A Solution of Rectangular Ridged Waveguide Using the Finite Element Method

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Abstract — **A numerical code based on the Finite Element Method (FEM) was developed to solve the ridged waveguide eigenvalue problem. In order to apply the FEM, the Galerkin Weak Formulation was used. The solution is obtained by using the Finite Element Method with quadratic triangular shape functions. The eigenvalue spectrum of the single and double rectangular ridged waveguide is shown and it is compared with other numerical approaches and the electrical field distribution of lower modes is also presented. The code was developed using C language.**

Key words — **Finite Element Method, Ridged Waveguide, Rectangular Ridged Waveguide, Second-order Triangular Element, Vector Field Distribution.**

I. INTRODUCTION

Ridged waveguides have been useful for several years in microwave systems requiring broadband operation and the ridged waveguide field problem has been investigated for many authors. In 1947 Cohn [1] obtained the ridged waveguide eigenvalues by using the transverse resonance technique. In 1955 Hopfner [2] extended the Cohn's work to other aspect ratios by inclusion of a first-order correction factor. Each of these previous investigations was primarily aimed at the solution for the TE_{n0} eigenvalue [3]. In order to performe a complete study of the ridged waveguide, in 1971, Montgomery [4] formulated an integral eigenvalue problem. In 1985 Utsumi [5] presented a variational method to obtain the approximate cut-off frequency and electromagnetic fields. Recently, 1999 Wu et al [6], investigated the ridged waveguide problem using a general spectral domain integral equation formulation.

In this work we investigated the field problem of the single and double rectangular ridged waveguides [7]-[8] with different aspect ratios and obtained the eigenvalue spectrum and the electric field distribution for arbitrary TE_{mn} and TM_{mn} modes by using the Finite Element Method (FEM) with quadratic triangular shape functions. The homogeneous Helmholtz equation is solved to yield a generalized matrix eigenvalue problem. Then the generalized eigenvalue problem is solved by applying Galerkin weak formulation method [9]-[10] and using Jacobi transformations method for evaluate the eigenvalues (cutoff frequency) and eigenvectors (fields) [13]-[14]. For numerical evaluation of eigenvalues (TE and TM modes) we have been used a numerical code, developed in C language, and the plots of the electric fields (eigenvectors) can be visualized by Scigraphica software, in the Linux environment, or Origin 5.0 for Windows environment.

The result of the investigation is the possibility of obtaining of any eigenvalue, at least in principle, with its associated field distribution by FEM application.

This paper is organized as follows: In Section II, the electromagnetic field problem is formulated. In Section III, a FEM for the guided wave propagation is outlined. In Section IV, the results are shown and discussed. Conclusions are in Section V.

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II. ELECTROMAGNETIC PROBLEM FORMULATION

In order to analyze the propagation characteristics of a lossless rectangular ridged waveguide we start from the Maxwell equations, where the electric field and magnetic field are denoted by $\vec{E}$ and $\vec{B}$, respectively:

$$\nabla \times \vec{E} = -j\,\omega\mu_0\,\vec{H}\,, \qquad \nabla \cdot \vec{E} = 0 \tag{1}$$

$$\nabla \times \vec{H} = j\,\omega\varepsilon_0\,\vec{E}\,, \qquad \nabla \cdot \vec{B} = 0 \tag{2}$$

where the harmonic variation $\exp(j\omega t)$ is assumed and $\omega = 2\pi f$ is the angular frequency of the electromagnetic wave. In (1), (2) μ_0 and ε_0 are the magnetic permeability and electric permittivity of vacuum, respectively. From (1) and (2), the vector Helmholtz equation can be derived as

$$\nabla \times \nabla \times \begin{Bmatrix} \vec{E} \\ \vec{H} \end{Bmatrix} = k_o^2 \begin{Bmatrix} \vec{E} \\ \vec{H} \end{Bmatrix}, \tag{3}$$

where $k_o = \omega\sqrt{\mu_o \varepsilon_o}$, is the free space wavenumber. By assuming that the z dependence can be given by $\exp(-j\beta z)$, where β is the propagation constant, and introducing the following notations

$$\vec{E} = \vec{E}_\perp + \hat{a}_z\,E_z\,, \tag{4}$$

$$\nabla = \nabla_\perp - j\beta\,\hat{a}_z, \tag{5}$$

that in a rectangular coordinates,

$E_\perp = E_x\hat{a}_x + E_y\hat{a}_y$, and $\nabla_\perp = \dfrac{\partial}{\partial_x}\hat{a}_x + \dfrac{\partial}{\partial_y}\hat{a}_y$,

where $\perp$ denotes the transverse parts of ∇ operator or field, for the transverse and parallel components of the wave equation, and $\hat{a}_z$ being a unit vector in the z direction, we can rewrite the vector Helmholtz equation as a pair of differential equations.

