Fourier Eigenfunctions, Uncertainty Gabor Principle And Isoresolution Wavelets

L.R. Soares, H.M. de Oliveira, R.J.S. Cintra and R.M. Campello de Souza

Abstract— **Shape-invariant signals under Fourier transform are investigated leading to a class of eigenfunctions for the Fourier operator. The classical uncertainty Gabor-Heisenberg principle is revisited and the concept of isoresolution in joint time-frequency analysis is introduced. It is shown that any Fourier eigenfunction achieve isoresolution. It is shown that an isoresolution wavelet can be derived from each known wavelet family by a suitable scaling.**

Index Terms—**Gabor-Heisenberg inequality, Fourier eigenfunctions, time-frequency analysis, isoresolution wavelets.**

I. PRELIMINARIES

The Fourier transform is often interpreted as a linear operator F. An interesting problem in this framework is to find out the so called eigenfunctions in the language of operators [1-3]. Let V be a vector space equipped with a linear transform, T:V $\rightarrow$ V, $v \mapsto T(v)$. Under the linear transform T, eigenfunctions are solutions of $T\{v\}=l v$, which corresponds here to $\mathsf{F}\{f(t)\}= l.f(w)$ for some $f \in L^2(\Re)$, l a scalar. They are a quite remarkable class of functions, which preserves the shape under Fourier transform: Both the signal and its spectrum (time and frequency representation) have the same shape.

In joint time-frequency representation [4, 5] this feature can represent a very good balance between the two domains. It is well known that the Gaussian pulse is a signal whose shape is preserved under the Fourier operator:

$$e^{-t^2/2} \leftrightarrow \sqrt{2p}\, e^{-w^2/2}. \qquad (1)$$

This can easily be derived by writing $\frac{1}{\sqrt{2p}}\int_{-\infty}^{+\infty} e^{-t^2/2} e^{-jwt}\,dt = F(w)$. Deriving this equation and using integral by parts, we find out $F'(w)=-w.F(w)$. The solution of the differential equation $F'(w)+w.F(w)=0$ under the initial condition $F(0)=1$ is $F(w)=e^{-w^2/2}$. It follows promptly that $l=\sqrt{2p}$.

L.R. Soares, H.M. de Oliveira, R.J.S. Cintra and R.M. Campello de Souza, Department of Electronics and Systems, Federal University of Pernambuco, Recife-PE,Brazil, E-mail: lusoares@ufpe.br, hmo@ufpe.br, rjsc@ee.ufpe.br, ricardo@ufpe.br. This paper is dedicated to Dr. Max Gerken (*in memoriam*).

The question is: Are there other eigenfunctions? This matter is addressed in the next section. It is worthwhile to bear in mind that some results in this paper are deliberately *non nova, sed nove*.

II. SHAPE-INVARIANT SIGNALS: EIGENFUNCTIONS OF THE FOURIER OPERATOR

Let E (respectively O) denote the functional that extracts the even (respectively odd) part of a signal.

Proposition 1. Let $f(t) \leftrightarrow F(w)$ be an arbitrary Fourier transform pair. Then the signal

$$h(t) := \sqrt{2p}\,\mathsf{E}[f(t)] + \mathsf{E}[F(t)]$$

is invariant under Fourier transform. Furthermore,

$$\mathsf{F}\{h(t)\} = \sqrt{2p}\; h(w).$$

Proof:
It follows from the definition of $h(.)$ that

$$2h(t) = \sqrt{2p}\,[f(t)+f(-t)] + [F(t)+F(-t)]. \quad (2)$$

Taking the Fourier transform,

$$2H(w) = \sqrt{2p}\,[F(w)+F(-w)] + [2pf(-w)+2pf(w)] \quad (3)$$

and the proof follows. ❑

Corollary. Each even function $f(t) \leftrightarrow F(w)$ induces a Fourier invariant $h(t)=\sqrt{2p}\,f(t)+F(t)$. ❑

For instance, the signals

$$h_1(t) = \sqrt{2p}\,\frac{1}{1+t^2} + pe^{-|t|},\; h_2(t) = \sqrt{2p}\,|t| - \frac{2}{t^2}$$

have spectra with similar shape. An extra remarkable example is

$$sec\,h\left(\sqrt{\frac{p}{2}}t\right) \leftrightarrow \sqrt{2p}\; sec\,h\left(\sqrt{\frac{p}{2}}w\right). \qquad (4)$$

