A Pseudo Inverse Solution Method Applied to an Adaptive Antenna

Fábio A. L. Gomes, Adrião Duarte D. Neto, Aldayr D. Araújo, Wilson da Mata
Departamento de Engenharia Elétrica - Universidade Federal do Rio Grande do Norte
Campus Universitário, Lagoa Nova, CEP 59072-970, Natal/RN – Brasil
adrianol@decom.fee.unicamp.br, adriao@leca.ufrn.br, aldayr@leca.ufrn.br, wilson@ct.ufrn.br

Abstract — *The adaptive antenna theory has been studied in several application fields such as radar, sonar, mobile communications, geophysics exploitation, etc. In all these cases, the different methods used to solve problems in the adaptive antenna approach generally use the same procedure, which is related to the increasing of the directivity of the antenna array and to the interfering signals adaptive cancellation. The research to obtain more efficient methods has motivated the adaptive antennas approach to use traditional Least Mean Square (LMS) method or state space variables methods such as Kalman Filter and State Observers. This work presents an algorithm to obtain the optimum gains of the Generalized Side Lobe Canceller Filter (GSC Filter) using a pseudo inverse matrix. Additionally, the importance of the regularization factor to the robustness of the method is verified. The method is recursive, with low computational complexity and fast calculation of the optimum gains. The method is applied to the cancellation of interfering signals in a non-stationary condition of an adaptive antenna array. A matrix regularization process is developed to get a robust method and a better matrix conditioning. The simulation results show the efficiency of the proposed method to cancel interfering signals in planar antennas. The simulation results were done with a 3x3 half wavelength elements in planar array for both signals (desired and interfering). A 40dB was considered for the relationship between the desired signal and the interfering signal. The irradiation diagrams were obtained in two dimensions for a fixed angular coordinate. The method seems to be efficient to generate null signals in the interfering signal direction. The LMS method was used to compare the performance with the proposed method.*

Key Words — Adaptive Filters, Adaptive Antenna, Pseudo Inverse Matrix.

1. Introduction

The theory of adaptive antennas [1]-[10] is the object of study in several application fields such an as radar, sonar, mobile communications, geophysical exploration, etc. In anyone of those cases, the different methods used to treat the problem of adaptive antennas possess a same objective, which consists of the increase of the directivity of the antennas arrangement and the adaptive cancellation of the interference. The research to obtain more efficient methods has motivated the adaptive antennas approach to use traditional Least Mean Square [3] - [5] or state space variables such as Kalman filter and State Observers [7]-[9].

This work presents a robust algorithm using a pseudo inverse matrix to obtain the optimum gains of an arrangement of antennas. This method intends to treat the problem in study, by obtaining the great determination of the GSC filter gain in matrix form, allowing to model the problem in a such way that the optimum gains could be treated by any commonly methods used in the solution of linear equations.

The work is organized in the following way. In section 2 a description of GSC will be presented. In section 3, a matrix representation will be shown for the GSC problem. In section 4, the method of the pseudo inverse matrix and the proposed algorithm are presented. In section 5, the results and conclusions are presented.

2. Generalized Side Lobe Canceller Filter – GSC

The structure in figure 1 is known as the Generalized Side Lobe Canceller Filter – GSC. The entrance vector sign u(n) is the resulting sign of an arrangement of antennas. The antennas are here considered like M isopropic sensors uniformly spaced.

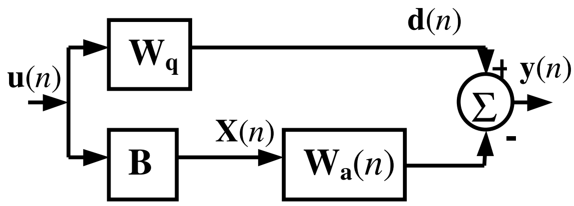

Figure 1: GSC structure

The output vector sign y(n) of the GSC filter is given by:

$$y(n) = \mathbf{w}^H \mathbf{u}(n) \qquad (1)$$

where H is the transposed conjugated matrix.

