Ratio of the Product-of-Two to One Variates: A Framework - Application to κ-μ and η-μ

Carlos Rafael Nogueira da Silva and Michel Daoud Yacoub

***Abstract*— The performance analyses of a number of fading scenarios require the knowledge of some of the statistics of the product and the ratio of the involved variates. With the advent of several enabling technologies, e.g., reconfigurable/large intelligent surfaces, massive MIMO, cooperative communications, device-to-device, vehicle-to-vehicle communications, and others, this is even more evident. In this paper, we provide a framework for the derivation of probability density functions and the cumulative distribution functions for the ratio of the product-of-two to one variates. The framework is then used to derive these statistics for all possible combinations of these variates taken from κ-μ and η-μ fading models. The expressions are obtained in terms of a simple-to-compute single-sum infinite series, with convergence attained very rapidly.**

***Keywords*— R/LIS, massive MIMO, D2D, V2V, product distribution.**

I. Introduction

Channel characterization has always been an important issue in telecommunications. This becomes even more so when the demand for services increases dramatically and the designers are faced with the very same question, which is how to enhance the capacity in order to comply with such a demand. As far as wireless networks are concerned, a better knowledge of the channel statistics is a stepping stone towards this. In this sense, a number of fading models arises, each of which attempting to fill the gap left by the previous ones with the inclusion of new physical phenomenon. As technologies evolve, the communication channel becomes more complex, hindering the usability of traditional fading models. This shifts the focus from single to composite fading models. As well known, the product of two fading variables models the multipath-shadowing channels, in which one variable represents the shadowing phenomenon and the other, the multipath fading. Typically, the lognormal distribution describes the shadowing phenomenon completely, although, because of its mathematical intricacy, other more tractable fading models are used in its place. On the other hand, the choice for a multipath fading model depends on the system design and physical medium. For instance, the κ-μ describes fading signal with a dominant component whereas the η-μ characterizes fading in a non-homogeneous medium with no dominant component.

Enabling technologies such as cascaded channel [1]–[3], reconfigurable/large intelligent surfaces (R/LIS) [3]–[5], multi-hop links [6], [7], massive MIMO systems [8]–[10] have physical models based on the product of fading variables, characterizing a composite fading signal. Hence, the behavior of composite channels and statistics of the product of random envelopes have been receiving great attention in the literature [11]–[13]. On the other hand, the overall system performance uses the ratio of some statistical metric. For instance, the signal-to-noise ratio (SNR) is the fundamental metric to evaluate bit error rate (BER), outage probability, channel capacity, and secrecy capacity, to name but a few. Therefore, the statistical characterization of the product and ratio of fading signals is important for the deployment of future generations of telecommunications.

In this paper, we develop a framework to derive the ratio involving three fading variables. This scenario finds application in the performance analysis of a wireless system in an interference environment assisted by R/LIS or a relay. Here, we derive the probability density function (PDF) and cumulative distribution function (CDF) of the ratio of two by one random variates. The general formulation is exercised for variates taken arbitrarily from the κ-μ and the η-μ fading models. Both PDFs and CDFs for all combinations of variates are obtained as single, relatively simple, infinite series that compute rapidly.

The remainder of the paper is divided as follows: Section II establishes the channel model and the framework for obtaining the PDF and CDF in terms of contour integrals and as series representation. Section III revisit the generalized fading models κ-μ and η-μ. Section IV exercises the framework by chosen the fading components arbitrarily from the κ-μ or the η-μ fading models; Section V provides final remarks.

II. Channel Model

Let $X_i > 0$, $i = 1, 2, 3$ be fading variates. Consider a channel characterized by the ratio of the product-of-two to one variables. Such a scenario may be found in an interference environment of a communication link assisted by a single relay. Let $Z > 0$ be the desired ratio as

$$Z = \frac{X_1 X_2}{X_3}, \tag{1}$$

in which all fading signals are independent and non-identically distributed.