Proposition 2. Let $f(t) \leftrightarrow F(w)$ be an arbitrary transform pair. Then the signal

$$h(t) := \sqrt{2p}\,\mathsf{O}[f(t)] - \mathsf{O}[F(t)]$$

is invariant under Fourier transform. Furthermore,

$$F\{h(t)\} = -\sqrt{2p}\; h(w)\,.$$

Proof:
Similar to proposition 1.

Corollary. Each odd function $f(t) \leftrightarrow F(w)$ induces a Fourier invariant $h(t) = \sqrt{2p}\, f(t) - F(t)$. ❑

Let us now focus on a particular and important class of Fourier invariant, which generates an orthogonal and complete set.

To begin with, let us denote by E_g a class of eigenfunctions of the Fourier operator defined according to the following:

Proposition 3. A signal $f(t)$ is in E_g *iff* the signal f satisfies the differential equation $f''(t)-t^2f(t)=kf(t)$, for some scalar $k\in C$.

Proof:
(⇒)

$$f(t) \leftrightarrow l f(w) \qquad \text{(hypothesis)}$$

The properties of time and frequency differentiation for F give:

$$f''(t) \leftrightarrow (jw)^2 l f(w),$$
$$(-jt)^2 f(t) \leftrightarrow l f''(w). \tag{5}$$

Adding both members[1], we derive

$$f''(t)-t^2f(t) \leftrightarrow l\,[f''(w) - w^2 f(w)]. \tag{6}$$

Thus, the signal $f''(t)-t^2f(t)$ has also its shape preserved, provided that f itself preserves its shape. Therefore, $f''(t)-t^2f(t)\in E_g$, that is, we are looking for signals such that $f''(t)-t^2f(t)=kf(t)$, since they have identical eigenvalues.

(⇐)
The signal $f(t)$ satisfies the differential equation $f''(t)-t^2f(t)=kf(t)$, $k\in C$. (hypothesis)

By taking F,

$$(jw)^2F(w)+F''(w)=k\,l\,F(w), \tag{7}$$

so that $F''(w)-w^2F(w)=k\,l\,F(w)$, i.e., its spectrum also obeys a similar differential equation. Therefore, f and F have identical shape, since they are solutions of the same differential equation. ❑

The key equation for shape-invariant signal is thus $f''(t)-t^2f(t)=kf(t)$. Let us try solutions of the form

$$f(t) = p(t).e^{-t^2/2}\,. \tag{8}$$

Therefore,

$$\left[p(t).e^{-t^2/2}\right]'' - t^2\, p(t).e^{-t^2/2} = kp(t).e^{-t^2/2}\,. \tag{9}$$

After simple algebraic manipulations, we derive

$$p''(t) - 2tp'(t) - [k+1]p(t) = 0. \tag{10}$$

A standard differential equation of the above form is [6]

$$p''(t) - 2tp'(t) + 2np(t) = 0\,,\ n \text{ integer.} \tag{11}$$

Thus, for a suitable choice $k = -(2n+1)$ (eigenvalues), the solutions $p(t)$ are exactly Hermite polynomials (12) [6], which form a complete orthogonal system.

$$p(t) = H_n(t)\,. \tag{12}$$

($H_0(t)=1$, $H_1(t)=2t$, $H_2(t)=-2+4t^2$, $H_3(t)=-12t+8t^3$, $H_4(t)=12-48t^2+16t^4$, etc.)

Proposition 4. Possible eigenvalues of the Fourier transform are the four roots of the unit (±1,±j) times $\sqrt{2p}$.

