Chebyshev Wavelets

R. J. de Sobral Cintra H. M. de Oliveira L. R. Soares

***Abstract*— In this note we introduce a new family of wavelets, named Chebyshev wavelets, which are derived from conventional Chebyshev polynomials. Properties of Chebyshev filter banks are investigated, including orthogonality and perfect reconstruction conditions. Chebyshev wavelets of 2nd kind have compact support, their filters possess good selectivity, but they are not orthogonal. The convergence into 2nd kind Chebyshev wavelets via the cascade algorithm is proved by the use of Markov chains theorems. Computational implementation of these wavelets and some clear-cut applications are presented. These wavelets are offered as a choice in wavelet analysis.**

***Keywords*— Wavelets, Filter banks, Chebyshev polynomials, Wavelet design.**

I. Introduction

Sturm-Liouville theory encompasses a multitude of engineering and physics problems [1]. One particular and interesting case is that one related to Chebyshev differential equations. Chebyshev polynomials of the first kind (Type I) of order m, $T_m(x)$, satisfies the equation $(1-x)\ddot{y} - x\dot{y} + n^2 y = 0$ and Chebyshev polynomials of second kind (Type II) of degree m, $U_m(x)$, satisfies $(1-x)\ddot{y} - 3x\dot{y} + n(n+2)y = 0$. Chebyshev polynomials form a complete set of orthogonal functions in the interval $[-1, 1]$ with weighting functions $(1-x^2)^{-1/2}$ and $(1-x^2)^{1/2}$, for the polynomials of first and second kind respectively. Some special values are $T_n(1) = 1$ and $T_{2n+1} = 0$; $U_n(1) = n+1$ and $U_{2n+1}(0) = 0$. Chebyshev polynomials also respect symmetry properties $T_n(-x) = (-1)^n T_n(x)$ and $U_n(-x) = (-1)^n U_n(x)$ [1]–[3].

Chebyshev polynomials have many applications in numerical computations, interpolation, series truncation and economization, to name a few. In the past few years, connections between orthogonal polynomials and wavelet analysis have been explored, particularly a wavelet decomposition in $L^2(-1, 1)$ has been proposed [3], [4].

Recently another approach has been investigated [5], [6]: the link between classical differential equation solutions — like Mathieu functions (elliptic cosine and sine) and Legendre polynomials — and wavelet design. Exploring this connections, in this paper we investigate the possibility of wavelet construction from Chebyshev polynomials. Can these polynomials be used as smoothing filters for wavelets? In order to answer this question, we use filter bank theory results.

The overview of our procedure is the following: (i) we start defining smoothing filters from Chebyshev polynomials, (ii) properties of filter banks based on these filters are explored, such as perfect reconstruction, orthogonality, and finally (iii) using these filter banks, we call the cascade iterative procedure for creating wavelets.

For the sake of notation, let us take the sequences $h[n]$ as the lowpass filters and $g[n]$ as the highpass filters (by convention $\sum_n h[n] = 1$ and $\sum_n g[n] = 0$). The matrix $\mathbf{H}$ is the convolution matrix. For the role of downsampling by two, it is adopted the operator $(\downarrow 2)$. The definiton symbol is $\triangleq$.

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II. Chebyshev Wavelets

In this section, we investigate the definition of filter banks based on Chebyshev polynomials and also their possible application for wavelet construction.

A. First Kind Chebyshev Filters

The well-known Chebyshev polynomials of 1st kind $T_m(\cdot)$ are defined by a simple recurrence formulation [1]

$$T_m(x) = 2xT_{m-1}(x) - T_{m-2}(x), \tag{1}$$

assuming that $T_0(x) \triangleq 1$ and $T_1(x) \triangleq x$.

Lowpass filters can be derived from these polynomials by simply assuming the variable change $x = \cos\omega$. Doing so, we have the new functions [2]

$$T_m(\cos\omega) = \cos(m\omega), \tag{2}$$

whose magnitude in the interval $[0, \pi]$ satisfies lowpass filter conditions for frequency response magnitude. In a naive way, one may take these polynomials to define smoothing (lowpass) filters to be used for wavelet generation through the cascade algorithm.

