Space-Time Convolutional Codes Over GF(p) for the Quasi-Static, Flat Rayleigh Fading Channel

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Abstract— **In this paper, we consider a space-time coded system consisting of a rate $R = 1/n$ linear convolutional encoder over GF(p), p a prime, followed by n mappers from GF(p) into a p-ary signal constellation, and by a transmitter with n transmit antennas. At each time, the n p-ary coded symbols are transmitted simultaneously from the n antennas. The convolutional codes are designed to provide the best error performance in the quasi-static, flat Rayleigh fading channel, according to the rank and the determinant criteria derived by Tarokh *et. al*. A spectral efficiency of $\log_2(p)$ b/s/Hz is achieved. Simple conditions on the generator matrices of a rate $R = 1/2$ convolutional code are given so that maximum diversity advantage is guaranteed. The linear structure of these convolutional codes reduces significantly the computer search effort with regard to the determinant criterion. New space-time codes with two transmit antennas are presented for the 5-PSK and 7-PSK modulations, with spectral efficiencies 2.32 and 2.81 b/s/Hz, respectively.**

Keywords— **Diversity, fading channels, multiple transmit antennas, space-time convolutional codes, wireless communications.**

I. Introduction

In a communication system employing n transmit antennas and m receive antennas, space-time codes (STC) have proved very effective for achieving both diversity and coding advantages at high spectral efficiencies in the fading channels. Since the inception of STC, by Tarokh *et. al.* [1], many researchers have been involved with the design of optimum STC, i.e., codes that achieve maximum diversity and coding gain. In this paper, we concentrate on the quasi-static, flat Rayleigh fading channel [1]. A remarkable result derived in [1] is that the best STC for this channel may be obtained by optimizing two design criteria, namely, the *rank* criterion and the *determinant* criterion. Some STC for two transmit antennas for the 4-PSK and 8-PSK modulations are provided in [1]. Until recently, STC had been constructed using hand design, and a systematic procedure, possibly algebraic, for constructing these codes had not been presented. The main difficulty for designing STC is that the two aforementioned criteria apply to the complex field of baseband modulated signals rather than a discrete domain (*e.g.*, the binary Galois field) in which the underlying codes could be designed. Recently, important papers providing a general design procedure of STC were written by Hammons and El Gamal [2] and by Blum [3], [4]. In [2], general binary rank criteria are developed that are sufficient to guarantee that the associated STC achieve maximum diversity. Blum's papers deal with binary convolutional codes that serve as STC. Necessary and sufficient conditions are given in [3], [4] for the STC to achieve maximum diversity gain. Also, Blum developed methods for calculating and bounding the coding gain (related to the determinant criterion).

In this paper, an alternative design procedure for achieving diversity and coding advantages is presented. We consider a space-time coded system consisting of a rate $R = 1/n$ linear convolutional encoder over GF(p), p a prime, followed by a serial-to-parallel converter, n mappers from GF(p) into a p-ary signal constellation, and a transmitter with n transmit antennas. This approach differs from those of [2] and [3], [4] since here the coded symbols are in one-to-one correspondence to the signals from the constellations. This makes the code design and the code search much easier to cope with. A spectral efficiency of $\log_2(p)$ b/s/Hz is achieved. Simple conditions on the generator matrices of a rate $R = 1/2$ convolutional code are given so that maximum diversity advantage is guaranteed. Under these conditions, and due to the linear structure of the convolutional codes, the computer search effort with regard to the determinant criterion is dramatically reduced. New space-time codes with two transmit antennas are presented for the 5-PSK and 7-PSK modulations, with spectral efficiencies 2.32 and 2.81 b/s/Hz, respectively.

