discrete memoryless channels are embedded.

Fact 1. Based on the error probability criterion, the communication system using QAM constellations achieve better performance than the communication system using PSK constellations under the constraint that the constellations have the same average energy.

Fact 2. Based on the error probability criterion, the performance of a binary digital communication system using soft-decision decoding (an 8-level quantizer, leading to a binary input, 8-ary output symmetric channel denoted by $C_{2,8}\,[8,2]$) achieves a coding gain of up to 2 dB when compared to the performance of a binary digital communication system using hard-decision decoding (a 2-level quantizer, leading to a binary symmetric channel (BSC), denoted by $C_{2,2}\,[2,2]$).

Although, the PSK and QAM constellations as well as the BSC and $C_{2,8}\,[8,2]$ channels, certainly have geometrical (topological) differences, not known presently, and which may help elucidating the better performance achieved in the previous two facts. In order to support this statement and to motivate going deeper into its fundamentals, consider the following case: it is known that the Slepian type of constellations (where the M-PSK is a particular case) are on the surface of an n-dimensional sphere, and that the QAM type of constellations are on the surface of a torus. Therefore, the most evident topological difference is the genus of the surface. To the best of our knowledge, there is no previous work on the identification of the surfaces characterizing the discrete memoryless channels, or equivalently, the surfaces on which theses channels may be embedded.

The aim of this paper is to provide the elements to realize the identification of the surfaces associated to DMC channels as well as in the characterization of its algebraic structures. In this direction, consider the binary input channel. Assume that an 8-level quantizer is used in the demodulator´s output, leading to a $C_{2,8}[8,2]$ channel having the complete bipartite graph $K_{2,8}$ as the associated graph. The $C_{2,2}[2,2]$ channel is associated to the graph $K_{2,2}$ which is embedded on a sphere with two regions, denoted by $S(2)$, thus on a surface of genus zero. In Section IV we show that the $C_{2,8}[8,2]$ channel may be embedded on the following surfaces: sphere with 8 regions, denoted by $S\,(8)$; torus with 6 regions, denoted by $T\,(6)$; two torus with 4 regions, denoted by $2T\,(4)$; and three torus with 2 regions, denoted by $3T\,(2)$. Note that the graph $K_{2,2}$ as well as the graph $K_{2,8}$ may be embedded on a sphere. Only this fact does not lead to a convincing answer, since the embedding of the $C_{2,2}[2,2]$ and $C_{2,8}[8,2]$ channels on a sphere are realized with two and eight regions, respectively. On the other hand, the $C_{2,8}[8,2]$ channel is used to transmit binary digits, and so the space has to be partitioned into two decision regions, that is, $C_{2,8}[8,2] \hookrightarrow 3T\,(2)$ is the only embedding representing this case. Since the $C_{2,2}[2,2]$ channel is embedded on $S\,(2)$, denoted by $C_{2,2}[2,2] \hookrightarrow S\,(2)$, and the $C_{2,8}[8,2]$ channel is embedded on $3T\,(2)$, denoted by $C_{2,8}[8,2] \hookrightarrow 3T\,(2)$, this leads to the conjecture that the error probability depends on the genus of the surface. Note that this conclusion is identical to the one achieved when considering the signal constellations.

From the intuitive point of view it is a simple matter to convince ourselves about the conjecture stated in the previous paragraph, however a proof of this statement is beyond the scope of this paper, see [10].

This paper is organized as follows. In Section II, we present the definitions, concepts, and results involving embedding of a graph on surfaces; the corresponding minimum and maximum genus of these embedding; the corresponding homology groups as well as the identification of the set of surfaces and the DMC channel classes with varying degrees. In Section III, by use of a systematic procedure we identify the topological structures of a compact surface, by use of the concept of a 2-cell embedding, of the graph associated to the DMC channel. As a result, the first homology group of the surface is equivalent to the symmetry group associated to the signal constellation. Certainly, a subgroup of it will act transitively on the signal constellation, and also will label the signals in the signal constellation. Furthermore, the model of the 2-cell embedding is a topological surface covered by polygonal regions, and when these regions have the same number of sides, results in a *regular tessellation* on the surface, a necessary condition to have a geometrically uniform signal constellation, [1]. In Section IV, we show by examples a variety of embedding of complete bipartite graphs associated

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conclusions are drawn.

II. Definitions, Concepts, and Results

In this section, we present the main concepts, definitions, and results related to the identification of the geometric and algebraic structures associated to a DMC channel.

