Classification of the perfect codes in the ∞-Lee metric

Claudio M. Qureshi and Sueli I. R. Costa

***Abstract*— This paper is concerned with perfect codes in the ∞-Lee metric. A complete classification and description of all (linear and non-linear) two-dimensional perfect codes in the ∞-Lee metric is presented. Moreover, in the linear case we construct a generator matrix for these codes which induces an isomorphism between the code and an abelian group of the form $\mathbb{Z}_a \times \mathbb{Z}_b$ with $a \mid b$ and we determine all possible group structures that can be represented for such codes. We also present an algorithm to obtain a q-ary perfect code with a given group structure and two methods of construction of perfect codes in the ∞-Lee metric from codes of smaller dimension. In particular, these methods allow us to construct interesting families of perfect ∞-Lee codes including n-dimensional cyclic perfect codes for all n.**

***Keywords*— Perfect codes, Lee metric, p-Lee metric, group isomorphism**

I. Introduction

Besides the Hamming metric, one of the metrics more often used in error-correcting codes is the Lee metric, in part due to several practical applications as those in [4], [8], [20], [22]. The most important theoretical open problem related to codes in the Lee metric is determining for what values of the parameters (n, e, q) there exists a perfect Lee code with those parameters (i.e. a n-dimensional perfect Lee code over the alphabet $\mathbb{Z}_q$ with packing radius e). In the seminal paper of Golomb and Welch [10], the authors obtain important results on the existence of perfect Lee codes over large alphabets (i.e. when $q \geq 2e+1$) and they conjecture that for dimension $n \geq 3$, the only perfect Lee codes over large alphabet are those with packing radius $e = 1$. In the referred paper, the authors construct perfect Lee codes with packing radius $e = 1$ for any dimension and prove that their conjecture holds for $n = 3$. At present it is known that the conjecture is true for dimension $n \leq 6$. There are several papers around the Golomb-Welch conjecture and other related problems as [2], [3], [1], [11], [18].

The Lee metric is part of a family of more general metrics called the p-Lee metrics (for $1 \leq p \leq \infty$). The use of these metrics in applications to coding and cryptography is relatively recent. For example, in [17] the authors study the complexity of various computational problems related to codes in the ℓ_p norm as the closest ans shortest vector problem (CVP and SVP). In [15] some decoding algorithms for codes in the ℓ_p metric are presented as well as generalizations of known results regarding the Lee metric. In addition to the above references, there are not many references in the literature on p-Lee metrics concerning error correcting codes, except for specific values of $p = 1$ (Lee metric) and $p = 2$ (Euclidean metric) which were extensively studied in the literature.

This paper is concerned with the case $p = \infty$. One motivation to study this case (∞-Lee metric) is because this metric captures much of the essence of perfect codes in other p-Lee metrics since any perfect code in the ∞-Lee metric is also perfect in the p-Lee metric for large enough p [5]. Another motivation is that for some specific values of p it is possible to prove that the only perfect codes in the p-Lee metrics are also perfect codes either in the Lee metric or in the ∞-Lee metric, this is the case for example for the two-dimensional 2-Lee perfect codes [6]. In contrast to other values of p, the determination of the paremeters (n, e, q) for which there exists a perfect code in the ∞-Lee metric is very simple and it was done in [15]. Having solved the problem of existence, the next step is to obtain these codes and characterize them from the geometric point of view (i.e. up to isometries) and from the algebraic point of view (i.e. up to group isomorphism). In Section 3 we describe completely the two-dimensional prefect codes. In Section 4 we consider some methods to construct a perfect code in the ∞-Lee metric from perfect codes in lower dimensions, in particular this leads us to generalize some of the results obtained for the two-dimensional case and to obtain some interesting families of perfect code as those described in Corollaries 27 and 28. In the conclusion we present some interesting problems related to this work.