A. General Framework for Obtaining the PDF

The PDF of the random variate Z may be found by the standard statistical procedures, which always lead to a double integral, using any of the available methods We maintain,

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however, that with the appropriate use of the direct/inverse Mellin transform pair in conjunction with the Cauchy residue theorem, we have been able to provide the required formulations through efficiently computable single-sum infinite series. The Mellin transform is defined as

$$f^*(s) = \int_0^\infty t^{s-1} f(t) dt, \tag{2}$$

provided convergence. The inverse transformation is obtained as

$$f(t) = \frac{1}{2\pi j} \oint_{\mathcal{L}} f^*(s) t^{-s} ds, \tag{3}$$

in which $j = \sqrt{-1}$, and $\mathcal{L}$ is a suitable contour on the complex plane[1]. From the formulations, it is evident that there is a clear relationship between the Mellin transform and the generalized moments of a positive fading distribution. Therefore, the Mellin transform of the PDF $f_R(r)$ of a fading signal $R > 0$ is obtained in terms of its moments as $f_R^*(s) = \mathbb{E}[R^{s-1}]$. More importantly, if the generalized moment is an analytic function, then its inverse is uniquely determined by $f_R^*(s)$. From (1), the generalized moment of the random variate Z applied at $s-1$ is

$$\mathbb{E}[Z^{s-1}] = \mathbb{E}[X_1^{s-1}]\mathbb{E}[X_2^{s-1}]\mathbb{E}[X_3^{1-s}], \tag{4}$$

Now, applying (4) in (3) results in the PDF of Z as a contour integration as

$$f_Z(z) = \frac{1}{2\pi j} \oint_{\mathcal{L}} \mathbb{E}[X_1^{s-1}]\mathbb{E}[X_2^{s-1}]\mathbb{E}[X_3^{1-s}] z^{-s} ds. \tag{5}$$

Provided that the moments are analytical, the PDF $f_Z(z)$ is uniquely determined by them. Integrating (5) from 0 to z defines the CDF of the random variate Z which is

$$F_Z(z) = \frac{1}{2\pi j} \oint_{\mathcal{L}} \mathbb{E}[X_1^{s-1}]\mathbb{E}[X_2^{s-1}]\mathbb{E}[X_3^{1-s}] \frac{z^{1-s}}{1-s} ds. \tag{6}$$

B. Series Representation

The Cauchy residue theorem [15] states that an integral of a function $f(s)$ over a closed contour can be found as the sum of the residues around the poles of $f(s)$ inside the contour of integration. Therefore, a series representation for the PDF in (5) is

$$f_Z(z) = \sum_i \text{res}_i \left(\mathbb{E}[X_1^{s-1}]\mathbb{E}[X_2^{s-1}]\mathbb{E}[X_3^{1-s}] z^{-s} \right), \tag{7}$$

C. The Reciprocal Distribution

Let Z be as defined before, then the reciprocal variate is its inverse, $Z' = 1/Z$. The first order statistics comes from the standard variable transformation and is given as

$$f_{Z'}(z) = z^{-2} f_Z(z^{-1}). \tag{8}$$

Hence, an expression for the ratio of one by product of two variates arises naturally.

[1]Please, refer to [14] for an appropriate contour

III. Generalized Fading Models

In the next section, we derive the PDF and CDF for the variate Z, as defined in (1), for several fading scenarios. In particular, the chosen models for the components of Z are the κ-μ and the η-μ distributions. These are general models which comprise several important fading distributions such as Nakagami-*m*, Rice, Hoyt, and Rayleigh. Using the proposed framework, we provide the statistics for all combination of X_1, X_2, and X_3 arbitrarily taken from the κ-μ or the η-μ distributions. First, let us briefly revisit some of the necessary statistics of these fading models.