We say that a graph G is *embedded* on a surface Ω, when its sides and vertices meet only in its sides and vertices, that is, there is no crossing of sides.

A *complete bipartite graph* with m and n vertices, denoted by $K_{m,n}$, is a graph consisting of two disjoint sets with m and n vertices, where each vertex of a set is connected by a side to every vertex of the other set.

Theorem II.1: [2] For $m, n \geq 3$, the Euler characteristic of the complete bipartite graph $K_{m,n}$ is given by

$$\gamma\left(K_{m,n}\right) = 2\{\left(m+n-mn/2\right)/2\}, \tag{1}$$

where $\{\alpha\}$ denotes the least integer greater than or equal to the real number α.

Definition II.1: [9] A graph G is called an *embedding* in a closed oriented variety Ω, if the geometric realization of G as a 1-complex is homeomorphic to a subspace of Ω. The components of the complement of G in Ω are called *regions*. A region which is homeomorphic to an open disc is called *2-cell*; if the entire region is a 2-cell, the embedding is said to be a *2-cell embedding*. It is known that if G is connected, then the minimum embedding is a 2-cell embedding.

The minimum and the maximum bounds of the genus of the surfaces associated to the embedding of the complete bipartite graph $K_{m,n}$ are:
(i) the minimum genus of an oriented surface is, [7]:

$$g_m\left(K_{m,n}\right) = \{\left(m-2\right)\left(n-2\right)/4\}, \text{ para } m, n \geq 2, \tag{2}$$

where $\{a\}$ denotes the least integer greater than or equal to the real number a.
(ii) the maximum genus of an oriented surface is, [5]:

$$g_M\left(K_{m,n}\right) = \left[\left(m-1\right)\left(n-1\right)/2\right], \text{ para } m, n \geq 1, \tag{3}$$

where $[a]$ denotes the greatest integer less than or equal to the real number a.
(iii) the minimum genus of a non-oriented surface is, [8]:

$$g\left(K_{m,n}\right) = \left[\left(m-2\right)\left(n-2\right)/2\right]. \tag{4}$$

Theorem II.2: [6] If a graph G has a 2-cell embedding on surfaces of genus g_1 and g_2, then for every integer k, $g_1 \leq k \leq g_2$, G has a 2-cell embedding on a surface of genus k.

Definition II.2: [4] A *bi-dimensional variety with border* is a Hausdorff space such that each point either has an open neighborhood homeomorphic to an open disc or it is homeomorphic to a half disc. The set of points having a neighborhood homeomorphic to an open disc is called the *interior* of the variety, and the set of points having a neighborhood homeomorphic to a half disc is called the *border* of the variety.

Let Ω_r, $r \geq 1$, be a bi-dimensional compact variety with r border components. Each border component is a 1-connected variety, that is, a circle. We consider that all embedding are a

We are concerned only with embedding of graphs on Ω_r which are a 2-cell embedding and that preserve the Euler characteristic of Ω_r. Since Ω_r is homeomorphic to a compact surface without border with r disjoint discs being removed, so the criterion to use in order to embed a graph G on Ω_r is to draw the interior of r regions of the model of the embedding of the graph on Ω. If the embedding $G \hookrightarrow \Omega$ is a 2-cell embedding, each region is homeomorphic to an open disc, equivalently, when removing the interior of a region coming from a 2-cell embedding, topologically we are removing an open disc of the interior of the corresponding region.

Definition II.3: [3] Let $F_{mn} \equiv \Omega\left(\alpha\right) = \cup_{i=1}^{\alpha} R^i$ be the model consisting of α regions spanned by the embedding of the graph $K_{m,n}$ on an oriented compact surface Ω. We call the *embedding with r border components* of the graph $K_{m,n}$, $r \leq \alpha$, on Ω_r, an embedding obtained by eliminating r regions of F_{mn}, denoted by $F_{mn}^r = \Omega_r\left(\alpha - r\right) = \cup_{i=1}^{\alpha-r} R^i$. We call the embedding $K_{m,n} \hookrightarrow \Omega$ *primary embedding* for the embedding on a surface with border $K_{m,n} \hookrightarrow \Omega_r$.

A DMC channel is represented by an input set X, an output set Y and a set of transition probabilities $p\left(y/x\right)$, $y \in Y$ and $x \in X$. This structure allows a natural representation of a DMC channel as a graph with vertices in the sets X and Y and sides (transitions) connecting the vertices in each one of these sets.