Smoothing filters $H(e^{j\omega})$ intended to be used for signal analysis [7] must hold some specific conditions, such as $|H(e^{j0})| = 1$ and $|H(e^{j\pi})| = 0$. In order to make Chebyshev polynomials useful for this kind of application, a slight modification on $T_m(\cdot)$ is carried out so as to meet these constraints. Taking only Chebyshev polynomials of odd order m, we can define the magnitude response of the smoothing filter as

$$|H_m^{(1)}(e^{j\omega})| \triangleq |T_m(\cos(\omega/2))|, \quad \text{for odd } m. \tag{3}$$

Observe that these functions are naturally normalized. Some examples can be seen in Figure 1.

In a previous work [5], wavelets based on Mathieu differential equations were defined. The mathematical structure of Mathieu wavelets naturally induces a linear phase assignment $e^{-jm\omega}$ for the smoothing filter. This kind of approach seems to be perfectly reasonable to be considered in this development. After this judicious phase adjustment, we have the following expression for the smoothing filter:

$$H_m^{(1)}(e^{j\omega}) \triangleq e^{-jm\omega/2} T_m(\cos(\omega/2)), \quad m \text{ odd}. \tag{4}$$

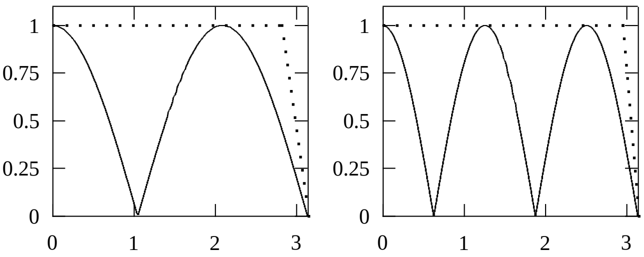


Fig. 1. Plot of $|T_m(\cos(\omega/2))|$, for $m = 3, 5$, $\omega \in [0, \pi]$.

Using Equation 2, we may easily recognize that

$$\begin{aligned} H_m^{(1)}(e^{j\omega}) =& e^{-jm\omega/2} T_m(\cos(\omega/2)) \\ =& e^{-jm\omega/2} \cos(m\omega/2) \\ =& \frac{1}{2}(1 + e^{-jm\omega}). \end{aligned} \tag{5}$$

Since $h_m^{(1)} \overset{\text{DTFT}}{\longleftrightarrow} H_m^{(1)}$, we can find $h_m^{(1)}$ by an application of the inverse discrete-time Fourier transform on $H_m^{(1)}$. That is,

$$h_m^{(1)}[n] = \begin{cases} 1/2, & n = 0, m, \\ 0, & \text{otherwise}. \end{cases} \tag{6}$$

We use this filter $h_m^{(1)}[n]$ to define reconstruction and decomposition filter banks. The relation among the highpass and lowpass filters of these two filter banks is well-established [8]–[10] namely:

$$h_r{}^{(1)}[n] = \sqrt{2} h_m^{(1)}[n], \qquad g_r{}^{(1)}[n] = \sqrt{2}(-1)^n h_r^{(1)}[m-n], \tag{7}$$

$$h_d{}^{(1)}[n] = \sqrt{2} h_r^{(1)}[m-n], \quad g_d{}^{(1)}[n] = \sqrt{2} g_r^{(1)}[m-n], \tag{8}$$

for $n = 0, \ldots, m$. Here, indexes r and d are used to denote reconstruction and decomposition filters, respectively.

1) Properties of Type I Chebyshev Filter Banks: The filter banks based on lowpass filters $h_m^{(1)}[n]$ share perfect reconstruction property. Let us use capital letters to denote z-transforms of time domain vectors. Therefore $H_r^{(1)}$ is the z-transform of the lowpass reconstruction filter $h_r^{(1)} \triangleq \sqrt{2} h_m^{(1)}$. In a similar way, we may define the reconstruction and decomposition filter bank z-transforms by $h_r^{(1)} \overset{z}{\longleftrightarrow} H_r^{(1)}$, $g_r^{(1)} \overset{z}{\longleftrightarrow} G_r^{(1)}$, $h_d^{(1)} \overset{z}{\longleftrightarrow} H_d^{(1)}$ and $g_d^{(1)} \overset{z}{\longleftrightarrow} G_d^{(1)}$.