II. Preliminaries

We consider the set $\mathbb{Z}_q = \{0, 1, 2, \ldots, q-1\}$ of integers modulo q with the distance given by $d(x, y) = \min\{|x - y|, q - |x - y|\}$ (Lee metric in $\mathbb{Z}_q$). This metric coincides with the metric in the circular graph whose vertices are the elements of $\mathbb{Z}_q$ and edges $\{i, i+1\}$ for i in $\mathbb{Z}_q$. Let p be a real number in $[1, \infty)$ or $p = \infty$, the p-Lee metric in $\mathbb{Z}_q^n$ is given by $d_p(x, y) = \begin{cases} \sqrt[p]{\sum_{i=1}^{n} d(x_i, y_i)^p} & \text{if } p \in [1, \infty), \\ \max_{i=1}^{n} d(x_i, y_i) & \text{if } p = \infty \end{cases}$ where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{Z}_q^n$ and d denote the Lee metric in $\mathbb{Z}_q$. The p-Lee metric in $\mathbb{Z}_q^n$ coincides with the quotient metric of $(\mathbb{Z}^n, \ell_1)$ over the subgroup $q\mathbb{Z}^n$, see [6]. When $p = 1$ we refer to the 1-Lee metric simply by Lee metric. We denote by $B_p(x, e)$ the ball with center $x \in \mathbb{Z}_q^n$ and radius $e \geq 0$ and define two radii e and e' as being equivalent when $B_p(x, e) = B_p(x, e')$ (note that since e is a real number we may have $B_p(x, e) = B_p(x, e')$ with $e \neq e'$ as in the example $B_2(0, \sqrt{2}) = B_2(0, \sqrt{3}) \subsetneq \mathbb{Z}_5^2$). Each equivalent class of radii is a left-closed interval of real numbers

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whose minimum and supreme element we denote by e^- and e^+ respectively. With this notation the equivalence class of e is given by $[e^-, e^+)$.

A n-dimensional q-ary code is a subset C of $\mathbb{Z}_q^n$. We refer the elements of C as the codewords. The minimum distance of C with respect to the p-Lee metric is given by $d_p(C) = \min\{d_p(x,y) : x, y \in \mathbb{Z}_q, x \neq y\}$. If $e \geq 0$ is such that the balls $B_p(c, e)$ are all disjoints when c runs over the codewords of C, but there exist distinct elements $c_1, c_2 \in C$ such that $B_p(c_1, e^+) \cap B_p(c_2, e^+) \neq \emptyset$ we say that e^- is the packing radius of C. We denote the packing radius of C with respect to the p-Lee metric by $e_p(C)$. A perfect code with respect to the p-Lee metric is a code for which the balls $B_p(c, e_p(C))$ cover the whole space $\mathbb{Z}_q^n$ when c runs over the elements of C. In this case we also say that C is an e-perfect code where $e = e_p(C)$. If $C \subseteq Z_q^n$ is an e-perfect code in the p-Lee metric for $p = 1$ or $p = \infty$ we have the relation $d_p(C) = 2e + 1$. In general, the minimum distance d_p is not determined by the packing radius e_p for $1 < p < \infty$, see [6].

We also consider codes in $\mathbb{Z}^n$ (defined as subsets of $\mathbb{Z}^n$) and we denote by d_p the metric ℓ_p given by $d_p(x,y) = \sqrt[p]{\sum_{i=1}^n |x_i - y_i|^p}$. The concept of perfect codes, packing radius and minimum distance of a code is analogous to the respective concepts over $\mathbb{Z}_q$. We say that a code $C \subseteq \mathbb{Z}^n$ is q-periodic for all $c \in C$ and $x \in \mathbb{Z}^n$ we have $c + qx \in C$, that is $C + q\mathbb{Z}^n \subseteq C$. We denote by $\pi : \mathbb{Z}^n \to \mathbb{Z}_q^n$ the natural projection (taking modulo q in every coordinate). There is a correspondence between q-periodic codes in $\mathbb{Z}^n$ and codes in $\mathbb{Z}_q^n$ induced by π, and this correspondence preserve linearity. This way of constructing linear codes (or lattices) in $\mathbb{Z}^n$ from a linear code in $\mathbb{Z}_q^n$ is the so called Construction A, see [7]. If a code $C \subseteq \mathbb{Z}_q^n$ verifies $d_p(C) \geq q$ then the corresponding code $\pi^{-1}(C) \subseteq \mathbb{Z}^n$ has the same minimum distance [21], in fact we have $d_p(\pi^{-1}(C)) = \min\{d_p(C), q\}$. In particular, since the ∞-Lee metric is upper bounded by q, we have the correspondence

$$\{\text{Perfect } q\text{-periodic codes in } (\mathbb{Z}^n, \ell_\infty)\} \underset{\pi^{-1}}{\overset{\pi}{\rightleftarrows}} \{\infty\text{-Lee Perfect codes in } \mathbb{Z}_q^n\}.$$

This correspondence preserve the packing radius and minimum distance. We remark that for $p < \infty$ a similar correspondence only exists for "large alphabets" (for example, when $q \geq 2e + 1$ if $p = 1$).