Definition II.4: [3] Let $X = \{x_1, \cdots, x_m\}$ and $Y = \{y_1, \cdots, y_n\}$ be two sets of vertices, with x_i and $y_j \in \mathbb{Z}$. A *channel class* $C_{m,n}$ is the set consisting of all channels with m vertices in X and n vertices in Y, satisfying the following properties:
$P1)$ Every vertex of X is connected to at least one vertex of Y and vice-versa;
$P2)$ $[x, y]$ is a transition of the $C_{m,n}$ channel if $x \in X$ is connected to $y \in Y$.

From Definition II.4, we see that the channel class $C_{m,n}$, seen as the corresponding graph class, contains the complete bipartite graph $K_{m,n}$. For the case in which $C_{m,n} \equiv K_{m,n}$, we have the following results:

Theorem II.3: [3] The first homology group of a surface Ω is given by:

$$H_1\left(\Omega\right) = \begin{cases} \mathbb{Z}^{2m}, & \text{if } \Omega \equiv mT \\ \mathbb{Z}^{2m+r-1}, & \text{if } \Omega \equiv mT_r \\ \mathbb{Z}^{r-1}, & \text{if } \Omega \equiv S_r; \\ \mathbb{Z}_2 \oplus \mathbb{Z}^{m+r-1}, & \text{if } \Omega \equiv mP_r \end{cases}$$

where $\mathbb{Z}^m$ denotes the direct sum of m copies of $\mathbb{Z}$.

Theorem II.4: [4] If $K_{m,n} \hookrightarrow \Omega\left(\alpha\right)$ is a minimum embedding on an oriented surface and $F_{mn} \equiv \Omega\left(\alpha\right)$ then the number of regions of F_{mn} is:

$$\alpha = \begin{cases} mn/2, \text{ if } m, n \equiv \left(0,0\right), \left(0,2\right), \left(1,2\right), \left(2,0\right), \left(2,1\right), \\ \left(2,2\right), \left(2,3\right), \left(3,2\right) mod4; \\ \left(mn/2\right) - 1, \text{ if } m, n \equiv \left(0,1\right), \left(0,3\right), \left(1,0\right), \left(3,0\right) mod4; \\ \left(mn/2\right) - 1/2, \text{ if } m, n \equiv \left(1,3\right), \left(3,1\right) mod4; \\ \left(mn/2\right) - 3/2, \text{ if } m, n \equiv \left(1,1\right), \left(3,3\right) mod4, \end{cases}$$

where $m, n \equiv \left(a_1, b_1\right), \cdots, \left(a_s, b_s\right) mod4$ means that $m \equiv a_1 mod4$ and $n \equiv b_1 mod4$, $\cdots$, or $m \equiv a_s mod4$ and $n \equiv$

Lemma II.1: [3] If $F_{mn} \in M_\alpha$ is such that $F_{mn} = \cup_{j=1}^{\alpha} R_{i_j}$ then $\sum_{j=1}^{\alpha} i_j = 2mn$, that is,

$$F_{mn} = R_{i_1} \cup R_{i_2} \cdots \cup R_{i_2} \Rightarrow i_1 + i_2 + \cdots + i_\alpha = 2mn.$$

In order to consider the channel class $C_{m,n}$ with all of its possible variations, assume that $P = \{p_1, \cdots, p_m\}$ e $Q = \{q_1, \cdots, q_m\}$ are the set of positive integers and that $X = \{x_1, \cdots, x_m\}$ and $Y = \{y_1, \cdots, y_n\}$ are the sets of vertices of the channel $C_{m,n}$. We denote by $val\, x$ the *degree* of the vertex x. If $valx_i = p_i$ and $valy_j = q_j$, $i \in \{1, \cdots, m\}$, $j \in \{1, \cdots, n\}$, then

(i) We call the *channel class with degrees* P *and* Q, the set

$$C_{m,n}[P,Q] = C_{m,n}[\{p_1, \cdots, p_m\}, \{q_1, \cdots, q_m\}] \quad (5)$$

consisting of all the channels in $C_{m,n}$ such that, for each $p_i \in P$ and each $q_j \in Q$ there are $x_i \in X$ e $y_j \in Y$ with $valx_i = p_i$ e $valy_j = q_j$.