$$\nabla_\perp \times \left(\nabla_\perp \times \vec{E}_\perp\right) - j\beta\left(\nabla_\perp E_Z + j\beta\,\vec{E}_\perp\right) = k_o^2\ \vec{E}_\perp, \tag{6}$$

$$\nabla_\perp \times \left[\left(\nabla_\perp E_Z + j\beta\,\vec{E}_\perp\right)\times \hat{a}_Z\right] = k_O^2\ \ E_Z\,\hat{a}_Z, \tag{7}$$

By introducing the variable transformation

$$\begin{aligned} e_x &= j\beta\, E_x, \\ e_y &= j\beta\, E_y, \\ e_z &= E_z. \end{aligned} \tag{8}$$

So, (7) and (8) can be written as

$$\nabla_\perp \times \left(\nabla_\perp \times \vec{e}_\perp\right) + \beta^2\left(\nabla_\perp e_z + \vec{e}_\perp\right) = k_O^2\vec{e}_\perp, \tag{9}$$

$$\beta^2\nabla_\perp \times \left(\nabla_\perp e_Z + \vec{e}_\perp\right)\times a_Z = \beta^2 k_O^2 e_Z\hat{a}_Z. \tag{10}$$

The coupled pair of differential equations (9) and (10) can now be solved for the square of the propagation constant β^2 of the homogeneous ridged waveguide, subject to the following boundary condition:

$$\begin{aligned} \hat{n}\times\vec{e}_\perp &= \vec{0} \\ e_z &= \vec{0} \end{aligned} \tag{11}$$

on perfect electric surfaces and

$$\begin{aligned} \left(\nabla_\perp e_z + \vec{e}_\perp\right)\cdot \hat{n} &= 0 \\ \nabla_\perp \times \vec{e}_\perp &= \vec{0} \end{aligned} \tag{12}$$

on perfect magnetic surfaces, where and $\hat{n}$ denotes the normal vector at each surface.

The corresponding variational functional [18] for the (8) can be written as

$$\Im(e) = \int_\Omega \beta^2\left[\left|\nabla_\perp e_Z + e_\perp\right|^2 - k^2\left|e_Z\right|^2\right] - k^2\left|e_\perp\right|^2 + \left|\nabla_r \times e_\perp\right|^2 d\Omega \tag{13}$$

For TM and TE modes analyses the boundary condition reduces to $\hat{n}\cdot\nabla e_z = 0$ for magnetic wall and $e_z = 0$ for electric wall. The cross-section shape and the parameters of ridged waveguide is shown in Fig. 1.

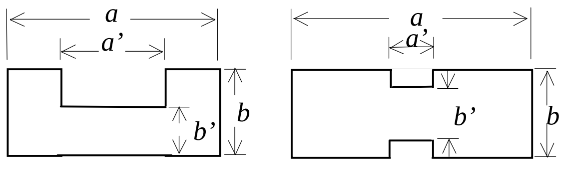

Fig. 1. Geometry of rectangular ridged waveguide: Cross-section and parameters of a single-ridge (a) and double-ridge (b) waveguide.

III. Finite Element Implementation

In the finite element method discussed here, the rectangular guide cross section is subdivided into set of triangular subregions and within each triangular subregion there is a point distribution to permit a quadratic approximation. This is in fact advantageous because we can use only a few triangles to describe the boundary shape and consequent accuracy will result, as it compared with linear approximation.

A. Finite Dimensional Approximation

In the present approach, the problem domain Ω is broken into triangles and it can be shown in Fig. 2, and within each triangle there is a point distribution with six points in order to permit a quadratic approximation. The waveguide cross section was meshed applying two kinds methods: regular, by generating a regular grid (Fig. 2, to the left), and GiD automatic mesh generator (to the right).

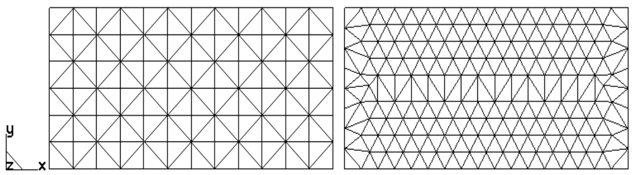

Fig. 2. A finite-dimensional discretization by second-order triangles of a waveguide cross section using a regular grid and by GiD automatic mesh generator.