Proof:
Let us denote by $F^{(n)}$ the operator corresponding to iterate n times the operator F. Let $t\leftrightarrow w\leftrightarrow w'\leftrightarrow W$ be the Fourier domain for the iterate Fourier transform. Observe that ($\forall f \in E_g$)

$$F^{(2)}\{f(t)\}=2pf(-w') \text{ and } F^{(4)}\{f(t)\}=4p^2f(W). \tag{13}$$

But

$$F^{(2)}\{f(t)\}=\lambda^2 f(w') \text{ and } F^{(4)}\{f(t)\}= \lambda^4 f(W). \tag{14}$$

From (13) and (14) it follows that $l/\sqrt{2p} \in C$ has order 4. ❑

We conclude that $\left\{y_n(t) := H_n(t).e^{-t^2/2}\right\}_{n=0}^{+\infty}$ are shape-invariant under Fourier operator associated to $l_n = (-j)^n\sqrt{2p}$. Therefore,

$$H_n(t).e^{-t^2/2} \leftrightarrow (-j)^n\sqrt{2p}\,H_n(w).e^{-w^2/2}\,. \tag{15}$$

Another interpretation can be derived evoking Rodrigues' formula [6]:

$$H_n(t) = (-1)^n e^{t^2}.\frac{d^n}{dt^n}e^{-t^2}\,. \tag{16}$$

The 2nd-order differential equation hold by invariant signals is

$$y''+(2n+1-x^2)y=0. \tag{17}$$

The above differential equation is exactly the celebrated Schrödinger equation for the harmonic oscillator [7].

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[1] N.B. Subtracting: $f''(t)+t^2f(t) \leftrightarrow -l\,[f''(w) + w^2f(w)]$.

III. CONSEQUENCES ON THE TIME-FREQUENCY PLANE

Let us now investigate certain consequences of eigenfunctions of the Fourier operator on the time-frequency plane [4, 8].

Let $f(t)$ be a finite energy signal E, with a transform $F(w)$. The time and frequency moments of f are defined by:

$$\overline{t^n} := \frac{\int_{-\infty}^{+\infty} f^*(t) t^n f(t) dt}{\int_{-\infty}^{+\infty} f^*(t) f(t) dt} = \frac{1}{E}\int_{-\infty}^{+\infty} t^n \, |f(t)|^2 \, dt,$$

and

$$\overline{w^n} := \frac{\int_{-\infty}^{+\infty} F^*(w) w^n F(w) dw}{\int_{-\infty}^{+\infty} F^*(w) F(w) dw} = \frac{1}{2pE}\int_{-\infty}^{+\infty} w^n \, |F(w)|^2 \, dw.$$

By analogy to Probability Theory, the term $|f(t)|^2/E$ denotes a "time-domain" energy density, where E is a normalising factor so as to make the whole integral of the density be equal to one. It is customary to deal with the energy spectral density $G(w)=|F(w)|^2$, whose integral over a frequency band gives the energy content of the signal within such a band. Let us suppose in the sequel, without loss of generality, that $E=1$ (energy normalised signals).

The "effective duration" (respectively "effective frequency width") of a signal $f(t)$ (respectively $F(w)$) is originally defined via:

$$\Delta t := \sqrt{\left[2p\,\overline{(t-\bar{t})^2}\right]} \quad \text{r.m.s. duration,} \qquad (18a)$$

$$\Delta f := \sqrt{\left[2p\,\overline{(f-\bar{f})^2}\right]} \quad \text{r.m.s. bandwidth,} \qquad (18b)$$

Δt and Δf correspond to the standard deviation (i.e., spreading measures). However, other common and much handier definitions are

$$\Delta_t := \sqrt{\overline{(t-\bar{t})^2}} \qquad \Delta_w := \sqrt{\overline{(w-\overline{w})^2}}. \qquad (19)$$

Clearly[2], $\Delta_t = \Delta t / \sqrt{2p}$, $\Delta_w = \sqrt{2p}.\Delta f$.

[2] $\Delta_w^2 = 2p\Delta f^2 = 2p.\left[2p\,\overline{(f-\bar{f})^2}\right] = \overline{(w-\overline{w})^2}$

A. *Revisiting the Gabor Principle*

By applying arguments from quantum mechanics [7], Gabor [9, 10] derived an uncertainty relation nowadays called Gabor-Heisenberg principle for signals: $\Delta t.\Delta f \geq 1/2$, proving that time and frequency cannot be exactly measured (simultaneously).