(ii) If the vertices of $C_{m,n}$ are such that $valx_1 = \cdots = valx_m = p$ e $valy_1 = \cdots = valy_n = q$, then

$$C_{m,n}[p,q] \quad (6)$$

denotes the *class of channels with degrees* p *and* q*;*

(iii) If $m = n$ and $valx_1 = \cdots = valx_m = valy_1 = \cdots = valy_n = p$,, then $C_m[p]$ denotes the *class of channels with degree* p.

Let $\langle \Omega(\alpha) \rangle$ be the set of surfaces for the embedding of the $C_{m,n}[P,Q]$ channel generated by the embedding of a surface without border $C_{m,n}[P,Q] \hookrightarrow \Omega(\alpha)$, that is,

$$\langle \Omega(\alpha) \rangle = \{\Omega(\alpha), \Omega_1(\alpha-1), \cdots, \Omega_{\alpha-1}(1)\}. \quad (7)$$

Lemma II.2: [3] Let $\langle \Omega(\alpha) \rangle$ be the set of surfaces generated by the embedding $C_m[p] \hookrightarrow \Omega(\alpha)$. If $C_m[p] \hookrightarrow kT(\alpha)$ and $C_m[p] \hookrightarrow hP(\beta)$ are minimum embedding, then the set of surfaces for the 2-cell embedding of the channel $C_m[p]$ is

$$Ş_m^p = \begin{cases} \left\{ \langle (k+i)\, T(\alpha - 2i) \rangle_{i=0}^{(\alpha-2)/2} \right\} \cup \\ \left\{ \langle (h+j)\, P(\beta - j) \rangle_{j=0}^{\beta-1} \right\} \text{ if } \alpha \text{ is even} \\ \left\{ \langle (k+i)\, T(\alpha - 2i) \rangle_{i=0}^{(\alpha-1)/2} \right\} \cup \\ \left\{ \langle (h+j)\, P(\beta - j) \rangle_{j=0}^{\beta-1} \right\} \text{ if } \alpha \text{ is odd.} \end{cases}$$

Proposition II.1: [3] The set of surfaces for the embedding of the class $C_4[4]$ is

$$Ş_4^4 = \left\{ \langle (1+i)\, T(8-2i) \rangle_{i=0}^{3} \right\} \cup \left\{ \langle (2+j)\, P(8-j) \rangle_{j=0}^{7} \right\}.$$

III. Identification of the Geometric and Algebraic Structures

The necessary steps to identify the geometric and algebraic structures of a DMC channel when considering an oriented surface are described next.

P1) Identify the graph which corresponds to the $C_{m,n}[P,Q]$ channel; this graph is a complete bipartite graph $K_{m,n}$, when all the transition probabilities are different from zero, otherwise it is a subgraph of it;

P2) Determine the minimum genus g_m and the maximum genus g_M of the surfaces for the embedding of the graph $K_{m,n}$ associated to the $C_{m,n}[P,Q]$ channel. From Theorem II.2, if

P3) For each γ, use Theorem II.4, and identify the model $F_{mn} \hookrightarrow \Omega(\alpha)$;

P4) For each model in P3 identify the set of surfaces generated by $\Omega(a)$, $\langle \Omega(\alpha) \rangle$, apply Lemma II.2 and obtain the set of surfaces $Ş_{m,n}$ for the embedding of the $C_{m,n}[P,Q]$ channel;

P5) Use Theorem II.3 to determine the set of algebraic structures associated to the $C_{m,n}[P,Q]$, channel, that is, the set $\hat{H}_{m,n}$ of the homology groups of the surfaces of $Ş_{m,n}$;

P6) Use Lemma II.1 and identify in P4 the set of regular tessellations with m identical regions, Ξ_m, with tm identical regions, Ξ_{tm}, and use Theorem II.3 to identify the corresponding set of algebraic structures;

Remark III.1: The set of non-oriented surfaces for the embedding of the DMC channel C is determined in an analogous way as for the oriented surfaces: the minimum genus for this embedding is given by (4), and we assume that the maximum genus of the surface for the embedding whose model consists of only one region is g_M given by (3), then $g_m P(\alpha), (g_m+1) P(\alpha-1), \cdots, g_M P(1)$ is the set of the non-oriented compact surfaces for the embedding of C.

