# Semi-blind Two-Stage Estimation of Carrier Frequency Offset for OFDM and OFDMA Systems

Tadeu N. Ferreira, Sergio L. Netto, Marcello L. R. de Campos, Paulo S. R. Diniz

*Abstract*— **Orthogonal Frequency-Division Multiplexing (OFDM) is a technique used in several wireless communication systems. One impairment present in OFDM-based systems is the Carrier Frequency Offset (CFO). The objective of this article is to propose a semi-blind two-stage CFO estimation algorithm for OFDM and for an Orthogonal Frequency-Division Multiple Access system. The proposed algorithm presents lower root-mean squared error metrics in all of the tested scenarios and a reduced complexity than other two-stage algorithms.**

*Keywords*— **OFDMA, CFO, two-stage estimation.**

## I. Introduction

Wireless communication systems demand progressively higher rates with an increasing set of applications. Orthogonal Frequency-Division Multiplexing (OFDM) is a widespread technique used in wireless systems. A straightforward extension of OFDM is the Orthogonal Frequency-Division Multiple Access (OFDMA) [1] [2], providing groups of orthogonal subcarriers for multiple users, such as in fifth-generation (5G) wireless systems and IEEE 802.16m (Mobile Wi-MAX) [1].

One impairment that damages the performance of OFDM and OFDMA is the Carrier Frequency Offset (CFO), which is present due to loss of synchronism between transmitter and receiver oscillators. CFO interference leads to loss of orthogonality among the sub-carriers, increasing the inter-carrier interference (ICI) of the system. CFO estimation was initially based on intuitive approaches, such as Moose's algorithm [3], or on Maximum Likelihood estimation [4]. Another approach to estimate CFO is a set of subspace-based algorithms, such as MUSIC (Multiple Signal Classification) [5]. In [6], two-stage blind algorithms are presented with combinations of MUSIC and ESPRIT (Estimation of Parameters via Rotational Invariance Techniques). In [7], a CFO estimator for the IEEE 802.16 standard is described, while in [8], [9] CFO is estimated in a mobile system environment.

The objective of this article is to propose a semi-blind algorithm for CFO estimation in OFDM and OFDMA. The proposed Moose-MUSIC algorithm uses the Moose's algorithm [3] as the first stage and MUSIC as the second stage. Semi-blind algorithms generally need fewer samples than their blind counterparts, which is observed in the proposed Moose-MUSIC. Moreover, Moose-MUSIC runs in less time than other two-stage algorithms.

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Section II shows the context of OFDM and OFDMA systems, as well as the CFO interference. In Section III, some CFO estimation algorithms are presented. In Section IV, the proposed Moose-MUSIC algorithm is described. Section V compares the computational complexity of the proposed Moose-MUSIC to its counterparts. Section VI shows the results of the performance evaluation of the proposed algorithms and other algorithms from the literature. In Section VII, the main conclusions of this work are presented.

## II. System Description

OFDM provides a transmission using multiple sub-carriers. It mitigates the effects of frequency-selective fading in communication systems. The sub-carriers are designed to be mutually orthogonal in order to avoid inter-carrier interference, caused by the frequency-selective fading. Consider a system with sampling period $T$. At time instant $t = kT, k \in \mathbb{N}$, an OFDM symbol $\mathbf{s}(k), k = 0, 1, 2, \ldots, N$ is transmitted, where $N$ is the number of OFDM sub-carriers. The orthogonal subcarriers are generated by an IDFT (Inverse Discrete Fourier Transform), which is applied to $\mathbf{s}(k) \in \mathbb{C}^N$, that is, $\mathbf{x}(k) = \mathbf{W}\mathbf{s}(k)$, where $\mathbf{W}$ is the IDFT matrix, and $\mathbf{x}(k)$ is the time-domain OFDM symbol. A cyclic prefix (CP) with length $N_{\mathrm{CP}}$ is then added to $\mathbf{x}(k)$, generating $\mathbf{x}_{\mathrm{CP}}(k)$.

The signal $\mathbf{x}_{\mathrm{CP}}(k)$ is transmitted through the channel with impulse response $\mathbf{h}(k)$. In the receiver, signal $\mathbf{y}_{\mathrm{CP}}(k) = \mathbf{x}_{\mathrm{CP}}(k) * \mathbf{h}(k) + \mathbf{n}_{CP}(k)$ is acquired, where $\mathbf{n}_{CP}(k)$ is a vector containing $N$ samples of additive white Gaussian noise and $N_{CP}$ repeated samples. Then, $\mathbf{y}(k)$ is generated by removing the CP from $\mathbf{y}_{\mathrm{CP}}(k)$, and, subsequently, $\mathbf{y}(k)$ is converted to the frequency domain by a DFT (Discrete Fourier Transform), i.e., $\mathbf{c}(k) = \mathbf{W}^H\mathbf{y}(k)$. Finally, a one-tap equalization is applied, generating an estimation of the transmitted symbol $\hat{\mathbf{s}}(k) = \mathbf{b}(k) \circ \mathbf{c}(k)$, where $\circ$ is the pointwise multiplication. The insertion of a CP creates the following equivalence:

$$\mathbf{y}(k) = \mathbf{H}_c(k)\mathbf{x}(k) + \mathbf{n}(k), \tag{1}$$

where $\mathbf{H}_c(k) \in \mathbb{C}^N$ is a circular convolution matrix. Due to the effect of the IDFT/DFT pair, $\mathbf{H}_c(k)$ is a diagonal matrix.