Every ball $B \subseteq \mathbb{Z}^n$ is associated with a polyomino $P_B \subseteq \mathbb{R}^n$ given by

$$P_B = \bigcup_{x \in B} x + [-1/2, 1/2]^n .$$

In this way, tilling Z^n by translated copies of B is equivalent to tilling $\mathbb{R}^n$ by translated copies of its associated polyomino P_B. This association give us an important geometric tool to study perfect codes over $\mathbb{Z}$. In the seminal paper of Golomb and Welch [10], the authors uses this approach to settle several results in perfect Lee codes over large alphabets.

The following specific notations and definition are used along this paper. For p ($1 \leq p \leq \infty$) we denote by
$PL^p(n, e, q) = \{C \subseteq \mathbb{Z}_q^n : C \text{ is } e\text{-perfect in the } p\text{-Lee metric}\}$,
$LPL^p(n, e, q) = \{C \in PL^p(n, e, q) : C \text{ is linear}\}$,
$CPL^p(n, e, q) = \{C \in LPL^p(n, e, q) : C \text{ is cyclic}\}$,
and for $C \in PL^\infty(n, e, q)$ we denote by f_C the corresponding error-correcting function, that is, the function $f_C : \mathbb{Z}_q^n \to C$ given by $f_C(x) = c \Leftrightarrow x \in B_p(c, e)$.

Definition 1: The trivial (perfect) codes in $\mathbb{Z}_q^n$ are the codes $\{0\}$ and $\mathbb{Z}_q^n$. A n-dimensional cartesian q-ary code is a code of the form $(2e+1)\mathbb{Z}_q^n$ for some $e \in \mathbb{N}$ such that $2e + 1 \mid q$.

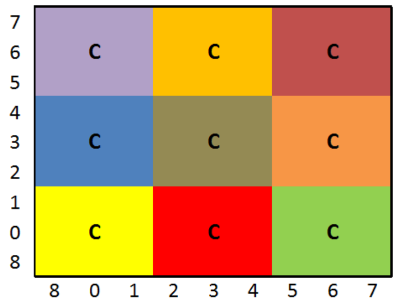

Fig. 1. The cartesian code $3\mathbb{Z}_9^2 \in LPL^\infty(2, 1, 9)$, the codewords are marked with C.

Definition 2: We say that a code $C \in PL^\infty(n, e, q)$ is standard if there exists a canonical vector e_i (i.e. a vector with an 1 in the i-st coordinate and 0 in the other coordinates) for some $i : 1 \leq i \leq n$ such that $C + (2e+1)e_i \subseteq C$. In this case we say that C is of type i (or a type i code).

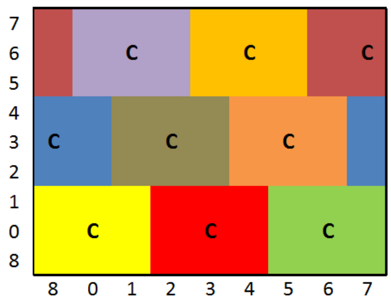

Fig. 2. The cyclic perfect code $C = \langle(2, 3)\rangle \in \mathbb{Z}_9^2 \in CPL^\infty(2, 1, 9)$ is a type 1 code but is not a type 2 code, the codewords are marked with C.

Remark 3: As we will see later, a code could be of no type or it could be of type i for different values of i (for example the cartesian codes are type i for $1 \leq i \leq n$).

III. Classification Theorem for Two-Dimensional Perfect Codes in the ∞-Metric

In this section we focus on two-dimensional perfect codes with respect the ∞-Lee metric and the main result is Theorem 14. The following results are summarized in Theorem 14 and the main tools used in the proofs are the sphere packing condition and some results from group theory such as the "product formula for subgroups" ($\#(H + K) = \#H \cdot \#K / \#(H \cap K)$ for H and K subgroups of a finite group $(G, +)$) and some standard formulas for calculating the order of an element of a group. It can also be helpful interpreting the results in term of tilling by polyominoes.

Lemma 4: A necessary and sufficient condition for the existence of a n-dimensional q-ary e-perfect code in the ∞-Lee metric is that $2e+1|q$. Moreover, if this condition is satisfied there exist a code in $LPL^\infty(n,q,e)$.