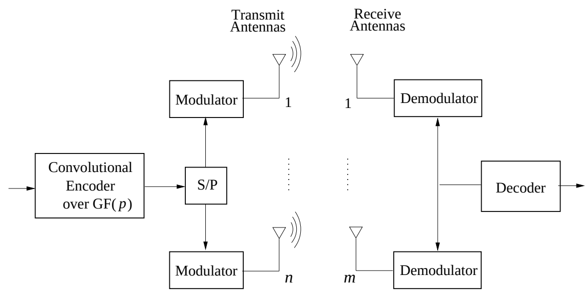

Fig. 1. Space-time system model.

II. System Model

We consider a communication system employing n transmit antennas and m receive antennas, as shown in Fig. 1. At the transmitter, data is first encoded by a rate $R = 1/n$ convolutional encoder over GF(p), p a prime, whose output is divided into n parallel streams. These streams are mapped into a p-ary signal constellation and transmitted simultaneously from the n antennas. A space-time system has been modeled by Tarokh *et. al.* [1] in such a way that the signal received by the jth antenna at time t, d_t^j, is given by

$$d_t^j = \sum_{i=1}^{n} \alpha_{i,j} c_t^i \sqrt{E_s} + \eta_t^j \quad (1)$$

where c_t^i is the signal transmitted from the ith antenna at time t, E_S is the average energy of the transmitted signal, η_t^j is a zero-mean complex white Gaussian noise with variance $N_0/2$ per dimension, and $\alpha_{i,j}$ denotes the fade present in the path from the ith transmit antenna to the jth receive antenna. For $1 \le i \le n$ and $1 \le j \le m$, these fades are modeled as independent samples of a zero-mean complex Gaussian random process with variance 0.5 per dimension. In practice, to achieve independent fades the antennas must be separated by a few wavelengths. For the quasi-static, flat-fading channel, it is assumed that the path fades remain constant during a frame and change independently from one frame to another. Also, assume the receiver perfectly knows the channel state information (i.e., the α's) and that the Viterbi algorithm with the Euclidean metric is used in the decoder. Under these conditions, the pairwise error probability, denoted by $P(\mathbf{c} \to \mathbf{e})$, and defined as the probability that a maximum-likelihood decoder decides erroneously in favor of codeword

$$\mathbf{e} = e_1^1 e_1^2 \cdots e_1^n e_2^1 e_2^2 \cdots e_2^n \cdots e_l^1 e_l^2 \cdots e_l^n$$

when the correct codeword is:

$$\mathbf{c} = c_1^1 c_1^2 \cdots c_1^n c_2^1 c_2^2 \cdots c_2^n \cdots c_l^1 c_l^2 \cdots c_l^n,$$

is upper bounded by [1]:

$$P(\mathbf{c} \to \mathbf{e}) \le \left(\prod_{i=1}^{r} \lambda_i \right)^{-m} \left(\frac{E_s}{4N_0} \right)^{-rm}, \qquad (2)$$

where r is the rank of the matrix:

$$B(\mathbf{c}, \mathbf{e}) \triangleq \begin{pmatrix} e_1^1 - c_1^1 & e_2^1 - c_2^1 & \cdots & e_l^1 - c_l^1 \\ e_1^2 - c_1^2 & e_2^2 - c_2^2 & \cdots & e_l^2 - c_l^2 \\ \vdots & \vdots & \ddots & \vdots \\ e_1^n - c_1^n & e_2^n - c_2^n & \cdots & e_l^n - c_l^n \end{pmatrix}$$

λ_i, $i = 1, \ldots, r$, are the nonzero eigenvalues of $A(\mathbf{c}, \mathbf{e}) \triangleq B(\mathbf{c}, \mathbf{e}) B(\mathbf{c}, \mathbf{e})^H$, and H denotes conjugate transpose.