To achieve perfect reconstruction, a filter bank must satisfy alias cancellation and present no distortion. To ensure alias cancellation, we must have [11]

$$H_r^{(1)}(z) H_d^{(1)}(-z) + G_r^{(1)}(z) G_d^{(1)}(-z) = 0. \tag{9}$$

Substituting these z-transforms by their corresponding explicit expressions and taking into account that m is odd, yields

$$\begin{aligned} &\frac{1}{\sqrt{2}}(1 + z^{-m}) \frac{1}{\sqrt{2}}(1 + (-z)^{-m}) + \\ &\frac{1}{\sqrt{2}}(-1 + z^{-m}) \frac{1}{\sqrt{2}}(1 - (-z)^{-m}) = \\ &\frac{1}{\sqrt{2}}(1 + z^{-m}) \frac{1}{\sqrt{2}}(1 - z^{-m}) - \\ &\frac{1}{\sqrt{2}}(1 - z^{-m}) \frac{1}{\sqrt{2}}(1 + z^{-m}) = 0, \end{aligned} \tag{10}$$

which asserts alias cancellation. Going further, to ensure perfect reconstruction it is also required that the filter banks introduce no distortion, that is, only a delay is allowed [12]:

$$H_r^{(1)}(z) H_d^{(1)}(z) + G_r^{(1)}(z) G_d^{(1)}(z) = 2z^{-l}. \tag{11}$$

Carrying over the substitutions, leads to

$$\begin{aligned} &\frac{1}{\sqrt{2}}(1 + z^{-m}) \frac{1}{\sqrt{2}}(1 + z^{-m}) + \\ &\frac{1}{\sqrt{2}}(-1 + z^{-m}) \frac{1}{\sqrt{2}}(1 - z^{-m}) = 2z^{-m}. \end{aligned} \tag{12}$$

Observe that the filter bank delay is m, exactly the order of the initially selected Chebyshev polynomial.

Another question to be examined is the orthogonality condition. A filter bank is orthogonal if it satisfies even-shift convolution ($*_2$) [10], [12]:

$$h[n] *_2 h[n] = \sum_k h[k] h[k - 2n] = \delta[n], \tag{13}$$

where $\delta[n]$ is the unit sample sequence. It can be shown that the lowpass filter $h^{(1)}[n] = \frac{1}{2}\begin{bmatrix}1 & 0 & \cdots & 0 & 1\end{bmatrix}$ fulfills this orthogonality test.

Although these two desirable properties — perfect reconstruction and orthogonality — are met, we will show that in a general manner the iterative process of the cascade algorithm using the filters $h_m^{(1)}[n]$ does not lead to wavelets. In other words, the limit of cascade algorithm is not a smooth function and the algorithm does not converge in L^2. The following handy theorem states a necessary and sufficient condition for iteration convergence [10], [13].

Theorem 1 (Smoothness) *Let $h[n]$ be a lowpass filter of length $m+1$ and $\mathbf{H}$ be its associated filter matrix. If the infinite matrix $\mathbf{T} = (\downarrow 2) 2\mathbf{H}\mathbf{H}^T$ has a centered submatrix $\mathbf{T}_{2m-1}$ of order $2m-1$, such that all its eigenvalues satisfy $|\lambda| < 1$ (except for a simple $\lambda = 1$), then the cascade algorithm converges in L^2 sense.*

According to the definition given in Theorem 1, by removing odd numbered rows of $2\mathbf{H}\mathbf{H}^T$ (i.e., applying the decimation-by-2 operator $(\downarrow 2)$), one can directly get $\mathbf{T}_{2m-1}$. For Chebyshev polynomials of 1st kind, we have derived the filter $\mathbf{h}_m^{(1)} = \frac{1}{2}\big[1 \ \underbrace{0\ 0\ \cdots 0\ 0}_{m-1 \text{ zeros.}} \ 1\big]$, thus the rows of $2\mathbf{H}\mathbf{H}^T$ are a stack of sequential single-shifted versions of the following vector:

$$\begin{aligned} &\frac{1}{2}\begin{bmatrix}1 & 0 & \cdots & 0 & 1\end{bmatrix} * \begin{bmatrix}1 & 0 & \cdots & 0 & 1\end{bmatrix} = \\ &\frac{1}{2}\big[1 \ \underbrace{0\ 0\ \cdots 0\ 0}_{m-1 \text{ zeros.}} \ 2 \ \underbrace{0\ 0\ \cdots 0\ 0}_{m-1 \text{ zeros.}} \ 1\big], \end{aligned} \tag{14}$$

where $*$ denotes usual convolution.