Corollary 5: There exists a non trivial perfect code over $\mathbb{Z}_q$ if and only if q is neither a power of 2 nor a prime number.

It is immediate to see that the only perfect codes C in $PL^\infty(1,e,q)$ are of the form $a+(2e+1)\mathbb{Z}_q$ where $q=(2e+1)t$. If we fixed a function $h:\mathbb{Z}_t\to\mathbb{Z}_q$ we can construct a two-dimensional q-ary perfect code as follows:

- Horizontal construction: $C_1(a,h)=\{(h(k),a+(2e+1)k):k\in\mathbb{Z}_t\}+(2e+1)\mathbb{Z}e_1$ where $e_1=(1,0)\in\mathbb{Z}_q^2$.
- Vertical construction: $C_2(a,h)\{(a+(2e+1)k,h(k)):k\in\mathbb{Z}_t\}+(2e+1)\mathbb{Z}e_2$ where $e_2=(0,1)\in\mathbb{Z}_q^2$.

Proposition 6: The codes obtained by horizontal (vertical) construction are e-perfect q-ary code type 1 codes (type 2 codes).

Remark 7: It can be prove easily that in fact, every standard two-dimensional perfect code can be obtained from a one-dimensional perfect code using horizontal or vertical construction.

The following geometric lemma is the analogous of the "Lema do estilingue" [19] and it is used in a similar way to prove the next proposition.

Lemma 8: Let e_i be the i-st canonical vector of $\mathbb{Z}_q^n$ and let $\pi_i:\mathbb{Z}_q^n\to\mathbb{Z}_q$ be the canonical projection (i.e. $\pi(x)$ is the i-st coordinate of x). Let $C\in PL^\infty(n,e,q)$ and f_C its correcting error function. Let x be an element of $\mathbb{Z}_q^n$.

- If $f_C(x)\neq f_C(x-e_i)$ then $\pi_i\circ f_C(x)=\pi_i(x)+e\cdot e_i$.
- If $f_C(x)\neq f_C(x+e_i)$ then $\pi_i\circ f_C(x)=\pi_i(x)-e\cdot e_i$.

Proposition 9: Every two-dimensional perfect code in the ∞-Lee metric is standard.

Remark 10: Proposition 9 cannot be generalized to greater dimensions. Let $C=\{000,500,105,605,150,651,155,656\}\subseteq\mathbb{Z}_{10}^3$. Then $C\in PL^\infty(3,2,10)$ is a three-dimensional non-standard perfect code.

Now, for linear perfect codes, a generator matrix is given in the next proposition.

Proposition 11: Let $C\in LPL^\infty(2,e,q)$ with $q=(2e+1)t$. Let $d_1=\gcd(2e+1,q)$ and $h_1=\frac{2e+1}{d_1}$. A generator matrix for C is given by

$$\begin{pmatrix} 2e+1 & 0\\ kh_1 & 2e+1\end{pmatrix}\in\mathcal{M}_{2\times2}(\mathbb{Z}_q)\quad\text{if } C \text{ is of type } 1,$$
$$\begin{pmatrix} 0 & 2e+1\\ 2e+1 & kh_1\end{pmatrix}\in\mathcal{M}_{2\times2}(\mathbb{Z}_q)\quad\text{if } C \text{ is of type } 2.$$

The next goal is determining when two linear q-ary perfect codes in dimension two are isomorphic (in the sense of group theory) and what isomorphism class of abelian groups are represented by perfect linear q-ary code in the ∞-Lee metric. In particular, we shall describe which of those are cyclic. Since the cardinality of a perfect q-ary code is determined by the packing radius, it suffices to classify isomorphism classes in the set $LPL^\infty(2,e,q)$ for a fixed value of e. Indeed, if $C\in LPL^\infty(2,e,q)$ and $q=(2e+1)t$ then by the packing sphere condition $\#C=t^2$. The following Lemma is a straightforward consequence of the Structure theorem for finitely generated abelian group.

Lemma 12: If C is an abelian group of order t^2 then there is an unique divisor $d|t$ such that $C\simeq\mathbb{Z}_{t/d}\times\mathbb{Z}_{dt}$.