Based on equation (2), Tarokh *et. al.* [1] arrived at two design criteria for the quasi-static, flat Rayleigh fading channel, namely,

- *The Rank Criterion:* In this criterion the parameter to be maximized is the minimum rank r of matrix $B(\mathbf{c}, \mathbf{e})$, over all distinct pairs of codewords $\mathbf{c}$ and $\mathbf{e}$. It is said that this is an r-space-time code. The diversity advantage is $rm \le nm$, with equality at full rank condition, that is, $r = n$.
- *The Determinant Criterion:* For a given diversity advantage rm, the goal of this criterion is to maximize the minimum geometric mean of the nonzero eigenvalues of the matrix $A(\mathbf{c}, \mathbf{e})$, $(\lambda_1 \lambda_2 \ldots \lambda_r)^{\frac{1}{r}}$, over all distinct pairs of codewords $\mathbf{c}$ and $\mathbf{e}$. This represents the coding gain.

Regarding the space-time system model of Fig. 1, in the next section we derive conditions on the generator matrix of a rate $R = 1/2$ convolutional code so that maximum diversity advantage, as presented in Section II, is guaranteed for the corresponding 2-space-time codes.

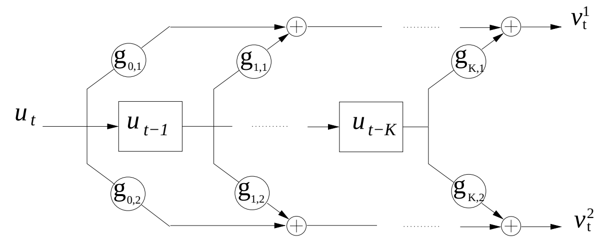

Fig. 2. Rate 1/2 convolutional encoder over GF(p).

III. Convolutional Space-Time Codes Over GF(p)

Let $U(D) = u_0 + u_1 D + u_2 D^2 + \cdots$ be the polynomial over GF(p), p a prime, representing a data sequence. This sequence is encoded by a rate $R = 1/n$ convolutional encoder over GF(p), which is the direct realization of the polynomial generator vector:

$$\mathbf{G}(D) = [G_1(D), G_2(D), \ldots, G_n(D)],$$

producing the encoded vector

$$\mathbf{V}(D) = U(D)\mathbf{G}(D) = \left[V^1(D), V^2(D), \ldots, V^n(D)\right],$$

where $V^i(D) = v_0^i + v_1^i D + v_2^i D^2 + \cdots + v_t^i D^t + \cdots$, for $i = 1, 2 \ldots, n$, are the n encoded sequences. The code generators are $G_i(D) = g_{0,i} + g_{1,i} D + g_{2,i} D^2 \cdots + g_{K,i} D^K$, for $i = 1, 2 \ldots, n$, where K is the memory of the encoder. An example of a rate 1/2 convolutional encoder over GF(p) is shown in Fig. 2.

At each time t, a data symbol from GF(p) produces a block of n symbols from GF(p) denoted by $(v_t^1, v_t^2, \ldots, v_t^n)$. Regarding the space-time system model in Fig. 1, these symbols are mapped into a p-ary signal constellation, as indicated in Fig. 3 for the 5-PSK and 7-PSK constellations. The n complex symbols, $(c_t^1, c_t^2, \ldots, c_t^n)$, are then transmitted by the n antennas. A sequence of blocks $(c_t^1, c_t^2, \ldots, c_t^n)$, for $t = 1, 2, \ldots, l$ forms a codeword $\mathbf{c}$ of the space-time code.

From now on, we shall focus our attention on the case of $n = 2$ transmit antennas. Thus, according to Tarokh's design criteria, the code search should aim at finding the "full rank" code $\mathcal{C}$ that maximizes the minimum determinant

$$\det \left(\sum_{i=1}^{l} (e_i^1 - c_i^1, e_i^2 - c_i^2)^* (e_i^1 - c_i^1, e_i^2 - c_i^2) \right), \qquad (3)$$

over all pairs of codewords $\mathbf{c}$ and $\mathbf{e}$, where * denotes complex conjugate. In the following, we explore the algebraic structure of the convolutional codes and present some guidelines to guarantee the full rank condition, yielding a reduced complexity code search.