B. Generalized Eigenmatrix Equation

In order to apply the FEM it is necessary to distinguish the exact and approximated solution of eigenvalue problem, therefore if U_e is used to denotes the exact solution of eigenvalue problem for e_z or h_z, satisfacting the equation

$$\nabla\left(\nabla U_e(x,y)\right) + k_c^2\, U_e = 0. \tag{14}$$

where $k_c^2 = k_o^2 - \beta^2$ is the square of cutoff wavenumber, being eigenvalue of the problem, and U is used to denote the solution obtained with the FEM, so that when U is substituted into (14), it generates a residual R, given by:

$$\nabla \cdot \left[\nabla U(x,y) \right] + k_c^2 U(x,y) = R \, . \tag{15}$$

In order to establish a numerical procedure, we force R to be zero using the following condition:

$$\int_\Omega W \, R \, d\Omega = 0, \tag{16}$$

where W is a weighting function and Ω represents the domain in with the condition is enforced. In our case, the expression in (16) is

$$\int_\Omega W \left[\nabla .(\nabla U(x,y)) + k_c^2 U(x,y) \right] d\Omega =$$

$$= \oint_{S(\Omega)} W \, \nabla U(x,y) \cdot d\vec{s} - \int_\Omega \nabla U(x,y) \cdot \nabla W \, d\Omega$$

$$+ \int_\Omega W \, k_c^2 \, U(x,y) \, d\Omega \, . \tag{17}$$

and S_i being the surface of Ω.

Equation (17) is the weak form of the formulation and the stiffness matrix is given by

$$\int_{S_i} \nabla N^T \, \nabla N U \, dxdy. \tag{18}$$

The stiffness matrix can be obtained after applying Galerkin method to (14), and separating from the boundary conditions, being $U(x,y) = \sum_{n=1}^{6} U_n \phi_n(x,y)$, and $\phi_n(x,y)$ an approximation function. Therefore, the elementar matrix system can be obtained by

$$\left[A_\nabla \right] \left[U \right] = -k_c^2 \left[A \right] \left[U \right], \tag{19}$$

where $\left[A_\nabla \right]$ and $\left[A \right]$ are submatrices to solve (14). Then, the assembly matrix can be denoted by

$$[SS]\{U\} = k_c^2 [R]\{U\}. \tag{20}$$

The Equation (20) denotes the generalized eigenvalue problem for which [SS] e [R] are normally real, symmetric, and positive definite matrices, {U} is the field values for the TE and TM modes, and k_c values are the eingenvalues (cutoff wavenumber), so the k_c can be obtained of a system of equations and the eigenvectors, as the corresponding solutions (i.e. fields).

The generalized eigenvalue problem can be reduced to a standard eigenvalue problem[10]-[11] and for this reason we have been used the Jacobi method [13]-[16] to finding the eigenvalues and eigenvectors.

IV. Results

In this section we have presented the cutoff wavenumbers, obtained by solving the generalized eigenvalue problem

$$-\left[A_\nabla \right] \begin{Bmatrix} h_z \\ e_z \end{Bmatrix} = k_c^2 \left[A \right] \begin{Bmatrix} h_z \\ e_z \end{Bmatrix}. \tag{21}$$

In order to verified the accurate of method was FEM numerical code implementation, this first applied to rectangular waveguide.

Considering a rectangular waveguide with dimensions a/b = 2, the computed eigenvalues are given in Table 1 and the corresponding mode field distributions (eigenvectors) are shown in Fig. 3 for TE_{10} and TE_{11}. These calculations were carried out using a quadratic approximation with 64 triangular elements over the waveguide cross section. The exact [7]-[8] eigenvalues are given by

$$k_c = \frac{\pi}{a} \sqrt{m^2 + (2n)^2} \tag{22}$$

TE_{mn}($m \neq 0$ or $n \neq 0$) and TM_{mn}($m \neq 0$ and $n \neq 0$) modes. We can remark too that the accuracy of the calculated eigenvalues deteriorates for the higher order modes since the latter require a finer mesh due to their more complex mode structure like it was observed by Volakis et al [9] and shown in the Table I, II and III. The Fig. 3 shows the fields plots (eigenvectors) for the various modes in a rectangular cross section using a regular grid (manually meshed) with 64 elements.

The Figure 3 shows only the results considering a rectangular waveguide of dimensions $a = 2b$ for testing the of code developed. The code permits to evaluate not only the TE and TM modes but also for the single ridged and double ridged waveguides. The results of visualization of fields (eigenvectors) for TE and TM modes in ridged waveguides have been presented at first time (at least knowing by the authors) in the literature using second-order triangular elements.