Remark 13: This mean that the question of what isomorphism class are represented by two-dimensional perfect codes in the ∞-Lee metric is equivalent to determining for what values of $d|t$ there exists $C\in LPL^\infty(2,e,q)$ such that $C\simeq\mathbb{Z}_{t/d}\times\mathbb{Z}_{dt}$.

Now, we summarize all the previous results in the following theorem.

Theorem 14: For two-dimensional perfect codes in the ∞-Lee metric over $\mathbb{Z}_q$ we have:
i. (Existence) The following statements are equivalent
• There exists a non-trivial code $C\in PL^\infty(2,e,q)$
• There exists a non-trivial code $C\in LPL^\infty(2,e,q)$
• $q=(2e+1)t$ where e and t are positive integers with $t>1$.
ii. (Characterization) Every two-dimensional q-ary type 1 perfect code C is standard and can be expressed as $C=C_1(a,h)$ for some $a\in\mathbb{Z}_q$ and some function $h:\mathbb{Z}_t\to\mathbb{Z}_q$. Moreover, if C is linear, a generating set for C is given by $\{(2e+1,0),(kh_1,2e+1)\}$ where $h_1=\frac{2e+1}{d_1},d_1=\gcd(2e+1,t)$ and $k\in\mathbb{Z}$ (that can be chosen such that $0\leq k<d_1$).
iii. (Structure) Let $q=(2e+1)t$. There exists a code $C\in LPL^\infty(2,e,q)$ isomorphic to $\mathbb{Z}_{t/d}\times\mathbb{Z}_{dt}$ if and only if $d\mid\gcd(2e+1,t)$. Moreover, if $d_1=\gcd(2e+1,t),h_1=\frac{2e+1}{d_1},k\in\mathbb{Z},d_2=\gcd(k,d_1),h_2=\frac{d_1}{d_2}$ and $C\in PLP^\infty(2,e,q)$ generated by $\{(2e+1,0),(kh_1,2e+1)\}\subseteq\mathbb{Z}_q^2$. Then, $C\simeq\mathbb{Z}_{t/h_2}\times\mathbb{Z}_{th_2}$. If $k'\in\mathbb{Z}$ is such that $kk'\equiv d_2\pmod{d_1}$ we have that $M=\begin{pmatrix}0 & (2e+1)h_2\\ h_1d_2 & (2e+1)k'\end{pmatrix}$ is a generator matrix for C and an explicit isomorphism $\phi:C\to\mathbb{Z}_{t/h_2}\times\mathbb{Z}_{th_2}$ is given by $\phi(x,y)M=(x\pmod{t/h_2}),y\pmod{th_2})$ for $x,y\in\mathbb{Z}$.

Remark 15: Composing with the linear isometry $\phi:\mathbb{Z}_q^2\to\mathbb{Z}_q^2$ given by $(x,y)\mapsto(y,x)$ we can obtain analogous results for type 2 codes. We also recall that, by Proposition 9, if C is a two-dimensional perfect code, then either C or $\phi(C)$ is a type 1 code (and the other is a type 2 code).

Proof: Existence follows from Lemma 4 and characterization follows from Prop. 9 and Prop. 11. For the last part (structure), we consider (by hypothesis) a generator matrix for C of the form $N=\begin{pmatrix}2e+1 & 0\\ kh_1 & 2e+1\end{pmatrix}$. First, we shall prove that the matrix $M=\begin{pmatrix}0 & (2e+1)h_2\\ h_1d_2 & (2e+1)k'\end{pmatrix}$ is also a generator matrix for C. For this, we consider the matrices $U=\begin{pmatrix}-k_1 & h_2\\ \frac{1-k_1k'}{h_2} & k'\end{pmatrix}$ and $V=\begin{pmatrix}-k' & h_2\\ \frac{1-k_1k'}{h_2} & k_1\end{pmatrix}$ where $k_1=\frac{k}{d_2}$.

Is straightforward to check that $UN = M$ and $VM = N$ (as matrices over $\mathbb{Z}_q$), which prove that M is a generator matrix for C. If $x, y \in \mathbb{Z}$ verify $(x, y)M = (0, 0) \in \mathbb{Z}_q^2$ is straightforward to check that this is equivalent to congruences $x \equiv 0 \pmod{t/h_2}$ and $y \equiv 0 \pmod{h_2 t}$, this proves that ϕ is an isomorphism of abelian group. We finish observing that $h_2 \mid \gcd(2e+1, t)$, so every $C \in LPL^{\infty}(2, e, q)$ is isomorphic to $\mathbb{Z}_{t/d} \times \mathbb{Z}_{td}$ for some $d \mid \gcd(2e+1, t)$, and for each choice of such d it is easy to construct a perfect code isomorphic to it. ■

Corollary 16: There exists a perfect code in the ∞-Lee metric such that $C \simeq \mathbb{Z}_a \times \mathbb{Z}_b$ and $a \mid b$ if and only if ab is a perfect square and b/a is an odd number.