A. The κ-μ Fading Model

The κ-μ distribution is a generalized fading model used to describe fading signal with a dominant component and multipath clusters. Let $R > 0$ be a fading signal with *rms* value given as $\hat{r}^2 = \mathbb{E}[R^2]$. Its moments are well known and are defined as

$$\mathbb{E}[R^k] = \mathcal{K}^k \frac{\Gamma\left(\frac{k}{2} + \mu\right)}{\Gamma(\mu)} {}_1F_1\left(-\frac{k}{2}; \mu; -\kappa\mu\right), \tag{9}$$

in which $\mathcal{K} = \hat{r}/\sqrt{(\mu(1+\kappa))}$, $\mu = \mathbb{E}^2[R^2]/\mathbb{V}[R^2] \times (1 + 2\kappa)/(1+\kappa)^2$ is related to the number of multipath clusters, κ is the ratio between the total power of the dominant component by the total power of the scattered waves, $\mathbb{E}[\cdot]$ and $\mathbb{V}[\cdot]$ are the expectation and variance operators respectively, $\Gamma(x)$ is the gamma function [16, Eq. (6.1.1)] and ${}_1F_1(\cdot,\cdot,\cdot)$ is the Kummer's confluent hypergeometric function [16, Eq. (13.1.2)].

B. The η-μ Fading Model

The η-μ distribution is a generalized fading model used to describe fading signals in a non-homogeneous medium with power imbalance between in-phase and quadrature components or with correlation between its components and multipath clusters. Let $R > 0$ be a fading signal with *rms* value as $\hat{r}^2 = \mathbb{E}[R^2]$. Its moments are well known and are defined as

$$\mathbb{E}[R^k] = \frac{\mathcal{E}^k \Gamma\left(\frac{k}{2} + 2\mu\right)}{\Gamma(2\mu)} {}_2F_1\left(-\frac{k}{4}, \frac{2-k}{4}; \mu + \frac{1}{2}; \frac{H^2}{h^2}\right), \tag{10}$$

in which $\mu > 0$ relates to the number of multipath clusters, $\mathcal{E} = \hat{r}/\sqrt{2\mu}$, the constants h and H varies in accordance with the chosen format such that in format 1 we have $h = (2 + \eta^{-1} + \eta)/4$ and $H = (\eta^{-1} - \eta)/4$ in which $\eta > 0$ is the power ratio between the in-phase and quadrature scattered waves, and for format 2 $h = 1/(1 - \eta^2)$ and $H = \eta h$ and $-1 < \eta < 1$ is the correlation coefficient between the in-phase and quadrature waves, and ${}_2F_1(a, b, c, x)$ is the Gauss' hypergeometric function [16, Eq. (15.1.1)].

IV. Ratio of the Product

This subsection derives the PDFs and the CDFs for all possible combinations of X_1, X_2 and X_3 arbitrarily chosen from the κ-μ or η-μ fading model. Interestingly, for every combination of X_1, X_2 and X_3, when evaluating the generalized moment of random variate Z in (5), two gamma functions provide the poles for computing the residues in (7). Thus, we can separate the sum of residues into two parts, and the PDF would, in general, be

$$\begin{aligned} f_Z(z) &= \sum_{j=0}^{\infty} \underset{s \to -\frac{j+a_1}{k_1}}{res} \Gamma(a_1 + k_1 s) f_1(s) \\ &+ \sum_{j=0}^{\infty} \underset{s \to -\frac{j+a_2)}{k_2}}{res} \Gamma(a_2 + k_2 s) f_2(s) \end{aligned} \tag{11}$$

in which $f_1(s)$ and $f_2(s)$ are

$$f_n(s) = \frac{\mathbb{E}[X_1^{s-1}]\mathbb{E}[X_2^{s-1}]\mathbb{E}[X_3^{1-s}]}{\Gamma(a_n + k_n s)} z^{-s}, \quad n \in \{1, 2\} \tag{12}$$

with the moment of X_i taken from (9) or (10) in accordance with the chosen model for each component. The residue around the poles of the gamma function are [17, Section 6.3.1]

$$\underset{s \to -\frac{j+a}{k}}{\text{res}} \Gamma(a + ks) f(s) = \frac{(-1)^j}{kj!} f\left(-\frac{a+j}{k}\right), \ j \in \mathbb{Z}_+, \tag{13}$$

provided that $-(a+j)/k$ is not a singularity of $f(s)$. Applying (13) in (11), the PDF for Z would have the general form

$$\begin{aligned} f_Z(z) &= \sum_{j=0}^{\infty} \frac{(-1)^j}{k_1 j!} f_1\left(-\frac{a_1 + j}{k_1}\right) \\ &+ \sum_{j=0}^{\infty} \frac{(-1)^j}{k_2 j!} f_2\left(-\frac{a_2 + j}{k_2}\right) \end{aligned} \tag{14}$$