IV. Examples of Embedded Discrete Memoryless Channels

The surfaces are denoted by S (sphere), T (torus), P (projective plane) or K (Klein bottle). We use the notation $k\Omega$ to mean a homeomorphic surface to the connected sum of k identical surfaces to Ω, and $k\Omega_r$ to denote that $k\Omega$ has r border components. In order to simplify notations, we further use:

1) $k\Omega_r(\alpha) \triangleq$ model with α regions on $k\Omega_r$;

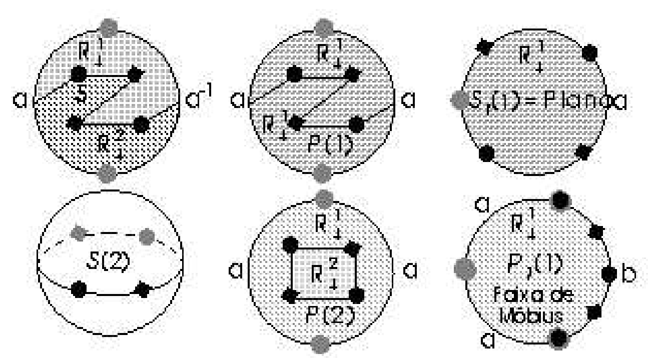
2) $R_t^i \triangleq i$-th region with t sides of the model F_{mn} for the embedding of the graph $K_{m,n}$.

A. BSC channel

The binary symmetric channel, BSC, is the simplest and the most employed channel in a communication systems. This channel is used for binary transmission. This channel is a complete bipartite graph $K_{2,2}$ and the set of surfaces for its embedding is:

$$Ş_{2,2} = \{S(2), S_1(1), P(2), P(1), P_1(1)\}. \quad (8)$$

The embedding corresponding to the set $Ş_{2,2}$ are shown in Fig. 1.



Making use of Theorem II.3, the set of algebraic structures associated to the BSC channel is

$$\hat{H}_{22} = \{0, \mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}\}.$$

The set Ş_{22} has only two regular tessellations, namely, $\Xi_{22} = \{S[2R_4], P[2R_4]\}$, both satisfying the characteristics for use as modulation or for the design of hard-decision quantizers.

B. Ternary channel with degree 3

Applying the procedures as established in Section III, we deduce that the set of surfaces in which the $C_3[3]$ channel may be embedded as a 2-cell embedding is

$$\text{Ş}_3^3 = \{\langle T(3)\rangle, 2T(1), \langle P(4)\rangle, \langle K(3)\rangle, \langle 3P(2)\rangle, 2K(1)\}.$$

Some of the embedding of the set Ş_3^3 are shown in Fig. 2.

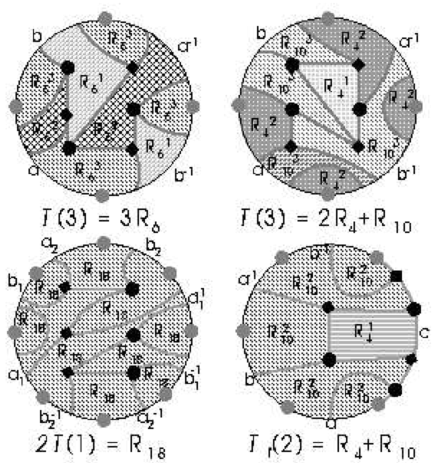


Fig. 2. Ş_3^3 channel

Consequently, the set of algebraic structures and of the regular tessellations associated to the $C_3[3]$ channel are, respectively,

$$\hat{H}_{3,3} = \{\mathbb{Z}^2, \mathbb{Z}^3, \mathbb{Z}^4, \mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}, \mathbb{Z}_2 \oplus \mathbb{Z}^2, \mathbb{Z}_2 \oplus \mathbb{Z}^3\}$$

and

$$\Xi_{3,3} = \{T[3R_6]\ P_1[3R_4], K[3R_6]\}.$$

As occurred with $\Xi_{2,2}$, the set $\Xi_{3,3}$ contains only regular tessellations for use as modulation and for the design of hard-decision quantizers.