Corollary 17: Let $C \in LPL^{\infty}(2, e, q)$ with $q = (2e+1)t$. Then, $C \simeq \mathbb{Z}_t \times \mathbb{Z}_t \Leftrightarrow C$ is a cartesian code (that is, $C = (2e+1)\mathbb{Z}_q^2$).

Corollary 18: There exists a linear two-dimensional q-ary perfect code C that is neither trivial nor cartesian if and only if $q = p^2 a$ where p is an odd prime number and a is a positive integer.

Example 19: The first value of q for which there exists a two-dimensional q-ary perfect code that is neither standard nor cartesian nor cyclic is for $q = 3^2 \cdot 2$. An example of such code has generators $\{(0, 9), (1, 3)\} \subseteq \mathbb{Z}_{18}^2$, see Figure 3.

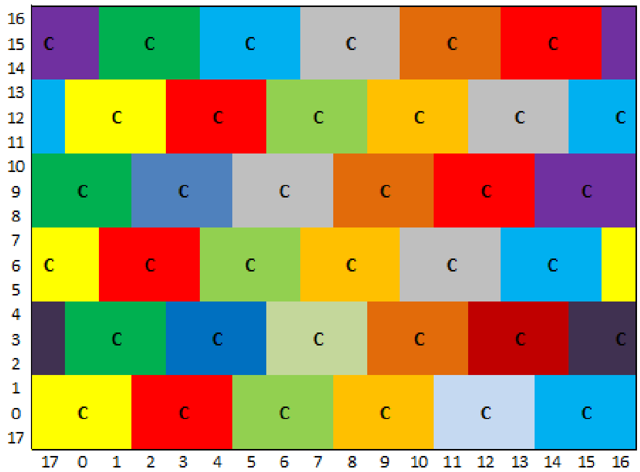

Fig. 3. For $q = 3^2 \cdot 2$ we have the perfect code $C = \langle (0, 9), (1, 3)\rangle \subseteq \mathbb{Z}_{18}^2$ which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{18}$, therefore this code is neither trivial nor cartesian nor cyclic.

Corollary 20: There exists a two-dimensional cyclic q-ary perfect code if and only if $q = p^2 a$ where p is an odd prime number and a is an odd positive integer.

Corollary 21: Let $q = (2e+1)t$. We have $CPL^{\infty}(2, e, q) \neq \emptyset \Leftrightarrow t \mid 2e+1$. In this case, a code $C = \langle (2e+1, 0), (kh_1, 2e+1)\rangle \in CPL^{\infty}(2, e, q)$ if and only if $\gcd(k, 2e+1) = 1$.

IV. Recursive construction of perfect codes in the ∞-Lee metric for all dimensions

In this section we give some construction of perfect codes in the ∞-Lee metric from perfect codes in smaller dimension. We use these construction to generalize some results given in the previous section and to construct some interesting families of perfect codes.

The simplest way to obtain ∞-Lee perfect codes is using cartesian product. The proof that it works is using the sphere packing condition and will be omitted.

Proposition 22 (Cartesian product construction): If $C_1 \in PL^{\infty}(n_1, e, q)$ and $C_2 \in PL^{\infty}(n_1, e, q)$ then $C_1 \times C_2 \in PL^{\infty}(n_1 + n_2, e, q)$. This construction preserves linearity.

Corollary 23: There exists a linear n-dimensional q-ary perfect code C that is neither trivial nor cartesian if and only if $q = p^2 a$ where p is an odd prime number and a is a positive integer.