The Gabor-Heisenberg uncertainty principle states a lower bound on the product $\Delta t.\Delta w$ (or $\Delta_t.\Delta_w$)

$$\Delta_t \,.\, \Delta_w \geq \frac{1}{2}. \qquad (20)$$

<u>Proposition 5.</u> The Gabor lower bound is only achieved by the first invariant signal (eigenfunctions of F operator).

<u>Sketch of the proof</u>:
From (20), the bound is achieved *iff* $f'(t)=\mathrm{k}t.f(t)$. This condition can be interpreted as: 'Derivative in time domain'≡'derivative in frequency domain'. Therefore

$$f''(t)=\mathrm{k}[f(t)+t.f'(t)]=\mathrm{k}f(t)+\mathrm{k}^2t^2f(t) \qquad (21)$$

or

$$f''(t)-(\mathrm{k}t)^2f(t)=\mathrm{k}f(t). \qquad (22)$$

And so $f''(t)-\mathrm{k}(1+\mathrm{k}t^2)f(t)=0$. The only solutions on E_g correspond to k=±1, i.e., $f''+(1-t^2)f=0$ or $f''-(1+t^2)f=0$. ❑

<u>Proposition 6.</u> Any real signal $f(t) \leftrightarrow F(w)$ such that $f, f', F, F' \in L^2(\Re)$ has finite resolutions.

<u>Proof</u>:
Applying the Parseval-Plancherel Theorem [6], it follows that

$$\int_{-\infty}^{+\infty} t^2 f^2(t) dt = \int_{-\infty}^{+\infty} [jtf(t)][jtf(t)]^* dt = \frac{1}{2p}\int_{-\infty}^{+\infty} |F'(w)|^2 \, dw, \qquad (23)$$

and

$$\int_{-\infty}^{+\infty} w^2 \, |F(w)|^2 \, dw = \int_{-\infty}^{+\infty} [jwF(w)][jwF(w)]^* dw = 2p\int_{-\infty}^{+\infty} [f'(t)]^2 dt. \qquad (24)$$

Therefore

$$\Delta_t{}^2 = \frac{\int_{-\infty}^{+\infty} |F'(w)|^2 \, dw}{\int_{-\infty}^{+\infty} |F(w)|^2 \, dw} < +\infty, \qquad (25)$$

$$\Delta_w{}^2 = \frac{\int_{-\infty}^{+\infty} |f'(t)|^2 \, dt}{\int_{-\infty}^{+\infty} |f(t)|^2 \, dt} < +\infty. \qquad (26)$$

❑

Δ is therefore given by the square root of the ratio between the energy of the signal derivative and the energy of signal itself. Thus, the resolution for the Fourier invariant signal *sech*(.) given by eqn(4) is

$$\Delta_t = \Delta_w = \sqrt{p/6} \cong 0.5235987766 \qquad (27)$$

since that

$$\int_{-\infty}^{+\infty} sec\,h^2(z)dz = 2,$$

$$\int_{-\infty}^{+\infty} tgh^2(z).sec\,h^2(z)dz = \frac{2}{3},$$

$$\int_{-\infty}^{+\infty}\left(\frac{2z}{p}\right)^2 sec\,h^2(z)dz = \frac{2}{3}. \qquad (28)$$

<u>Proposition 7</u>. [9]. Time-frequency uncertainty of Fourier Eigenfunctions[3] $\{y_n(t)\}$ reach quantized values of the Gabor-Heisenberg lower bound, i.e.

$$\Delta t \,.\, \Delta f = \frac{1}{2}\,.\,(2n+1), \qquad (27a)$$

$$\Delta_t \,.\, \Delta_w = \frac{1}{2}(2n+1). \qquad (27b)$$

❑

That is why Gabor functions are relevant in some problems (e.g. [11]).

IV. THE CONCEPT OF ISORESOLUTION WAVELET

The concept of isoresolution analysis is introduced in this section. According to the Gabor principle, if one increases resolution in one domain, the resolution must decrease in the other domain so as to guarantee the lower bound given by (20). When analysing signals in joint time-frequency plane, frequently, there is no grounds to assure a better resolution in a domain than in the other domain. As an interesting property, any Fourier eigenfunction achieves isoresolution as it can be seen by:

<u>Proposition 8</u>. Fourier-invariant signals perform an isoresolution, that is, $\Delta_w=\Delta_t$.