The optimum weight vector $\mathbf{w}_0$ is calculated by the minimization of the output power sign, in accordance to L linear restrictions, described by the following equation:

$$\mathbf{B}^H \mathbf{w} = \mathbf{g} \qquad (2)$$

The matrix **B** is a restriction matrix a matrix with dimension M x L. **g** is the gains vector with dimension Lx1.

The component non adaptive of the GSC is represented by:

$$\mathbf{w_q} = \mathbf{Bq} = (\mathbf{B^H B})^{-1}\mathbf{g} \quad (3)$$

For the whole GSC structure, the weight expression is defined by:

$$\mathbf{w} = \mathbf{w_q} - \mathbf{B}_a \mathbf{w_a} \quad (4)$$

where $\mathbf{B_a}$ is the blocking matrix **B,** that means, $\mathbf{B_a}$ is a null space of the matrix $\mathbf{B}^T$. The vector $\mathbf{w}_a$ is the adaptive component.

3. Matrix Analysis

The determination of the gains of an arrangement of antennas can be reduced to a solution of a linear equations system with M unknown and of lineal equations of incognito *M* and *M* equations. For that, the optimum gains can be obtained by minimizing the function:

$$y(n) = d(n) - \mathbf{w}_a^H \cdot \mathbf{x}(n) \quad (5)$$

where

$$d(n) = \mathbf{w}_q^H \cdot \mathbf{u}(n)$$

$$\mathbf{x}(n) = \mathbf{B}^H \cdot \mathbf{u}(n)$$

The objective is to have a null output signal, defined in (5) for an interference u(n).

$$0 = d(n) - \mathbf{w}_a^H \cdot \mathbf{x}(n) \quad (6)$$

The interference signal matrix $\mathbf{X}(n)$ and wished output signals $\mathbf{D}(n)$ are defined by:

$$\mathbf{X}(n) = [\mathbf{x}(1)\ \mathbf{x}(2) \cdots \mathbf{x}(n)\ ] \quad (7)$$

$$\mathbf{D}(n) = [d(1)\ d(2) \cdots d(n)] \quad (8)$$

Substituting these matrixes in (6), we have:

$$\mathbf{w}_a^H(n) \cdot \mathbf{X}(n) = \mathbf{D}(n) \quad (9)$$

or

$$\mathbf{X}^H(n) \cdot \mathbf{w}_a(n) = \mathbf{D}^H(n) \quad (10)$$

The equation A (10) still isn't the ideal solution for an on-line processing, because it has the following inconveniences: (i) it can have no solution, that means, the vector $\mathbf{D}^H$ doesn't belong to the column space of $\mathbf{X}^H$, or the *rank* de $\mathbf{X}^H$ is less than number of lines; (ii) according to the definitions given by (7) e (8), the number o columns of the matrixes o $\mathbf{X}(n)$ e $\mathbf{D}(n)$ grows linearly with *n* and, consequently, the necessary memory for the storage of these matrixes will also grow lineally with the time; (iii) if the equation (10) has more than one solution, that means, for the rank of $\mathbf{X}^H$ smaller than the number of columns, it becomes necessary to determine the optimum solution among those possible.

3.1 Guaranteeing the Existence of the System Solution

A generic linear system of equations is given by:

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b} \quad (11)$$

where $\mathbf{A}$ is a matrix *m* x *n* of coefficients, $\mathbf{x}$ is a *n* x 1 vector of unknown terms and $\mathbf{b}$ is a o *m x 1* vector of independent terms. It is demonstrated that a necessary and enough condition for a linear equations system (11) to have a solution is that the $\mathbf{b}$ vector belongs to the space column of the matrix $\mathbf{A}$. This means that $\mathbf{b}$ is a linear combination of the columns of $\mathbf{A}$ [11]. Therefore, to guarantee that the system (10) has a solution, it is enough to project the vector $\mathbf{D}^H$ on the space column of $\mathbf{X}^H$. That is equal to solve the system in its normal form or to solve the following system:

$$\mathbf{X}(n) \cdot \mathbf{X}^H(n) \cdot \mathbf{w}_a(n) = \mathbf{X}(n) \cdot \mathbf{D}^H(n) \quad (12)$$