Corollary 24: If $q = (2e+1)t$ and $d_1, d_2, \dots, d_k$ are divisors (not necessarly distinct) of $\gcd(2e+1, t)$, there exists a code $C \in LPL^{\infty}(2k, e, q)$ such that $C \simeq \mathbb{Z}_{\frac{t}{d_1}} \times \mathbb{Z}_{\frac{t}{d_2}} \times \dots \mathbb{Z}_{\frac{t}{d_k}} \times \mathbb{Z}_{d_1 t} \times \mathbb{Z}_{d_2 t} \times \dots \mathbb{Z}_{d_k t}$ and a code $C \in LPL^{\infty}(2k+1, e, q)$ such that $C \simeq \mathbb{Z}_{\frac{t}{d_1}} \times \mathbb{Z}_{\frac{t}{d_2}} \times \dots \mathbb{Z}_{\frac{t}{d_k}} \times \mathbb{Z}_t \times \mathbb{Z}_{d_1 t} \times \mathbb{Z}_{d_2 t} \times \dots \mathbb{Z}_{d_k t}$.

Remark 25: As we will show next, there are others linear perfect codes whose group structure is not of the form given in Corollary 24 (for example cyclic perfect codes in every dimension).

The next construction is exclusively for linear codes, this lead us to construct a linear perfect q-ary code from other codes of smaller dimension.

Proposition 26 (Linear construction): Let C and (n, e, q)-perfect code in the ∞-Lee metric with $q = (2e+1)t$ and $x \in \mathbb{Z}_q^n$ is such that $tx \in C$, then $\widetilde{C} = C \times \{0\} + (x, 2e+1)\mathbb{Z} \in LPL^{\infty}(n+1, e, q)$.

Sketch of the proof: We denote by $\|x\|_{\infty} = d_{\infty}(x, 0)$. As $tx \in C$ every codeword $v \in \widetilde{C}$ can be written as $v = (c + xk, (2e+1)k)$ with $c \in C$ and $0 \leq k < t$ and we have

$$\|(c + xk, (2e+1)k)\|_{\infty} = \max\{\|c + xk\|_{\infty}, \|(2e+1)k\|_{\infty}\}. \tag{1}$$

If $k = 0$, then $\|(c + xk, (2e+1)k)\|_{\infty} = \|c\|_{\infty} \geq 2e+1$ if $c \neq 0$ (because C have packing radius e). If $0 < k < t$, then $\|(2e+1)k\|_{\infty} \geq 2e+1$ and by Equation (1) we have $\|(c + xk, (2e+1)k)\|_{\infty} \geq 2e+1$. We conclude that C has packing radius at last e. We want to calculate the cardinality of C, that is

$$\#C = \frac{\#C \times \{0\} \cdot \#(x, 2e+1)\mathbb{Z}}{\#C \times \{0\} \cap (x, 2e+1)\mathbb{Z}}. \tag{2}$$

We have $\#C \times \{0\} = \#C = t^n$. Let θ be the additive order of tx in $\mathbb{Z}_q^n$ (i.e. the least positive integer θ such that $\theta tx = 0$). It is straightforward to check that the order of $(x, 2e+1)$ in $\mathbb{Z}_q^{n+1}$ is $t\theta$ and that $C \times \{0\} \cap (x, 2e+1)\mathbb{Z} = (tx, 0)\mathbb{Z}$. Using Equation (2) we have $\#C = \frac{t^n \cdot t\theta}{\theta} = t^{n+1}$ and by the sphere packing condition the code $\widetilde{C} \subseteq \mathbb{Z}_q^{n+1}$ is perfect with packing radius e. ■

Corollary 27: If $q = (2e+1)t$ with $t^{n-1} \mid 2e+1$ and $n \geq 1$, then the q-ary cyclic code

$$\mathcal{C}_{n,e,q} = \left\langle \left(\frac{2e+1}{t^{n-1}}, \frac{2e+1}{t^{n-2}}, \ldots, \frac{2e+1}{t}, 2e+1 \right) \right\rangle$$

is a cyclic perfect code in $CPL^{\infty}(n,e,q)$.

Corollary 28: If $q = t^n$ where t is an odd number, then the q-ary code $C = \langle (1, t, t^2, \ldots, t^{n-1}) \rangle \in CPL^{\infty}(n,e,q)$ for the packing radius $e = (t^{m-1} - 1)/2$.

V. Conclusion

In this work we have studied the problem of describing the perfect codes in the ∞-Lee metric over the alphabet $\mathbb{Z}_q$ and which isomorphism classes of abelian groups can be represented for such codes. Theorem 14 in Section 3 solves this problem for the two-dimensional case and several corollaries are obtained from this. For greater dimensions, we presented two recursive constructions of perfect codes in the ∞-Lee metric obtained interesting families of perfect codes (such as those that appear in Corollaries 27 and 28) and extending some results obtained for the two-dimensional case. It should be interesting to obtain extensions of Theorem 14 to higher dimensions. Regarding Proposition 9, as remarked it cannot be extended to higher dimensions, although we can prove that an analogous result holds for linear perfect codes in dimension 3. We wonder if with this extra condition the result can be extended to higher dimension.