Since the element 1 in this resulting vector is separated from the element 2 by a even number of zeros $m-1$, the odd-line elimination of $2\mathbf{H}\mathbf{H}^T$ will make every column of $\mathbf{T}_{2m-1}$ have a single element 1 or a pair of $1/2$, as it can be seen below:

$$\mathbf{T}_{2m-1} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & \ddots & \ddots & & & \ddots & & & \ddots & \ddots & & & \\ 0 & 0 & 0 & \cdots & 2 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 2 & \cdots & 0 & 0 & 0 \\ & \ddots & \ddots & & & \ddots & & & \ddots & \ddots & & & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}. \tag{15}$$

By explicit computation of the eigenvalues, the search for an m which makes the matrix $\mathbf{T}_{2m-1}$ meet the conditions of Theorem 1 returned only one favorable case, for $m < 256$. This exception is $m = 1$. It is interesting to remark that when setting $m = 1$, the resulting $h_1^{(1)}[n] = \frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}$ is the Haar filter bank, which makes the cascade algorithm generate the Haar wavelets. Limited to our computational results, this is the only choice of Chebyshev polynomial that produces a wavelet.

B. Second Kind Chebyshev Wavelets

Now we examine another class of polynomials, namely the Chebyshev polynomials of 2nd kind. This family of polynomials is also built from the same recurrence relation used to derive the 1st kind ones. However, different initial conditions are set:

$$U_m(x) = 2xU_{m-1}(x) - U_{m-2}(x), \tag{16}$$

for $U_0(x) = 1$ and $U_1(x) = 2x$. A variety of interesting properties and theorems on these polynomials can be found in [1], [2].

Following similar steps and derivations as in the previous subsection, we investigate the use of $U_m(x)$ in the definition of lowpass filters. This time, our aim is to construct new wavelets.

First, we adopt a usual variable change $x = \cos\omega$, yielding to [2, p.776]:

$$U_m(\cos\omega) = \frac{\sin(m+1)\omega}{\sin\omega}. \tag{17}$$

Now we may consider the use of the modulus of these functions as the magnitude response of lowpass filters. However, one may not directly proceed in such a way, since $|U_m(\cos\omega)|$ does not promptly satisfy lowpass filter conditions ($|H(e^{j0})| = 1$ and $|H(e^{j\pi})| = 0$). To make this possible, a simple rule-of-thumb adjustment can be used. Just as in the former 1st kind polynomial case, a scaling on the argument of $U_m(\cdot)$ by $1/2$ solves the problem, and makes $|H(e^{j\pi})| = 0$. The restriction of oddness for m must be checked, otherwise the proposed $\frac{1}{2}$-scaling on frequency cannot work.

In contrast with Chebyshev polynomials of 1st kind, the polynomials of 2nd kind are not naturally normalized. The maximum value of $U_m(\cos\omega)$ is located at the peak of the main lobe (vicinity of zero) and can be computed without effort:

$$\lim_{\omega\to 0} U_m(\cos(\omega)) = \lim_{\omega\to 0}\frac{\sin((m+1)\omega)}{\sin(\omega)} = m+1. \tag{18}$$

Then a scaling factor of $\frac{1}{m+1}$ must be taken into consideration to normalize the filter response. This adjustment redefines the magnitude of the frequency response to

$$|H_m^{(2)}(e^{j\omega})| \triangleq \frac{1}{m+1}|U_m(\cos(\omega/2))|, \qquad \text{for odd } m. \tag{19}$$

This ensures that $|H_m^{(2)}(e^{j0})| = 1$. Illustrations of the frequency response magnitude of $H_m^{(2)}(e^{j\omega})$ are shown in Figure 2.

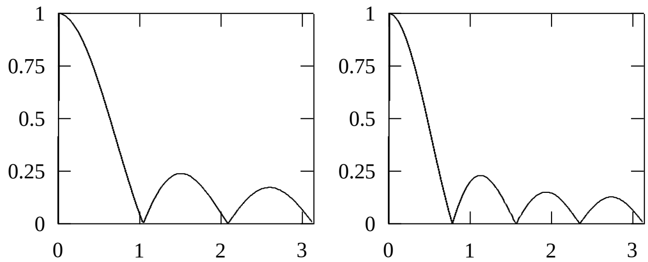


Fig. 2. Plot of $|U_m(\cos(\omega/2))|$, for $m = 5, 7$, $\omega \in [0, \pi]$.