The equation (12) can be written as:

$$\Phi(n) \cdot \mathbf{w}_a(n) = \mathbf{z}(n) \quad (13)$$

where $\Phi(n)$ is a correlation matrix of temporal interference signals given by:

$$\Phi(n) = \mathbf{X}(n) \cdot \mathbf{X}^H(n) \quad (14)$$

and $\mathbf{z}(n)$ is a cross matrix correlation between the interference signal and the wished signal expressed by:

$$\mathbf{z}(n) = \mathbf{X}(n) \cdot \mathbf{D}^H(n) \quad (15)$$

The matrix $\Phi(n)$ can be rewritten by:

$$\Phi(n) = \sum_{i=1}^{n} \mathbf{x}(i) \cdot \mathbf{x}^H(i) \quad (16)$$

where $\mathbf{x}(i)$ is the *i-th* column of $\mathbf{X}(n)$. Similarly, the vector of crossed correlation $\mathbf{z}(n)$ can be rewritten as:

$$\mathbf{z}(n) = \sum_{i=1}^{n} \mathbf{x}(i) \cdot d^*(i) \quad (17)$$

3.2 Recursive Calculation of the Temporary Correlation Matrix and the Temporary Crossed Correlation Vector

As mentioned for the solution of equation (10) and, in special, for the corresponding normal equation (13), it is necessary to storage the signals $\mathbf{x}(n)$ and $d(n)$, what makes unfeasible the use of those equations in a processing on-line. Therefore, it makes necessary a recursive method for the calculation of the matrix $\Phi(n)$ without necessarily the storage of the signals $\mathbf{x}(n)$ and $d(n)$. From the equation (16), it is obtained:

$$\Phi(n) = \sum_{i=1}^{n} \mathbf{x}(i) \cdot \mathbf{x}^H(i)$$
$$= \sum_{i=1}^{n-1} \mathbf{x}(i) \cdot \mathbf{x}^H(i) + \mathbf{x}(n) \cdot \mathbf{x}^H(n) \quad (18)$$
$$= \Phi(n-1) + \mathbf{x}(n) \cdot \mathbf{x}^H(n)$$

This equation is the sought recursive expression. Similarly, $\mathbf{z}(n)$ is given by:

$$\mathbf{z}(n) = \sum_{i=1}^{n} \mathbf{x}(i) \cdot d^*(i)$$
$$= \sum_{i=1}^{n-1} \mathbf{x}(i) \cdot d^*(i) + \mathbf{x}(n) \cdot d^*(n) \quad (19)$$
$$= \mathbf{z}(n-1) + \mathbf{x}(n) \cdot d^*(n)$$

The equations (18), (19) and (13) give, in this order, the recursive algorithm for the calculus of the adaptive gains $\mathbf{w}_a(n)$, using the least mean square method.

3.3 Pseudo Inverse Solution

The solution of a linear equation system (11) is, in general form, given by:

$$\bar{\mathbf{x}} = \mathbf{A}^+ \cdot \mathbf{b} \quad (20)$$

where $\mathbf{A}^+$ is a matrix *n x m* called *pseudo inverse* of $\mathbf{A}$. By equation (20), the pseudo inverse represents a generalization of the matrix inversion. It is important to say that any matrix has its pseudo inverse. Even the null matrix $\mathbf{0}_{mxn}$, given by $\mathbf{0}^+_{mxn} = \mathbf{0}_{nxm}$ [11].

The solution of equation system (20), using the pseudo inverse, corresponds to the minimization of the functional,

$$J(n) = \left\| \mathbf{A} \cdot \bar{\mathbf{x}} - \mathbf{b} \right\| \quad (21)$$

by the least mean square criteria, with the solution vector attending the following property: $\left\| \bar{\mathbf{x}} \right\| \le \left\| \tilde{\mathbf{x}} \right\|$. Where $\tilde{\mathbf{x}}$ represents any other solution for (20).