A. The κ-μ times κ-μ over κ-μ case

Let $X_1 > 0$, $X_2 > 0$, and $X_3 > 0$ be fading variates following the κ-μ distribution with parameters $\{\kappa_i, \mu_i, \hat{r}_i\}$ with $i \in \{1, 2, 3\}$, and their generalized moments are given in (9) with the appropriate subscripts. Now applying (9) in (4) and then in (5), the PDF of the random variable Z is obtained as a contour integral. In this scenario, the two gamma functions with poles inside the contour of integration are $\Gamma(\mu_1 + (s-1)/2)$ and $\Gamma(\mu_2 + (s-1)/2)$. Therefore, a series representation for the PDF in this scenario arises from (14) with $a_i = \mu_i - 1/2$, $i = \{1, 2\}$ and $k_1 = k_2 = 1/2$, and $f_1(s)$ and $f_2(s)$ are as in (12) in which the moments are replaced with (9). After some algebraic manipulations the PDF for Z is given in (15) in which ${}_1\tilde{F}_1(a, b, x) = {}_1F_1(a, b, x)/\Gamma(b)$ is the regularized form of Kummer's hypergeometric function. By integrating (15) the CDF for Z is obtained as in (16).

B. The η-μ times η-μ over η-μ case

Here, we consider that X_1, X_2, and X_3 follow the η-μ fading model with parameters $\{\eta_i, \mu_i, \hat{r}_i\}$ with $i \in \{1, 2, 3\}$. Applying (10) with the appropriate subscript in (5) defines the contour representation for this PDF. The series representation is obtained as in (14) with $a_i = 2\mu_i - 1/2$, $i = \{1, 2\}$ and $k_1 = k_2 = 1/2$. After some algebraic manipulations, the PDF is given as in (17) and the CDF is at (18).

C. The κ-μ times κ-μ over η-μ case

Let us consider both X_1 and X_2 taken from the κ-μ model with parameters $\{\kappa_i, \mu_i, \hat{r}_i\}$ with $i = \{1, 2\}$ while X_3 follows the η-μ distribution with parameters $\{\eta_3, \mu_3, \hat{r}_3\}$. The series representation is derived from (14) with $a_i = \mu_i - 1/2$, $i = \{1, 2\}$ and $k_1 = k_2 = 1/2$. The PDF of Z in this scenario is given in (19) and its CDF is in (19).

D. The κ-μ times η-μ over κ-μ case

Here, we consider the scenario involving the product of a κ-μ random variate and an η-μ variate divided by another κ-μ. Thus, X_1 and X_3 are κ-μ with parameters κ_1, μ_1 and $\hat{r}_1$ and κ_3, μ_3 and $\hat{r}_3$, respectively while X_2 follows the η-μ distribution with parameters η_2, μ_2, and $\hat{r}_2$. The series representation is obtained from (14) with $a_1 = \mu_1 - 1/2$, $a_2 = 2\mu_2 - 1/2$ and $k_1 = k_2 = 1/2$. After manipulations, the PDF and CDF, in series format, is given in (21) and (22), respectively.

E. The κ-μ times η-μ over η-μ case

Here, X_2 and X_3 follow the η-μ fading model with parameters $\{\eta_i, \mu_i, \hat{r}_i\}$ with $i = \{2, 3\}$ and X_1 follows the κ-μ model with parameters κ_1, μ_1, and $\hat{r}_1$. In this scenario, the series in (14) will have $a_1 = \mu_1 - 1/2$, $a_2 = 2\mu_2 - 1/2$ and $k_1 = k_2 = 1/2$. After some algebraic manipulation, the PDF and CDF for this scenario are given in (23) and (24) respectively.