The final, but crucial, step concerns phase assignment. Again let us take a linear phase convenient choice [5]. Consequently, the Chebyshev lowpass filters are completely specified by

$$H_m^{(2)}(e^{j\omega}) \triangleq \frac{1}{m+1}e^{-jm\omega/2}U_m(\cos(\omega/2)). \tag{20}$$

Using now the fact that $U_m(\cos(\omega)) = \sin((m+1)\omega)/\sin(\omega)$, we can write the following:

$$H_m^{(2)}(e^{j\omega}) = \frac{1}{m+1}e^{-jm\omega/2}\frac{\sin((m+1)\omega/2)}{\sin(\omega/2)}. \tag{21}$$

Surprisingly, this is the exact formulation of the moving average filters or rectangular window (!) [14]. The impulse response $h_m^{(2)}[n]$ of these filters are promptly derived:

$$h_m^{(2)}[n] = \begin{cases} 1/(m+1), & n = 0, \ldots, m, \\ 0, & \text{otherwise.}\end{cases} \tag{22}$$

1) Properties of the Type II Chebyshev Filter Banks: Taking Equation 22 as a starting point, we are now in a position to carry on some investigation on Type II Chebyshev filter banks.

Based on $h_m^{(2)}[n]$ and using similar definitions for the reconstruction and decomposition filters as done before (Equations 7 and 8), we may find the following z-transforms for

$h_r^{(2)}[n]$, $g_r^{(2)}[n]$, $h_d^{(2)}[n]$ and $g_d^{(2)}[n]$:

$$H_r^{(2)}(z) = \frac{\sqrt{2}}{m+1}\sum_{i=0}^{m} z^{-i}, \tag{23}$$

$$G_r^{(2)}(z) = \frac{\sqrt{2}}{m+1}\sum_{i=0}^{m} (-1)^i z^{-i}, \tag{24}$$

$$H_d^{(2)}(z) = \frac{\sqrt{2}}{m+1}\sum_{i=0}^{m} z^{-i}, \tag{25}$$

$$G_d^{(2)}(z) = \frac{\sqrt{2}}{m+1}\sum_{i=0}^{m} -(-1)^i z^{-i}. \tag{26}$$

Let us begin examining perfect reconstruction questions. As stated before, a filter satisfying both alias cancellation and no distortion has

$$H_r^{(2)}(z)H_d^{(2)}(-z) + G_r^{(2)}(z)G_d^{(2)}(-z) = 0, \tag{27}$$

$$H_r^{(2)}(z)H_d^{(2)}(z) + G_r^{(2)}(z)G_d^{(2)}(z) = 2z^{-l}, \tag{28}$$

respectively. After some tedious manipulation, we find that alias cancellation is completely fulfilled:

$$\begin{aligned}&\frac{\sqrt{2}}{m+1}\sum_{i=0}^{m} z^{-i}\frac{\sqrt{2}}{m+1}\sum_{i=0}^{m}(-z)^{-i} + \\ &\frac{\sqrt{2}}{m+1}\sum_{i=0}^{m}(-1)^i z^{-i}\frac{\sqrt{2}}{m+1}\sum_{i=0}^{m} -(-1)^i(-z)^{-i} = \\ &\frac{2}{(m+1)^2}\left(\sum_{i=0}^{m} z^{-i}\sum_{i=0}^{m}(-1)^i z^{-i} - \sum_{i=0}^{m}(-z)^{-i}\sum_{i=0}^{m} z^{-i}\right) \\ &= 0.\end{aligned} \tag{29}$$

However, after an application of Equation 28, we find that

$$\begin{aligned}&H_r^{(2)}(z)H_d^{(2)}(z) + G_r^{(2)}(z)G_d^{(2)}(z) = \\ &\qquad \frac{8}{(m+1)^2}\left(\frac{1-z^{-(m+1)}}{1+z^{-2}}\right)^2 z^{-1}.\end{aligned} \tag{30}$$

Since this is clearly not in the form $2z^{-l}$, we conclude that such a filter bank introduces some distortion.

It is easy to see that $h^{(2)}[n]$ does not verify Equation 13, therefore there is no orthogonality. It remains to examine whether this filter bank class produces a convergent smoothing (regular) wave or not. In the appendix, we show the sketch of a proof of the following lemma.

Lemma 1 *Filter banks based on odd order Chebyshev polynomials of 2nd kind satisfy Theorem 1.*

Figure 3 displays some results derived by the iterative cascade algorithm, depicting the formation of a wavelet function with compact support.