Lemma 1: Consider a rate $R = 1/2$ convolutional space-time code over GF(p), p a prime, with a generator matrix

$$\mathbf{G}(D) = \begin{bmatrix} g_{1,1} D + \ldots + g_{K,1} D^K \\ g_{0,2} + \ldots + g_{K-1,2} D^{K-1} \end{bmatrix}^T,$$

where $g_{K,1} \neq 0$ and $g_{0,2} \neq 0$. Then, 1) all branches in the code trellis departing from the same state coincide in the first symbol and differ in the second symbol, and 2) all transitions arriving at the same state coincide in the second symbol and differ in the first symbol.

Proof: We shall use the fact that, for field elements u, u', and g:

$$u \cdot g = u' \cdot g \quad \Rightarrow \quad u = u', \tag{4}$$

and

$$u \cdot g \neq u' \cdot g \quad \Rightarrow \quad u \neq u'. \tag{5}$$

To prove 1), let us consider the current state as being $\sigma_s = [u_1, \ldots, u_K]^1$. When $u_0(f)$ is the input symbol, the state transition is $\sigma_s \rightarrow \sigma_f = [u_o(f), u_1, \ldots, u_{K-1}]$. Since $g_{0,1} = 0$, the coded symbol corresponding to the first antenna will be given by:

$$v^1 = \sum_{\mu=1}^{K-1} u_\mu \cdot g_{\mu,1} + u_K \cdot g_{K,1}, \tag{6}$$

which is a constant in GF(p) that does not depend on $u_0(f)$, but on σ_s. This shows that the first symbol of all branches departing from the same state are equal. Now for the same state transition $\sigma_s \rightarrow \sigma_f$, since $g_{K,2} = 0$, the symbol corresponding to the second antenna will be given by:

$$v^2 = u_0(f) \cdot g_{0,2} + \sum_{\mu=1}^{K-1} u_\mu \cdot g_{\mu,2} = u_0(f) \cdot g_{0,2} + V_2, \tag{7}$$

where V_2 is a constant in GF(p). Thus, in GF(p), v^2 will be different for different values of $u_0(f)$.

To prove 2), let us consider the final state as being $\sigma_f = [u_0, u_1, \ldots, u_{K-1}]$. Then, for the state transition $\sigma_s = [u_1, \ldots, u_{K-1}, u_K(s)] \rightarrow \sigma_f$, the symbol corresponding to the first antenna is given by:

$$v^1 = \sum_{\mu=1}^{K-1} u_\mu \cdot g_{\mu,1} + u_K(s) \cdot g_{K,1} = V_1 + u_K(s) \cdot g_{K,1}, \tag{8}$$

where V_1 is a constant in GF(p). So, in GF(p), v^1 will be different for different values of $u_K(s)$. For the same transition $\sigma_s \rightarrow \sigma_f$, the symbol corresponding to the second antenna will be given by:

$$v^2 = u_0 \cdot g_{0,2} + \sum_{\mu=1}^{K-1} u_\mu \cdot g_{\mu,2}, \tag{9}$$

which is a constant in GF(p) that does not depend on $u_K(s)$, only on σ_f. This completes the proof. ■

Theorem 1: Consider a rate $R = 1/2$ convolutional space-time code over GF(p), p a prime. If the generator matrix $\mathbf{G}$ is in accordance with Lemma 1, then the full rank condition is guaranteed.

Proof: Assume the generator matrix $\mathbf{G}$ is in accordance with Lemma 1, and consider an arbitrary pair of codewords $\mathbf{c}$ and $\mathbf{e}$ of lenght $2l$ symbols. Then, equation (3) results in:

$$\det_l \triangleq \det\left(\begin{bmatrix} 0 & 0 \\ 0 & |a_1|^2 \end{bmatrix} + \begin{bmatrix} f & d \\ d^* & h \end{bmatrix} + \begin{bmatrix} |b_l|^2 & 0 \\ 0 & 0 \end{bmatrix} \right), \tag{10}$$

[1]Note that here we drop the index t used in Fig. 2 for the sake of simplicity.