3.4 Pseudo Inverse Application in the Solution of Adaptive Antennas Problems

In general, to obtain the pseudo inverse means to calculate the eigen values and the eigen vectors. However, if the matrix $\mathbf{X}^H$ has L.I. columns, the solution of (10), using the pseudo inverse, is given by:

$$\mathbf{w}_{a,opt} = \mathbf{X}^{H+}(n) \cdot \mathbf{D}^H(n) \quad (22)$$

where $\mathbf{X}^{H+}$ is the pseudo inverse of $\mathbf{X}^H$, given by

$$\mathbf{X}^{H+}(n) = \left(\mathbf{X}(n) \cdot \mathbf{X}^H(n)\right)^{-1} \cdot \mathbf{X}(n) \quad (23)$$

Substituting (14) and (15) in (22), it gives:

$$\mathbf{w}_{a,opt} = \Phi^{-1}(n) \cdot \mathbf{z}(n) \quad (24)$$

Admitting that $\Phi(n)$ has an inverse, the solution of the system (13) gives the solution for the pseudo inverse method. It remains to determine in which situation this matrix admits an inverse matrix.

3.5 Matrix Regularization $\Phi(n)$

There are two possibilities for this problem: (i) A sufficient condition to guarantee that $\Phi(n)$ has an inverse is that the signals $\mathbf{x}(n)$, composing the columns of $\mathbf{X}(n)$, are sufficient rich in directions. This means, they are formed by the sum, at least, $\mathbf{K}$ signals from different directions, $\mathbf{K} > M - L$, where *M* is the number of antenna elements the antenna arrangement and L the number of restrictions, given by (2). It must be noted that $\mathbf{x}(n) = \mathbf{B}^H \cdot \mathbf{u}(n)$ has *L* signals, unless $\mathbf{u}(n)$, to guarantee that $\mathbf{x}(n)$ is sufficient rich in directions, the signals $\mathbf{u}(n)$ must have, at least, $M+1$ signals. Consequently, to say that $\mathbf{x}(n)$ is sufficiently rich in directions it is the same as to affirm that the entrance signals $\mathbf{u}(n)$ are too. (ii) To outline the need of the entrance signals be sufficiently rich in directions, it should be made use of the regularization theory, so that the solution of the system (13) becomes:

$$\mathbf{w}_{a,opt} = \left(\Phi(n) + \eta \cdot \mathbf{I}\right)^{-1} \cdot \mathbf{z}(n) \quad (25)$$

where η is the regularization factor.

4. Algorithm

The ideas discussed up to now about the use of the pseudo inverse for the calculation of the optimum gains of an arrangement of antennas can be summarized by the following three equations:

$$\Phi(n) = \Phi(n-1) + \mathbf{x}(n) \cdot \mathbf{x}^H(n) \quad (26)$$

$$\mathbf{z}(n) = \mathbf{z}(n-1) + \mathbf{x}(n) \cdot d^*(n) \quad (27)$$

$$\mathbf{w}_{a,opt} = \left(\Phi(n) + \eta \cdot \mathbf{I}\right)^{-1} \cdot \mathbf{z}(n) \quad (28)$$

However, two aspects need to be taken in consideration before the algorithm to be proposed. The first aspect is that in a non stationary stochastic process, the last samples should have little influence in the calculation of the new estimates of the gains of the arrangement of antennas. That can be obtained, multiplying those samples by a positive number, smaller than the unit. So, the equations (26) and (27) can be written as follow:

$$\Phi(n) = \lambda \cdot \Phi(n-1) + \mathbf{x}(n) \cdot \mathbf{x}^H(n) \quad (29)$$

$$\mathbf{z}(n) = \lambda \cdot \mathbf{z}(n-1) + \mathbf{x}(n) \cdot d^*(n) \quad (30)$$

where, the scalar λ, is called forgetfulness factor.