C. Quaternary channel with degree 4

The $C_4[4]$ channel is represented by the complete bipartite graph $K_{4,4}$. The set of surfaces in which $C_4[4]$ may be embedded as a 2-cell embedding is

$$\text{Ş}_4^4 = \left\{\langle (1+i)\,T(8-2i)\rangle_{i=0}^3\right\} \cup \left\{\langle (2+j)\,P(8-j)\rangle_{j=0}^7\right\}.$$

The set of algebraic structures and the set of regular tessellations associated to the $C_4[4]$ channel are, respectively,

Channel C_m	$\#\text{Ş}_m$	$\#\hat{H}_m$	$\#\Xi_m$	$\#\Xi_{tm}$	N max
C_4	56	15	8-3	2-1	2
C_5	114	23	12-4	6-1	2
C_6	261	35	20-5	13-3	3
C_7	469	48	27-6	23-3	3
C_8	800	63	38-7	42-6	4

TABLE I

CARDINALITIES OF THE SETS ASSOCIATED TO THE C_m CHANNEL

and

$$\Xi_{4,4} = \{T_4[4R_4], 2T_2[4R_6], 3T[4R_8], K_4[4R_4], 3P_3[4R_4], 2K_2[4R_6], 5P_1[4R_6], 3K[4R_8], T[8R_4]^{[2]}, K[8R_4]^{[2]}\}$$

Besides the typical tessellations for use as modulation, there are two tessellations which can be used for the design of hard-decision quantizers. In order to have a precise idea of the number of elements related to the $C_m[m]$, channels for $m = 4, 5, 6, 7, 8$, observe in Table I, the cardinalities of the sets associated to these channels.

In Table I, the meaning of the notation 8-3 is that there are 8 regular tessellations associated to C_4, however, only 3 are distinct, that is, they are models for the embedding of C_4 on 8 distinct surfaces of Ş_4, all with 4 regions with 4, 6 or 8 sides. The column 'N max' refers to the maximum number of quantizer levels. For instance, the number 4, in the last column, means that there exists at least one regular tessellation in Ş_8 with 32 square regions; in this case there are two, namely, $9T[32R_4]^{[4]}$ and $9K[32R_4]^{[4]}$.

D. The $C_{2,8}[8,2]$ channel

The 8-level quantizer for the binary input channel corresponds to the $C_{2,8}[8,2]$ channel. This channel and one of its minimum embedding on the sphere are shown in Fig. 3.

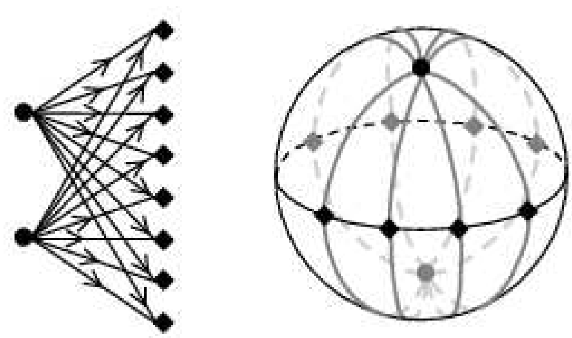


Fig. 3. $C_{2,8}[8,2]$ channel, and one of its minimum embedding on the sphere

The minimum genus of the embedding on an oriented surface for this channel is on the sphere with 8 square regions. Therefore, a regular tessellation. Hence, the design of this quantizer can be realized by using the model as shown in Fig. 3. In this case, the set of oriented compact surfaces without border in which the $C_{2,8}[8,2]$ channel is embedded as a 2-cell embedding is

V. Conclusions

In this paper the geometric and algebraic structures associated to a discrete memoryless channel were identified. The geometric structure comes from the characterization of the genus of the corresponding surface in which the DMC channel is embedded. The algebraic structure comes from the homology group of the corresponding surface in which the DMC channel is embedded. Furthermore, the regular tessellations for the design of modulation schemes and quantizers arise of the corresponding embedding.

The following properties were obtained for the regular tessellations associated to the $C_{m,n}[p]$, channel:
(i) the regular tessellations which are on oriented surfaces have the same algebraic structures;
(ii) there exists only one regular tessellation Ξ^{p}_{tm} on an oriented compat surface, and its algebraic structure is different from the one obtained in (i);
(iii) the regular tessellations which are on non-oriented surfaces have the same type of algebraic structure;
(iv) there exists at least one regular tessellation Ξ^{p}_{tm} on aa non-oriented compact surface.

We have shown that the models with two regions employed for the embedding of the $C_{2,2}[2,2]$ and $C_{2,8}[8,2]$ channels on oriented compact surfaces without border with two regions occur, respectively, on $S\,(2)$ and $3T\,(2)$. Consequently, the performance improvement observed in Fact 1 and Fact 2, is related to the genus of the surface where the corresponding channel is embedded.

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