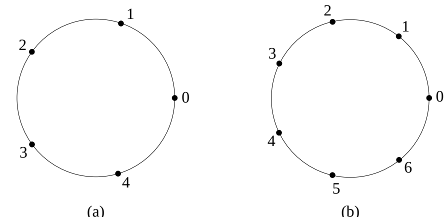

Fig. 3. Constellations 5-PSK (a) and 7-PSK (b).

where

$$a_i = \left(e_i^2 - c_i^2\right), \quad b_i = \left(e_i^1 - c_i^1\right), \quad f = \sum_{i=2}^{l-1} |b_i|^2,$$

$$h = \sum_{i=2}^{l-1} |a_i|^2, \quad \text{and} \quad d = \sum_{i=2}^{l-1} a_i b_i^*. \tag{11}$$

From Lemma 1, we have that $a_1 > 0$ and $b_l > 0$. By Schwarz inequality,

$$\sum_i |a_i|^2 \times \sum_i |b_i|^2 \geq \left| \sum_i a_i b_i^* \right|^2, \tag{12}$$

and thus $fh \geq |d|^2$. Since $f \geq 0$ and $h \geq 0$, we must have that:

$$\det_l = \underbrace{|a_1|^2 |b_l|^2}_{>0} + \underbrace{|a_1|^2 f}_{\geq 0} + \underbrace{|b_l|^2 h}_{\geq 0} + \underbrace{fh - |d|^2}_{\geq 0} > 0, \tag{13}$$

which guarantees the full rank condition. ■

IV. Code Search Results

In this section, we present some examples of 2-space-time codes generated by a rate $R = 1/2$ linear convolutional encoder over GF(5) and GF(7). The restrictions derived in the previous section are applied to the convolutional encoder (i.e., $g_{1,1} \neq 0$, $g_{0,2} \neq 0$ and $g_{0,1} = g_{K,2} = 0$) to guarantee the full rank condition. The minimum determinant is then maximized. We begin with examples of unit-memory encoders.

Example 1: Consider the 5-PSK constellation in Fig. 3(a). Following the restrictions of the previous section, the convolutional encoder over GF(5) has generator matrix $\mathbf{G}(D) = [g_{1,1}D, g_{0,2}]$. These codes have diversity gain 2, with spectral efficiency of 2.32 b/s/Hz. In performing a code search, we made $g_{1,1}$ and $g_{0,2}$ to vary over all nonzero elements in GF(5), resulting in 16 different codes. The minimum determinant of $A(\mathbf{c}, \mathbf{e})$ for each code is listed in the matrix $D(g_{1,1}, g_{0,2})$ below.

$$D(g_{1,1}, g_{0,2}) = \begin{pmatrix} 1,38 & 2,23 & 2,23 & 1,38 \\ 2,23 & 1,38 & 1,38 & 2,23 \\ 2,23 & 1,38 & 1,38 & 2,23 \\ 1,38 & 2,23 & 2,23 & 1,38 \end{pmatrix}$$

Clearly, the best code is obtained with, for instance, $g_{1,1} = 1$ and $g_{0,2} = 2$. This code has coding gain 2.23. The trellis of this code is shown in Fig.5.

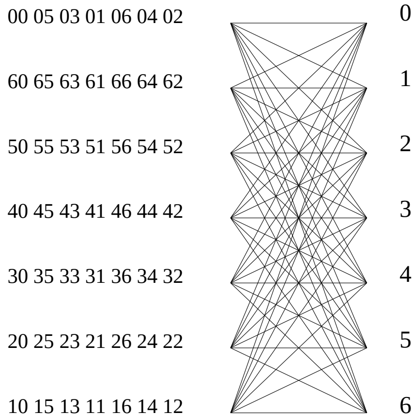

Fig. 4. 2-space-time code for 7-PSK, 2.81 b/s/Hz, (convolutional encoder: GF(7), $R = 1/2$, $K = 1$).