A second aspect refers to the equation (28). It is necessary to determine an adapted value for the regularization factor η. Reminding that the trace of a positive semi-defined symmetrical matrix, that is the case of $\Phi(n)$, is larger or equal to its maximum eigen value. It can be demonstrated that η can be determined by the following expression [11]:

$$\eta = \frac{tr[\Phi(n)]}{\alpha \cdot M_t} \tag{31}$$

where M_t is a dimension matrix $\Phi(n)$, and α a positive scalar greater or equal to the unit.

Being taken here in consideration all the factors discussed, the algorithm proposed for the estimate of the optimum gains of an arrangement of antennas is synthesized below:

Table 1: Summary of the algorithm proposed using the pseudo inverse

Known Parameters:

$\mathbf{x}(n)$ (inputs)

$d(n)$ (wished outputs)

M (Number of elements of an arrangement of antennas)

L (Number of linear restrictions)

Initialization:

$\Phi(0) = \mathbf{0}$

$\mathbf{z}(0) = \mathbf{0}$

For n = 1,2,3, ...

$$\Phi(n) = \lambda \cdot \Phi(n-1) + \mathbf{x}(n) \cdot \mathbf{x}^H(n)$$

$$\mathbf{z}(n) = \lambda \cdot \mathbf{z}(n-1) + \mathbf{x}(n) \cdot d^*(n)$$

$$\eta = \frac{tr[\Phi(n)]}{\alpha \cdot (M - L)}$$

$$\mathbf{w}_{a,opt} = (\Phi(n) + \eta \cdot \mathbf{I})^{-1} \cdot \mathbf{z}(n)$$

5. Results

To show the efficiency of the algorithm developed for the adjustment of the gains of an arrangement of antennas, some simulations were accomplished showing, first, the irradiation diagrams and after, the gains optimization. Those diagrams were obtained using the function $20 \cdot \log_{10}\left(\left| \mathbf{w}^H{}_{opt} \cdot \mathbf{s}(\phi,\theta) \right|\right)$, where $\mathbf{w}_{opt}$ are the optimized gains and $\mathbf{s}(\phi,\theta)$ are the propagation vectors in the directions (ϕ,θ). The LMS method, already used [3,4], will serve as reference. In all the simulations, a planar arrangement of antennas was used with 3 x 3 elements, half wave length apart. On the arrangement two signals arrive: a wished signal and an interference signal. The relationship signal/interference is 40 dB. For better visualization of the irradiation diagrams, these were traced in 2D, for $\phi = \theta$.

In the first simulation, the method used was the one presented in this work, synthesized in the table 1. The relationship signal/interference is -40 dB. It was also analyzed the influence of the regularization factor η on the conditioning of the correlation matrix $\Phi(n)$. For the calculation of the regularization factor, the equation (31), it will be considered, $M = 3x3 = 9$, $L = 1$ and $\alpha = 10$. Figure 2 shows the irradiation diagrams before and after the optimization using the methods of the pseudo inverse and the LMS. The wished signal is in the $\phi_d = 20^0, \theta_d = 30^0$ direction. The interference signal is in the $\phi_I = -20^0, \theta_I = -30^0$ direction.

The figures 3 and 4 show the conditional numbers of $\Phi(n)$ and $\Phi(n) + \eta \cdot \mathbf{I}$ as a function of the sample number. The forgetfulness factor λ was fixed to 0.5. The figure 5 exhibits the regularization factor variation η as a function of the sample number.

Analyzing the illustration in figure 2, it is observed that the optimized radiation diagrams for normalized LMS and for the pseudo inverse are almost coincident. It is also verified that both methods introduced a null exactly in the direction of the interference signal, $\theta_i = -30^0$. At the same time, the gain in the wished direction, $\theta_d = 30^0$, is close to 0 dB, meaning that the signal was preserved.

The figures 3, 4 and 5 show the efficiency of the regularization factor η in respect to the problem conditioning.

A second simulation was done to verify the effect of an additional gaussian white noise to the obtained results. For that, a gaussian white noise was added to the entrance assuming a relationship signal/noise (SNR) of -20 dB. In that simulation, the constant α assumed was 50.