F. The η-μ times η-μ over κ-μ case

Now, we consider that X_1 and X_2 follow the η-μ model with parameters $\{\eta_i, \mu_i, \hat{r}_i\}$ with $i = 1, 2$ and X_3 follow the κ-μ distribution with parameters κ_3, μ_3 and $\hat{r}_3$. The series representation in this scenario arises from (14) with $a_i = 2\mu_i - 1/2$, $i = \{1, 2\}$ and $k_1 = k_2 = 1/2$. After the some manipulations, the PDF and CDF in this scenario are given in (25) and (26) respectively.

V. Conclusions

In this paper, we provided a framework for the derivation of the first order statistics for the ratio of the product-of-two by one variates and its reciprocal. This scenario may arise in the analysis of multi-hop communications immerse in an interference environment. The framework was exercised for the variates taken arbitrarily from κ-μ and the η-μ fading models, which, by themselves, comprise several important fading models such as Nakagami-*m*, Rice, Hoyt, and Rayleigh. The expressions were given in series representations involving a single sum. We maintain that they compute easily and converge to any required accuracy with the appropriate number of terms in the series. Unfortunately, due to lack of space, we have not been able to use these in applications examples, which may be chosen among several important enabling technologies such as reconfigurable/large intelligent surfaces, massive MIMO, cooperative communications, device-to-device, vehicle-to-vehicle communications, and others.

$$\begin{aligned} f_Z(z) = \frac{2}{z} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{k=1}^{2} \Gamma\left((-1)^k (\mu_1 - \mu_2) - j\right) \Gamma(j + \mu_k + \mu_3) \left(\frac{z\mathcal{K}_3}{\mathcal{K}_1\mathcal{K}_2}\right)^{2(j+\mu_k)} \\ \times\ {}_1\tilde{F}_1(-j - \mu_k; \mu_3; -\kappa_3\mu_3) \prod_{i=1}^{2} {}_1\tilde{F}_1(j + \mu_k; \mu_i; -\kappa_i\mu_i) \end{aligned} \tag{15}$$

$$\begin{aligned} F_Z(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{k=1}^{2} \frac{\Gamma\left((-1)^k (\mu_1 - \mu_2) - j\right) \Gamma(j + \mu_k + \mu_3)}{j + \mu_k} \left(\frac{z\mathcal{K}_3}{\mathcal{K}_1\mathcal{K}_2}\right)^{2(j+\mu_k)} \\ \times\ {}_1\tilde{F}_1(-j - \mu_k; \mu_3; -\kappa_3\mu_3) \prod_{i=1}^{2} {}_1\tilde{F}_1(j + \mu_k; \mu_i; -\kappa_i\mu_i) \end{aligned} \tag{16}$$

$$\begin{aligned} f_Z(z) = \frac{2}{z\prod_{i=1}^{3}\Gamma(2\mu_i)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{k=1}^{2} \Gamma\left(2(-1)^k (\mu_1 - \mu_2) - j\right) \Gamma(j + 2(\mu_k + \mu_3)) \left(\frac{z\mathcal{E}_3}{\mathcal{E}_1\mathcal{E}_2}\right)^{2(j+2\mu_k)} \\ \times\ {}_2F_1\left(-\left(\frac{j}{2} + \mu_k\right), -\left(\frac{j-1}{2} + \mu_k\right); \frac{1}{2} + \mu_3; \frac{H_3^2}{h_3^2}\right) \prod_{i=1}^{2} {}_2F_1\left(\frac{j}{2} + \mu_k, \frac{1+j}{2} + \mu_k; \frac{1}{2} + \mu_i; \frac{H_i^2}{h_i^2}\right) \end{aligned} \tag{17}$$