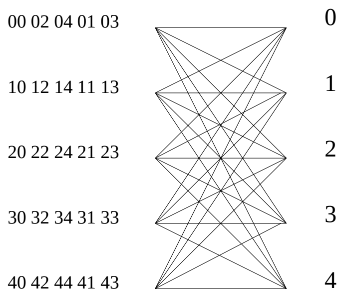

Fig. 5. 2-space-time code for 5-PSK, 2.32 b/s/Hz, (convolutional encoder: GF(5), $R = 1/2$, $K = 1$).

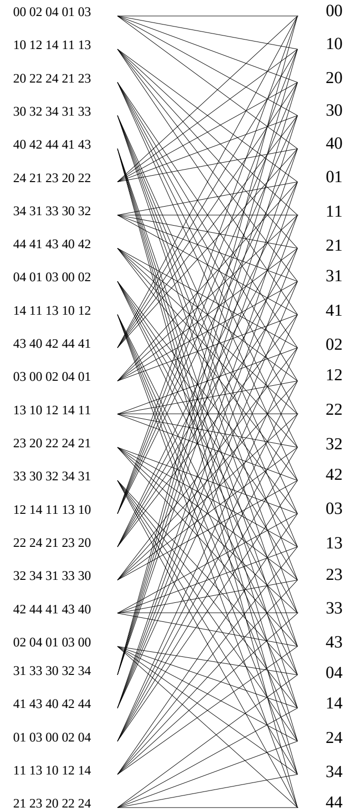

Fig. 6. 2-space-time code for 5-PSK, 2.32 b/s/Hz, (convolutional encoder: GF(5), $R = 1/2$, $K = 2$).

In this and in the following trellises, each pair of symbols to the left of the trellis indicates the signal transmitted over the first and the second antennas, respectively, and the label to the right of the trellis indicates the state of the encoder.

Note that $D(g_{1,1}, g_{0,2})$ is symmetric, which can be used to reduce our code search. It can be shown that similar simplification may be obtained with higher memory codes.

Example 2: Now consider the signal constellation 7-PSK in Fig. 3(b). Again, we consider a unit-memory encoder satisfying the conditions of Section III. We thus have diversity gain 2, with spectral efficiency of 2.81 b/s/Hz. Similarly to Example 1, we obtained the matrix $D(g_{1,1}, g_{0,2})$ as

$$D(g_{1,1}, g_{0,2}) = \begin{pmatrix} 0,75 & 1,35 & 1,35 & 1,35 & 1,35 & 0,75 \\ 1,35 & 0,75 & 1,35 & 1,35 & 0,75 & 1,35 \\ 1,35 & 1,35 & 0,75 & 0,75 & 1,35 & 1,35 \\ 1,35 & 1,35 & 0,75 & 0,75 & 1,35 & 1,35 \\ 1,35 & 0,75 & 1,35 & 1,35 & 0,75 & 1,35 \\ 0,75 & 1,35 & 1,35 & 1,35 & 1,35 & 0,75 \end{pmatrix}$$

The best code is obtained with, for instance, $g_{1,1} = 6$ and $g_{0,2} = 5$. This code has coding gain 1.35. The trellis of this code is shown in Fig.4.