Example 1 *Take the Chebyshev 2nd kind filter of order 3,* $h_3^{(2)} = \frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$. *Constructing the centered submatrix of* $\mathbf{T} = (\downarrow 2)2\mathbf{H}\mathbf{H}^T$, *we have:*

$$\mathbf{T}_5 = \frac{1}{8}\begin{bmatrix}2 & 1 & 0 & 0 & 0\\ 4 & 3 & 2 & 1 & 0\\ 2 & 3 & 4 & 3 & 2\\ 0 & 1 & 2 & 3 & 4\\ 0 & 0 & 0 & 1 & 2\end{bmatrix}.$$

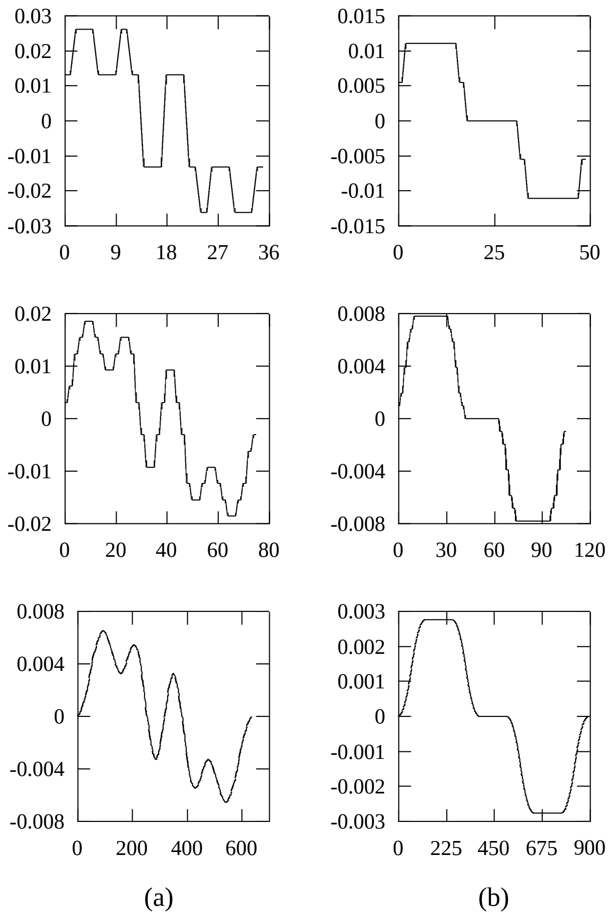


Fig. 3. Second order Chebyshev wavelets in 2, 3 and 4 iterations, (a) for $m = 5$ and (b) for $m = 7$.

Since all eigenvalues — 1, $\frac{1}{2}$, $\frac{1}{4}$ *and 0 (double) — are less than one (except one), the regularity is assured.*

C. Implementing Chebyshev Wavelets

The filters proposed in this work were simulated with the use of MATLAB Wavelet Toolbox [8]. Some standard sample signals were analized to illustrate the behavior of the proposed wavelet and potential applications.

In Figure 4, we display the Chebyshev wavelet analysis of the step signal: a naive, but ellucidative example. Figure 5 brings two practical examples. Firstly, we examine a 3-level decomposition of a standard frequency breakdown signal. A noisy signal was also analyzed in a 2-level decomposition, illustrating potential uses of these wavelets in waveshrinkage [15].

III. Final Remarks

Impelled by a classical differential equation problem, we introduced a new family of functions for signal analysis via wavelet approach. Based on the Chebyshev polynomials (type I and II) and on the results derived in [5], we defined simple filter banks.

We showed that Chebyshev polynomials of 1st kind are not naturally suitable wavelet construction via the cascade