<u>Proof</u>:
Supposing that $f \in E_g$, then $F(w)=l_n f(w)$. Therefore

$$\frac{\int_{-\infty}^{+\infty} F(w)w^2F^*(w)dw}{\int_{-\infty}^{+\infty} |F(w)|^2\,dw} = \frac{\int_{-\infty}^{+\infty} w^2\,|l_n|^2\,f^2(w)dw}{\int_{-\infty}^{+\infty} |l_n|^2\,f^2(w)dw}$$

and the proof follows. ❑

This is an interesting property for signalling on the joint time-frequency plane.

[3] $y_n(t) = H_n(t)\,exp\left(-t^2/2\right)exp\left(jw_0 t + f_0\right)$.

It is suggested here the changing of the time-frequency resolution by a proper scaling that allows for identical resolution in both domains.

<u>Proposition 9</u>. If $y(t)$ has effective duration Δ_t and effective bandwidth Δ_w, then the scaled version $y\left(\sqrt{\Delta_t/\Delta_w}\,t\right)$ achieves isoresolution.

<u>Proof</u>:
Scaled versions $y(at)$, $a\neq0$, have resolutions $\Delta_t/|a|$ and $|a|.\Delta_w$, so $|a|$ can be appropriately chosen.

❑

The essential idea of isoresolution can be placed in the wavelet structure. Normally, the basic wavelet of a family $\left\{\frac{1}{\sqrt{|a|}}\,y\left(\frac{t-b}{a}\right)\right\}$ holds the admissibility condition but often does not achieve isoresolution. We propose here to redefine the basic wavelet of a family so as to achieve isoresolution.

Take as a model the standard Mexican hat wavelet, $y^{Mhat}(t)$, defined by

$$2(t^2-1)\frac{e^{-t^2/2}}{\sqrt[4]{p}\,\sqrt{3}} \leftrightarrow -2\sqrt{\frac{2}{3}}\sqrt[4]{p}\,w^2e^{-w^2/2}. \qquad (28)$$

The isoresolution Mexican hat wavelet can be found applying proposition 9:

$$\sqrt{\frac{7}{15}}\,y^{Mhat}\left(\sqrt{\frac{7}{15}}\,t\right) \qquad (29)$$

For any isoresolution wavelet, the scaling by $a>1$ or $a<1$ corresponds to unbalance resolution in a different way.

Table 1 displays both time and frequency resolution for few known continuous wavelets: Gaussian derivatives, Mexican hat, Morlet, frequency *B*-Spline, Shannon and Haar [12]. Gaus1 is an invariant wavelet therefore it achieves isoresolution, in accordance to proposition 8. It is valuable to mention that compact support wavelets (in time or frequency) cannot attain isoresolution, since no signal can simultaneously be time and frequency limited [13].

TABLE 1. RESOLUTION OF FEW STANDARD CONTINUOUS WAVELETS.

Wavelet name	**Time resolution** Δ_t	**Frequency resolution** Δ_w	**Isoresolution factor** $\sqrt{\Delta_t/\Delta_w}$
Gaus1	1.500000	1.500000	1.000000
Mexican hat	1.166667	2.500000	0.683130
Morlet	0.500002	25.499997	0.140028
fbsp 2-1-0.5	+∞	14.475133	-
Shannon 1-0.5	+∞	13.159733	-
Haar	0.333333	+∞	-

V. Perspectives and Closing Remarks

Eigenfunctions of the Fourier operator were investigated and the Gabor principle was revisited defining the concept of isoresolution, i.e, a signal with the same time and frequency resolution. The $\{y_n(t)\}$ functions (see eqn(15)) turn up as a very appealing choice for designing representations such as wavelets. It is time to try finding new wavelets starting with the equation (11). Since they are solutions of a wave equation (2nd order differential equation), our approach (Mathieu [14], Legendre [15], Chebyshev [16]) can be useful to construct new wavelets: The Quantum Wavelets, or Gabor-Schrödinger wavelets. The construction of new wavelets based on these complete, orthogonal, domain shape-invariant system is currently being investigated. The idea is to adapt the concept of isoresolution in orthogonal multiresolution analysis [17, 18].

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