$$\begin{aligned} F_Z(z) = \frac{1}{\prod_{i=1}^{3}\Gamma(2\mu_i)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{k=1}^{2} \frac{\Gamma\left(2(-1)^k (\mu_1 - \mu_2) - j\right) \Gamma(j + 2(\mu_k + \mu_3))}{j + 2\mu_k} \left(\frac{z\mathcal{E}_3}{\mathcal{E}_1\mathcal{E}_2}\right)^{2(j+2\mu_k)} \\ \times\ {}_2F_1\left(-\left(\frac{j}{2} + \mu_k\right), -\left(\frac{j-1}{2} + \mu_k\right); \frac{1}{2} + \mu_3; \frac{H_3^2}{h_3^2}\right) \prod_{i=1}^{2} {}_2F_1\left(\frac{j}{2} + \mu_k, \frac{1+j}{2} + \mu_k; \frac{1}{2} + \mu_i; \frac{H_i^2}{h_i^2}\right) \end{aligned} \tag{18}$$

$$\begin{aligned} f_Z(z) = \frac{2}{z\Gamma(2\mu_3)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{k=1}^{2} \Gamma\left((-1)^k (\mu_1 - \mu_2) - j\right) \Gamma(j + \mu_k + 2\mu_3) \left(\frac{z\mathcal{E}_3}{\mathcal{K}_1\mathcal{K}_2}\right)^{2(j+\mu_k)} \\ \times\ {}_2F_1\left(-\frac{1}{2}(j + \mu_k), \frac{1}{2}(1 - j - \mu_k); \frac{1}{2} + \mu_3; \frac{H_3^2}{h_3^2}\right) \prod_{i=1}^{2} {}_1\tilde{F}_1(j + \mu_k; \mu_i; -\kappa_i\mu_i) \end{aligned} \tag{19}$$

$$\begin{aligned} F_Z(z) = \frac{1}{\Gamma(2\mu_3)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{k=1}^{2} \frac{\Gamma\left((-1)^k (\mu_1 - \mu_2) - j\right) \Gamma(j + \mu_k + 2\mu_3)}{j + \mu_k} \left(\frac{z\mathcal{E}_3}{\mathcal{K}_1\mathcal{K}_2}\right)^{2(j+\mu_k)} \\ \times\ {}_2F_1\left(-\frac{1}{2}(j + \mu_k), \frac{1}{2}(1 - j - \mu_k); \frac{1}{2} + \mu_3; \frac{H_3^2}{h_3^2}\right) \prod_{i=1}^{2} {}_1\tilde{F}_1(j + \mu_k; \mu_i; -\kappa_i\mu_i) \end{aligned} \tag{20}$$

$$\begin{aligned} f_Z(z) = \frac{2}{z\Gamma(2\mu_2)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{k=1}^{2} \Gamma\left((-1)^k (\mu_1 - 2\mu_2) - j\right) \Gamma(j + k\mu_k + \mu_3) \left(\frac{z\mathcal{K}_3}{\mathcal{E}_2\mathcal{K}_1}\right)^{2(j+k\mu_k)} \\ \times\ {}_1\tilde{F}_1(j + k\mu_k; \mu_1; -\kappa_1\mu_1)\ {}_1\tilde{F}_1(-j - k\mu_k; \mu_3; -\kappa_3\mu_3)\ {}_2F_1\left(\frac{1}{2}(j + k\mu_k), \frac{1}{2}(1 + j + k\mu_k); \frac{1}{2} + \mu_2; \frac{H_2^2}{h_2^2}\right) \end{aligned} \tag{21}$$