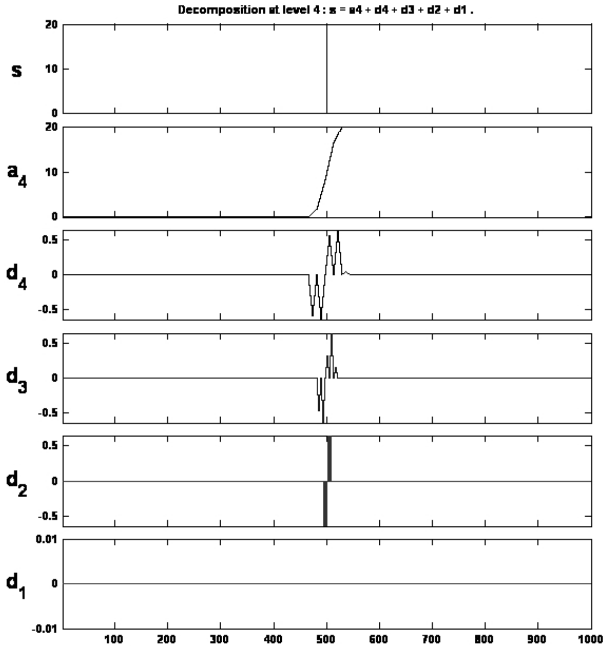
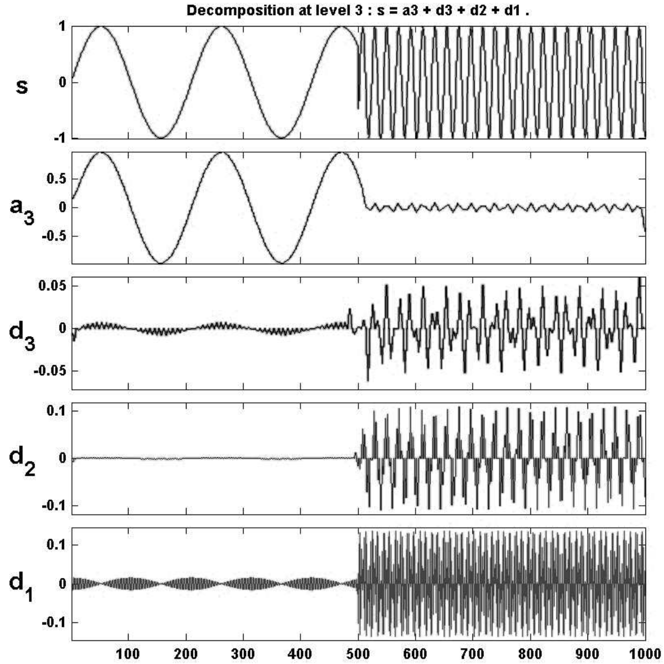
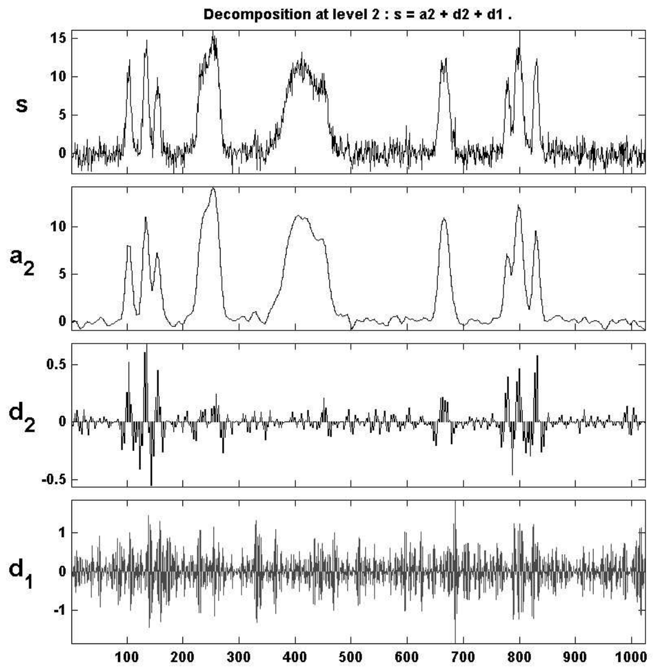


Fig. 4. An ellucidative example of Chebyshev wavelet decomposition: an analysis of the step function.



(a)



(b)

Fig. 5. (a) Analysis of a signal with a frequency breakdown (3-level decomposition), (b) Denoising of noisbump signal (2-level decomposition). Both analysis were done with a wavelet generated by the Chebyshev polynomial of 2nd kind for $m = 3$. These test signals are part of MATLAB wavelet toolbox.

algorithm. But on the other hand, we demonstrated that the Chebyshev polynomials of 2nd kind are adequate for such an iterative process.

We also observed unexpected results, like the connection between the magnitude of frequency response of the filter based on Chebyshev polynomial of 2nd kind and the well-known moving average filter.

The main properties of these filter banks were examined in detail. In particular, a convergence proof for the iterative process with Chebyshev Type II filter banks was presented.