Example 3: Here we extend ours examples above utilizing convolutional encoders of memory order $K = 2$ over GF(5) and GF(7), respectively. That is, the convolutional encoder has generator matrix $\mathbf{G}(D) = [g_{1,1}D + g_{2,1}D^2, g_{0,2} + g_{1,2}D]$. For GF(5), one best code is generated by $g_{1,1} = 2$, $g_{2,1} = 1$, $g_{0,2} = 2$, and $g_{1,2} = 4$ (see trellis in Fig. 6). The coding gain of this code is 4.47. The diversity gain and the spectral efficiency are the same as those in Example 1. For GF(7), the multipliers set as $g_{1,1} = 3$, $g_{2,1} = 2$, $g_{0,2} = 1$, and $g_{1,2} = 3$ produce one best code. The trellis description for this code is shown in Fig. 7. It has a coding gain of 3.74, while the diversity gain and the spectral efficiency are the same as those in Example 2.

The new space-time codes presented in this paper could be found with a reduced computer search, due to the linearity of the convolutional codes, the restrictions derived in Section III, and the symmetry of the matrix D. Particularly, an exhaustive code search for the cases of the previous examples would need to check a huge number ($O(p^{(p^K \; 2p)})$) of codes, while here we only had 10 candidates for Example 1 and only a few thousand candidates to search from in the most complex case. This advantage becomes more significant as the memory order increases, allowing good codes to be obtained with relatively low computer effort.

V. Final Comments

In this paper, we considered space-time convolutional codes over GF(p) for the quasi-static, flat Rayleigh channel. The codes were designed to provide the best performance according to the rank and the determinant criteria derived by Tarokh *et. al.*. Simple conditions on the generator matrices of a rate $1/2$ convolutional code were given that guarantee maximum diversity advantage for 2 transmit antennas.

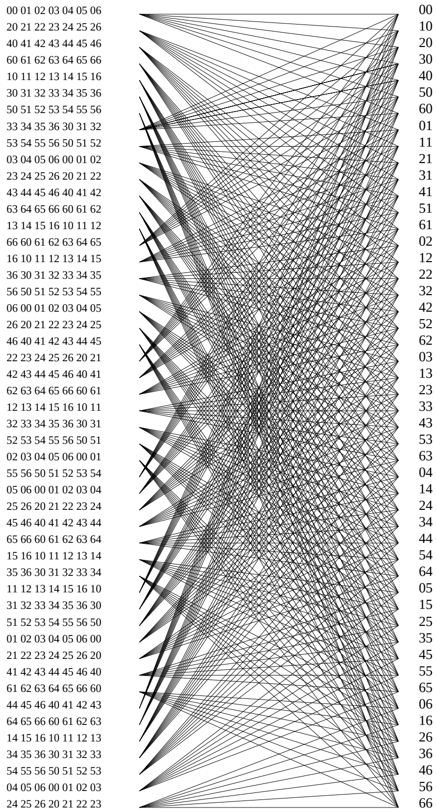

Fig. 7. 2-space-time code for 7-PSK, 2.81 b/s/Hz, (convolutional encoder: GF(7), $R = 1/2$, $K = 2$).

New space-time convolutional codes for 2 transmit antennas for the 5-PSK and 7-PSK constellations were presented. Spectral efficiencies of 2.32 and 2.81 b/s/Hz, respectively, were achieved.

The fact that p is not a power of 2 may pose as a problem if transmission of binary data is required. We tackle this problem by associating a different word of n_p p-ary symbols to each word of n_2 bits, where $p^{n_p} > 2^{n_2}$. The spectral efficiency is $n_2/n_p < \log_2(p)$ b/s/Hz, that is, there is a loss of

$$1 - \left(\frac{n_2}{n_p \log_2(p)}\right).$$

Clearly this loss is minimized by maximizing the relation $\frac{n_2}{n_p \log_2(p)}$, where $p^{n_p} > 2^{n_2}$. For instance, consider that, in the case of the 5-PSK constellation, each word of $n_2 = 9$ bits is associated to a different word of $n_p = 4$ symbols. Then, the spectral efficiency will be of 2.25 b/s/Hz, producing a loss of only 3%.

Acknowledgments

This work was supported by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), as an M.Sc. scholarship, and by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) under grants 141577/2001-0, 301593/96-5 , and 462450/00-7.

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