In the Tables I-IV we have presented the TE modes, obtained using MEF, for double ridged waveguide (a/b = 2) in comparison with analytical results [20]. The Figures 6 and 7 show the lowest TM modes for single and double ridged waveguides.

Investigating the Tables I-IV we can observed the good accuracy and according of results, in comparison to literature [9,20], using MEF code to calculate the wavenumbers cutoff for rectangular waveguide and rectangular ridged waveguide. It was possible verified that the most accuracy values can be calculated using a mesh containing few elements, not only for second-order triangular elements as well as for first-order elements, and a good visualization of TE and TM modes is related to generation mesh way. The best cutoff values and visualization for both modes (TE and TM) was obtained using a regular grid with 64 triangular elements over the waveguide cross section.

TABLE I: CUTOFF WAVENUMBERS FOR A RETANGULAR WAVEGUIDE: COMPARISON BETWEEN ANALITYCAL AND FEM CALCULATIONS FOR TE AND TM MODES USING A REGULAR GRID OF SECOND-ORDER FINITES ELEMENTS

$k_c a$ (a/b=2)				
Analytical[12]			FEM Calculation - regular grid 64 triangle elements	
TE	TM			
10		3.142	3.146	
20		6.283	6.267	
01		6.283	6.267	
11	11	7.025	7.003	7.107
12	12	12.953	12.839	13.414
21	21	8.886	8.847	9.008
31	31	11.327	11.260	11.518

TABLE II: COMPARISON BETWEEN ANALITYCAL AND FEM CALCULATIONS USING VARIOUS MESHES ON FIRST-ORDER ELEMENTS

Mode		Exact[12]	$k_c a$ (a/b=2) FEM Calculations Triangle Elements - 1st order			
TE	TM		162	300	402	695
10		3.142	3.234	3.224	3.093	3.122
20		6.283	6.394	5.928	6.249	6.194
01		6.283	6.427	6.440	6.251	6.302
11	11	7.025	7.269	6.913	6.990	7.002
12	12	12.953	13.078	12.892	12.870	13.070
21	21	8.886	9.247	8.971	8.845	8.792

TABLE III: COMPARISON BETWEEN ANALITYCAL AND FEM CALCULATIONS USING VARIOUS MESHES ON SECOND-ORDER ELEMENTS

Mode		Exact[12]	$k_c a$ (a/b=2) FEM Calculations Triangle Elements - 2nd order			
TE	TM		64	144	256	300
10		3.142	3.146	3.093	3.140	3.207
20		6.283	6.265	6.249	6.276	5.892
01		6.283	6.268	6.251	6.277	6.394
11	11	7.025	7.004	6.990	7.019	6.845
12	12	12.953	12.819	12.870	12.908	13.065
21	21	8.886	8.846	8.845	8.873	8.839

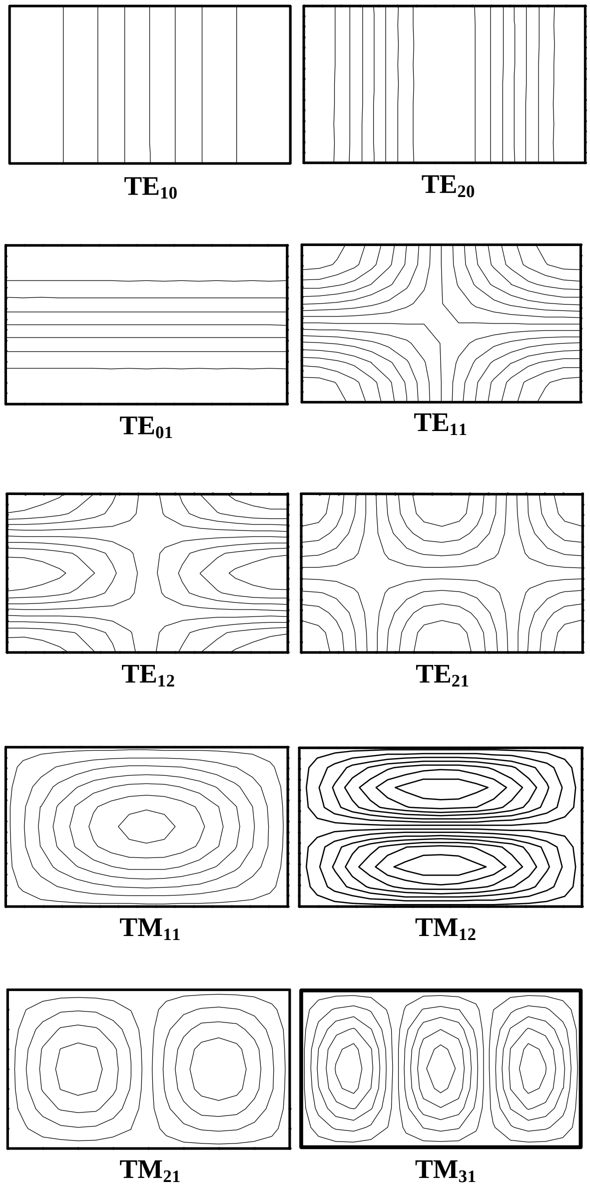

Fig. 3. Calculated mode electric fields in a rectangular waveguide with dimensions a/b = 2.