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\begin{aligned}
F_Z(z) = \frac{1}{\Gamma(2\mu_2)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{k=1}^{2} \frac{\Gamma\left((-1)^k(\mu_1 - 2\mu_2) - j\right)\Gamma(j + k\mu_k + \mu_3)}{j + k\mu_k} \left(\frac{z\mathcal{K}_3}{\mathcal{E}_2\mathcal{K}_1}\right)^{2(j+k\mu_k)} \\
\times\ {}_1\tilde{F}_1(j + k\mu_k; \mu_1; -\kappa_1\mu_1)\ {}_1\tilde{F}_1(-j - k\mu_k; \mu_3; -\kappa_3\mu_3)\ {}_2F_1\left(\frac{1}{2}(j + k\mu_k), \frac{1}{2}(1 + j + k\mu_k); \frac{1}{2} + \mu_2; \frac{H_2^2}{h_2^2}\right)
\end{aligned} \tag{22}
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\begin{aligned}
f_Z(z) = \frac{2/z}{\prod_{i=2}^{3}\Gamma(2\mu_i)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{k=1}^{2} \Gamma\left((-1)^k(\mu_1 - 2\mu_2) - j\right)\Gamma(j + k\mu_k + 2\mu_3)\ {}_1\tilde{F}_1(j + k\mu_k; \mu_1; -\kappa_1\mu_1) \\
\times \left(\frac{z\mathcal{E}_3}{\mathcal{E}_2\mathcal{K}_1}\right)^{2(j+k\mu_k)} {}_2F_1\left(-\frac{j + k\mu_k}{2}, \frac{1 - j - k\mu_k}{2}; \frac{1}{2} + \mu_3; \frac{H_3^2}{h_3^2}\right) {}_2F_1\left(\frac{j + k\mu_k}{2}, \frac{1 + j + k\mu_k}{2}; \frac{1}{2} + \mu_2; \frac{H_2^2}{h_2^2}\right)
\end{aligned} \tag{23}
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\begin{aligned}
F_Z(z) = \frac{1}{\prod_{i=2}^{3}\Gamma(2\mu_i)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{k=1}^{2} \frac{\Gamma\left((-1)^k(\mu_1 - 2\mu_2) - j\right)\Gamma(j + k\mu_k + 2\mu_3)}{j + k\mu_k}\ {}_1\tilde{F}_1(j + k\mu_k; \mu_1; -\kappa_1\mu_1) \\
\times \left(\frac{z\mathcal{E}_3}{\mathcal{E}_2\mathcal{K}_1}\right)^{2(j+k\mu_k)} {}_2F_1\left(-\frac{j + k\mu_k}{2}, \frac{1 - j - k\mu_k}{2}; \frac{1}{2} + \mu_3; \frac{H_3^2}{h_3^2}\right) {}_2F_1\left(\frac{j + k\mu_k}{2}, \frac{1 + j + k\mu_k}{2}; \frac{1}{2} + \mu_2; \frac{H_2^2}{h_2^2}\right)
\end{aligned} \tag{24}
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$$
\begin{aligned}
f_Z(z) = \frac{2}{z\prod_{i=1}^{2}\Gamma(2\mu_i)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{k=1}^{2} \Gamma\left(2(-1)^k(\mu_1 - \mu_2) - j\right)\Gamma(j + 2\mu_k + \mu_3) \left(\frac{z\mathcal{K}_3}{\mathcal{E}_1\mathcal{E}_2}\right)^{2(j+2\mu_k)} \\
\times\ {}_1\tilde{F}_1(-j - 2\mu_k; \mu_3; -\kappa_3\mu_3) \prod_{i=1}^{2} {}_2F_1\left(\frac{j}{2} + \mu_k, \frac{1 + j}{2} + \mu_k; \frac{1}{2} + \mu_i; \frac{H_i^2}{h_i^2}\right)
\end{aligned} \tag{25}
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$$
\begin{aligned}
F_Z(z) = \frac{1}{\prod_{i=1}^{2}\Gamma(2\mu_i)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{k=1}^{2} \frac{\Gamma\left(2(-1)^k(\mu_1 - \mu_2) - j\right)\Gamma(j + 2\mu_k + \mu_3)}{j + 2\mu_k} \left(\frac{z\mathcal{K}_3}{\mathcal{E}_1\mathcal{E}_2}\right)^{2(j+2\mu_k)} \\
\times\ {}_1\tilde{F}_1(-j - 2\mu_k; \mu_3; -\kappa_3\mu_3) \prod_{i=1}^{2} {}_2F_1\left(\frac{j}{2} + \mu_k, \frac{1 + j}{2} + \mu_k; \frac{1}{2} + \mu_i; \frac{H_i^2}{h_i^2}\right)
\end{aligned} \tag{26}
$$

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