Analyzing the results after the second simulation, we can conclude that the pseudo inverse method, as observed in previous simulations, introduced a null signal to the interference signal direction, preserving the wished signal (Figure 6). On the other hand, it is verified that the addition of the noise provoked a small oscillation on the conditional number of the temporary correlation matrix and also to the regularization factor. The method showed to be efficient and capable to trace even in the presence of the noise.

6. Conclusions

In this work was presented the development of the pseudo inverse method of the applied together with the

GSC, proceeding the adjustment of the gains of an arrangement of antennas. The algorithm focused the adaptive antennas problem purely under a matrix treatment way, allowing to solve it with no big knowledge in discreet linear control. It was also discussed the qualitative and quantitative aspects of the obtained solution showing how the regularization theory allows to transform a system not well conditioned in a system well conditioned.

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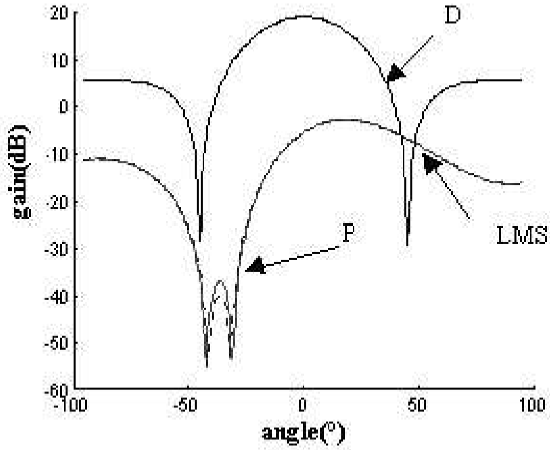

Figure 2: Radiation diagrams: (D) non optimized diagram, (LMS) optimized diagram using LMS method, (P) optimized diagram using pseudo inverse.

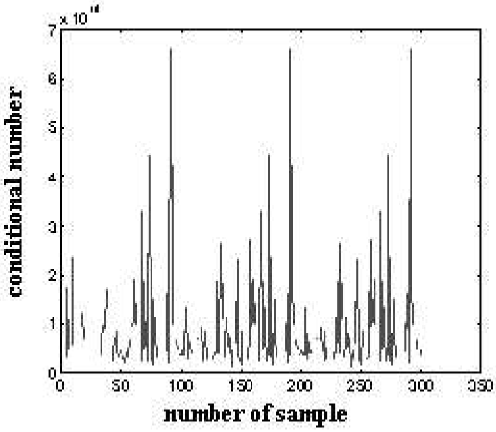

Figure 3: Conditional number of the correlation matrix $\Phi(n)$ as a function of the sample number.

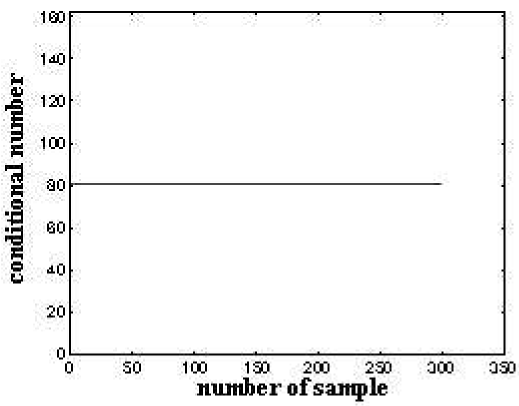

Figure 4: Conditional number of the correlation matrix after the addition of the regularization term $\eta \cdot \mathbf{I}$.

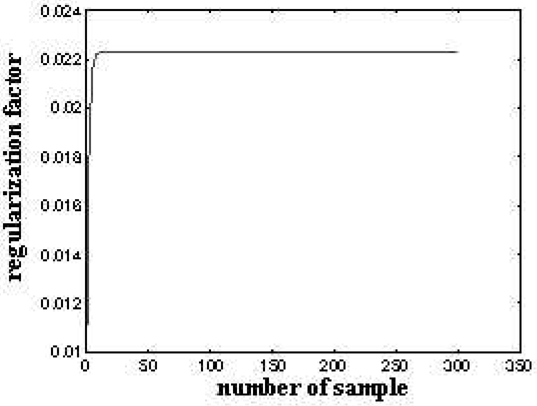

Figure 5: Evolution of the regularization factor η function of the sample number.