TABLE IV: CUTOFF WAVELENGHT OF H_{10} MODE

b'/b = 0.25			*b'/b=0.5*		
Theor. [20]		MEF	Theor. [20]		MEF
a'/a	λ_c/a	λ_c/a ($2\pi/k_c a$)	a'/a	λ_c/a	λ_c/a ($2\pi/k_c a$)
0.20	3.349	3.059	0.25	2.604	2.569
0.50	3.609	3.471	0.50	2.666	2.606

TABLE V: CUTOFF WAVELENGHT OF H_{20} MODE

b′/b = 0.25			*b′/b=0.5*		
Theor. [20]		MEF	Theor. [20]		MEF
a′/a	λ_c/a	λ_c/a ($2\pi/k_c a$)	a′/a	λ_c/a	λ_c/a ($2\pi/k_c a$)
0.20	0.884	0.883	0.25	0.942	0.943
0.50	1.157	1.134	0.50	1.095	1.088

TABLE VI: CUTOFF WAVELENGHT OF H_{30} MODE

b′/b = 0.25		
Theor. [20]		MEF
a′/a	λ_c/a	λ_c/a ($2\pi/k_c a$)
0.20	0.762	0.764
0.50	0.647	0.649

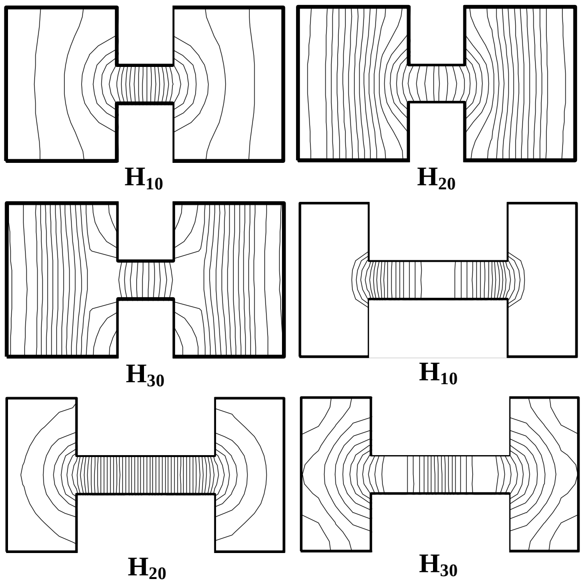

Fig.6. Calculated mode electric fields in the rectangular double ridged waveguides (*a/b* = 2) with *a′/a* = .0.2 and *a′/a* = 0.5 (*b′/b* = 0.25).

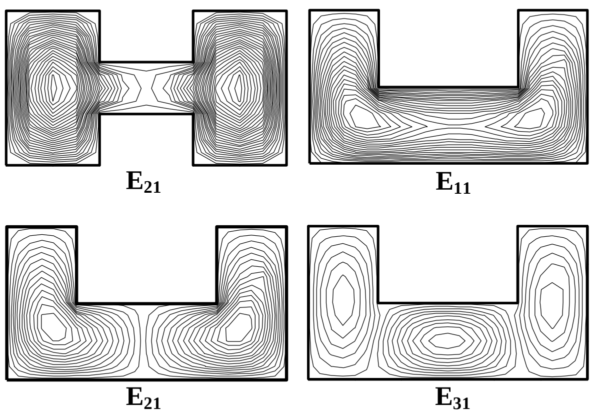

Fig.7. Calculated fields for the lowest TM modes of the rectangular (*a/b* = 2) double ridged and single ridged waveguides.

V. CONCLUSION

The finite-element method of solving ridged and homogeneous waveguides problems by quadratic triangular shape function appears to be capable of higher accuracy and reliably produces complete sets of propagating modes at little computational cost. The new method for calculating the TE and TM modes in the rectangular ridged waveguides have been tested and presented at first time. This method is capable of extension to inhomogeneously filled guides and cavity resonator, as well as to other field problems involving lossy or not. Some of these possibilities are now under examination.

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