References

[1] J. Astola, ´´On perfect Lee codes over small alphabets of odd cardinality". Discrete Applied Mathematics, vol. 4, pp. 369-380. 1975.

[2] B. F. Al-Bdaiwi, P. Horak, L. Milazzo. ´´Enumerating and decoding perfect linear Lee codes". Des. Codes. Crypt., vol. 52 no. 2, pp. 155-162, 2009.

[3] B. F. Al-Bdaiwi, P. Horak, L. Millazo, ´´Perfect 1-error correcting Lee codes". Des.Codes Cryptogr. 52: 155-162, 2009.

[4] M. Blaum, J. Bruck, A. Vardy, Interleaving schemes for multidimensional cluster errors, IEEE trans. Inform. Theory, vol. IT-53, pp. 808-814, 2007.

[5] S. I. R. Costa, A. Campello, G. C. Jorge, J. E. Strapasson, C. Qureshi. ´´Codes and lattices in the lp metric". Information Theory Workshop (ITW), 2014 IEEE. 2014.

[6] A. Campello, G. C. Jorge, J. E. Strapasson, S. I. R. Costa, ´´Perfect codes in the ℓ_p metric", see ref, 2015.

[7] J. H. Conway, N. J. A. Sloane, ´´Sphere packings, lattices and groups", Springer-Verlag, New York, 3rd Ed, 1998.

[8] T. Etzion, E. Yaakobi, ´´Error-Corretion of Multidimensional Burst", IEEE trans. Inform. Theory, vol. IT-55, pp. 961-976, 2009.

[9] S. Gravier, M. Mollard, Ch. Payan, ´´On the existence of three-dimensional tiling in the Lee metric", Eur. J. Comb. 19: 567-572, 1998.

[10] S. W. Golomb, L. R. Welch, ´´Perfect Codes in the Lee metric and the packing of polyominoes", SIAM Journal Applied Math. 18: 302-317, 1970.

[11] P. Horak, ´´On Perfect Lee Codes", Discrete Mathematics 309: 5551-5561. 2009.

[12] P. Horak, B. F. Al-Bdaiwi, ´´Non-periodic tiling of Rn by crosses", Discrete Comput.Geom. 47: 1-16. 2012.

[13] P. Horak, O. Grosek, ´´A new approach towards the Golomb-Welch conjecture", Eur. J. Comb. 38: 12-22. 2014.

[14] P. Horak, V. Hromada, ´´Tiling R5 by Crosses", Discrete Comput.Geom. 51, 269-284. 2014.

[15] G. C. Jorge, A. Campello, S. I. R. Costa, ´´q-ary lattices in the ℓ_p norm and a generalization of the Lee metric", International Workshop on Coding and Cryptography (WCC 2013), 2013.

[16] T. K. Moon, ´´Error Correction Coding: Mathematical methods and algorithms", Hoboken, N.J.: Wiley-Interscience, 2005.

[17] C. Peikert, ´´Limits on the hardness of lattice problems in ℓ_p norms", Computational Complexity, vol. 17(2): 300-351, 2008.

[18] K. A. Post, "Nonexistence theorems on perfect Lee-codes over large alphabets", Information and Control, 29: 369-380, 1975.

[19] C. Qureshi, S. I. R. Costa, ´´Classificação dos códigos bidimensionais na métrica de Lee", XXXI Simpósio brasileiro de telecomunicações, 2013.

[20] R. M. Roth, P. H. Siegel, ´´Lee-metric BCH codes and their application to constrained and partial-response channels", IEEE trans. Inform. Theory, vol. IT-40, pp. 1083-1096. 1994.

[21] J. A. Rush, N. J. A. Sloane, ´´An improvement to the Minkowski-Hlawka bound for packing superballs", Mathematikal, 34: 8-18, 1987.

[22] K. U. Schmidt, ´´Complementary sets, generalized Reed-Muller codes, and power control for OFDM", IEEE trans. Inform. Theory, vol. IT-53, pp. 808-814, 2007.