Potential applications of Chebyshev polynomials and wavelets are particularly motivated by problems that deal with signal/pattern detection or denoising. Currently we are investigating the possibility of use of these wavelets in biomedical signal analysis, in particular electrogastrography signals (pattern recognition) [16].

Finally we may call attention that the Chebyshev polynomials are in fact particular cases of the more general Gegenbauer (ultraspherical) polynomials, which can be an attractive tool for investigating new wavelet constructions. Moreover, it is expected that Gegenbauer polynomials based wavelets should exhibit a broader range of flexibility.

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APPENDIX I
SKETCH OF THE CONVERGENCE PROOF

We have that $h_m^{(2)}[n] = \frac{1}{m+1}\left[\underbrace{1 \ \ 1 \ \cdots \ 1 \ \ 1}_{m+1 \text{ ones}}\right]$. The rows of matrix $2\mathbf{H}\mathbf{H}^T$ have the following pattern

$$
\begin{aligned}
&2\frac{1}{m+1}\begin{bmatrix}1 & \cdots & 1\end{bmatrix} * \frac{1}{m+1}\begin{bmatrix}1 & \cdots & 1\end{bmatrix} = \\
&\frac{2}{(m+1)^2}\begin{bmatrix}1 & 2 & \cdots & m & m+1 & m & \cdots & 2 & 1\end{bmatrix},
\end{aligned} \tag{31}
$$

a triangular-shaped vector. The matrix $(\downarrow 2)2\mathbf{H}\mathbf{H}^T$ is therefore described by:

$$
\mathbf{T}_{2m-1} = \frac{2}{(m+1)^2} \cdot \begin{bmatrix}
2 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
4 & 3 & \cdots & 0 & 0 & 0 & 0 & 0 & & 0 & 0 \\
 & & \ddots & & & \ddots & & & \ddots & & \\
6 & 7 & \cdots & m-1 & m-2 & m-3 & m-4 & m-5 & \cdots & 0 & 0 \\
4 & 5 & \cdots & m+1 & m & m-1 & m-2 & m-3 & \cdots & 1 & 0 \\
2 & 3 & \cdots & m-1 & m & m+1 & m & m-1 & \cdots & 3 & 2 \\
0 & 1 & \cdots & m-3 & m-2 & m-1 & m & m+1 & \cdots & 5 & 4 \\
0 & 0 & \cdots & m-5 & m-4 & m-3 & m-2 & m-1 & \cdots & 7 & 6 \\
 & & \ddots & & & \ddots & & & \ddots & & \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 3 & 4 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 2
\end{bmatrix}. \tag{32}
$$

One can check that such a specific matrix has the stochastic property: every column sums one. This can be done by separately analyzing even and odd columns, noting the fact that each column has even or odd elements only. The sum of the columns of the even (s_e) and odd (s_o) elements can be calculated by:

$$
\begin{aligned}
s_e =& m+1+2\sum_{k=1}^{\frac{m-1}{2}} 2k \\
=& m+1+2\frac{m-1}{2}\frac{m+1}{2} = \frac{(m+1)^2}{2}.
\end{aligned} \tag{33}
$$

$$
s_o = 2\sum_{k=0}^{\frac{m-1}{2}} 2k+1 = \frac{(m+1)^2}{2}. \tag{34}
$$

Consequently, $\mathbf{T}_{2n-1}$ is a stochastic matrix.

The following theorem, derived from celebrated Perron-Frobenius theorem [18, p.53], is useful for showing that $\mathbf{T}_{2m-1}$ satisfies the conditions of Theorem 1.

Theorem 2 (Eigenvalues of Irreducible Stochastic Matrix) *Let $\mathbf{M}$ be an irreducible Markov matrix. Then the eigenvalue $\lambda = 1$ of $\mathbf{M}$ is simple. If $\mathbf{M}$ is aperiodic, then $|\lambda| < 1$ for all other eigenvalues λ of $\mathbf{M}$.*

It remains to show that $\mathbf{T}_{2m-1}$ is (a) irreducible and (b) aperiodic. The first condition is easily verified, because $\mathbf{T}_{2m-1}$ is a band-like matrix with non null elements within the band. In Markov chain terminology, we can say that if all states can be reached from each other, then $\mathbf{T}_{2m-1}$ is irreducible. Moreover, the diagonal of matrix $\mathbf{T}_{2m-1}$ has all elements different from zero, then all states have a self-loop. This guarantees that the periodicity of the Markov matrix equals to one (aperiodicity).