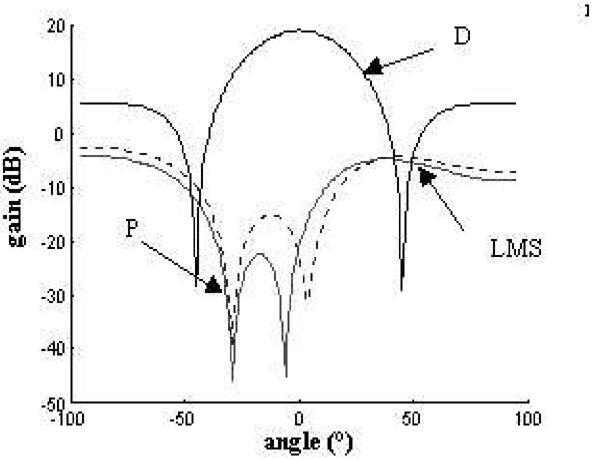

Figure 6: Radiation diagram including a white noise to the entrance signals of the arrangement of antennas.

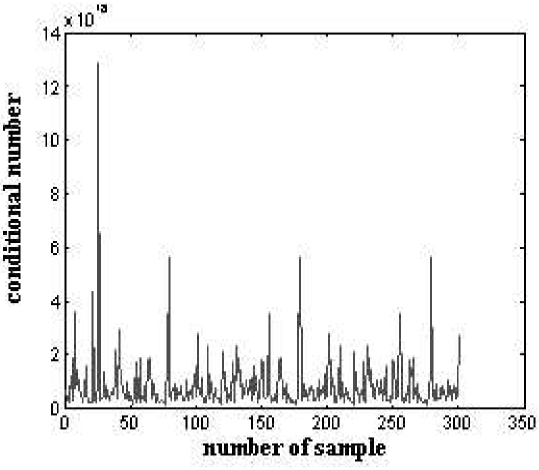

Figure 7: Conditional number of temporal correlation matrix before the regularization.

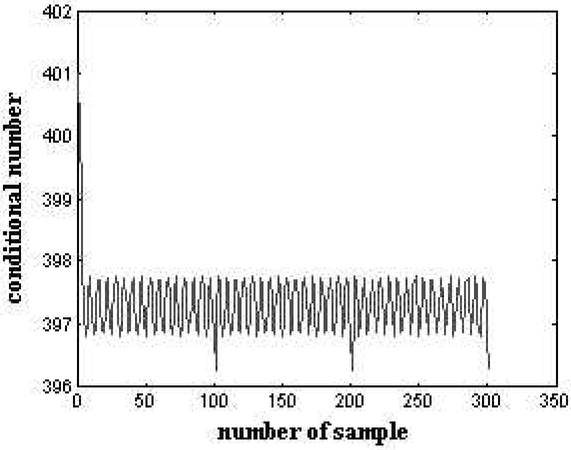

Figure 8: Conditional number of regularized temporal correlation matrix.

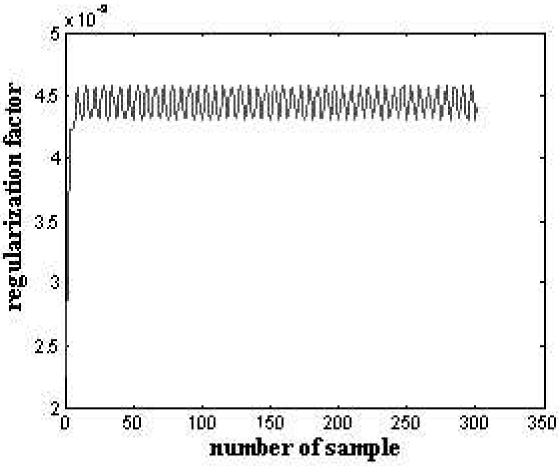

Figure 9: Regularization factor after the addition of the white noise.