### A. Carrier Frequency Offset

When there is an impairment of the frequency generated by the oscillators on the transmitter in relation to their counterparts on the receiver, then a CFO $\theta$ appears. In this article, CFO denominates a deterministic and cumulative phase impairment [3], [6]. Most articles from the literature on CFO estimation consider the random variation of phase as a generic Phase Noise.

The received signal is given by $\mathbf{y}(k) = \exp{(jk\theta)}\mathbf{E}\ \mathbf{x}(k) \circ \mathbf{h}(k) + \mathbf{n}(k)$, where

$$\mathbf{E} = \mathrm{diag}(1, \exp{(j\theta)}, \ldots, \exp{(j(N-1)\theta)}), \qquad (2)$$

where $\mathrm{diag}(\mathbf{a})$ is a matrix with $\mathbf{a}$ as its diagonal and 0 elsewhere.

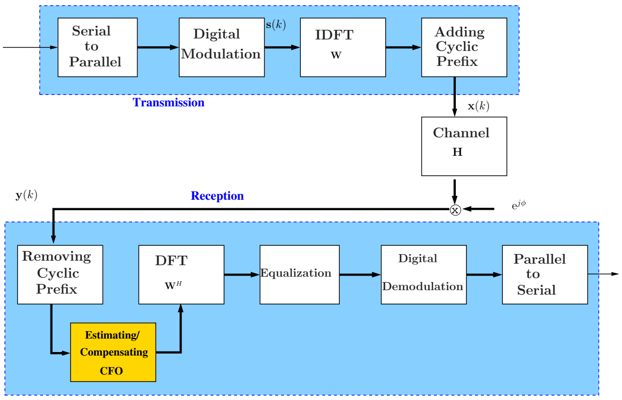


Fig. 1. Diagram representing the simulated OFDM system with CFO interference.

When the CFO is present in an OFDM system, there is loss of the mutual orthogonality, which is provided by the IDFT/DFT pair and by the CP. Then, the CFO should be estimated and compensated, as shown in Fig. 1.

After compensation, the orthogonality is restored, and the following relation is still valid: $\mathbf{c}(k) = \mathbf{W}^H\mathbf{y}(k)$.

## III. CFO Estimation Algorithms

In this section, two CFO estimation algorithms are described. In Moose's CFO estimation algorithm [3], the CFO estimation is performed using a few symbols, while MUSIC [5] is a blind estimator based on the use of the noise subspace. Both Moose's algorithm and MUSIC are employed later on for designing our proposed algorithm.

### A. Moose's CFO Estimation Algorithm

In Moose's CFO estimation algorithm [3], a known training sequence is transmitted at discrete time instants $k = 0, \ldots, N_T - 1$, where $N_T$ is the number of training OFDM symbols. Then, for each sub-carrier [3]:

$$\hat{\theta} = \frac{1}{jN(N_T-1)} \sum_{k=1}^{N_T-1} \frac{\mathbf{y}_i(k)\mathbf{s}_i(k-1)}{\mathbf{y}_i(k-1)\mathbf{s}_i(k)}, \qquad (3)$$

where $\mathbf{y}_i(k)$ denotes the $i^{\text{th}}$ element of $\mathbf{y}(k)$, and $\mathbf{s}_i(k)$ is the $i^{\text{th}}$ entry of $\mathbf{s}(k)$.

Moose's algorithm provides a low-complexity estimation of the CFO. On the other hand, its estimation presents a resolution problem as the CFO increases, since it is affected by the presence of a residual ICI.

### B. MUSIC CFO Estimation Algorithm

MUSIC is a traditional blind technique used to estimate Direction-of-Arrival in antenna array systems. It was later adapted to the CFO estimation scenario [5]. MUSIC requires the use of virtual sub-carriers, that is, unused sub-carriers in the systems, where there is no transmitted data, and they contain noise samples. The noise subspace of data is important to the application of MUSIC [5].

Consider that data is transmitted on only $P$ out of the $N$ sub-carriers. Then, there are $N-P$ virtual sub-carriers in the system. Consider $\mathbf{s}_P(k) \in \mathbb{C}^P$ comprising the $P$ effectively transmitted symbols, and $\mathbf{W}_P$ containing the $P$ columns of $\mathbf{W}$ corresponding to the non-virtual sub-carriers. The other ones are assembled as the columns of $\mathbf{W}^{\perp}$. The time-domain transmitted signal $\mathbf{x}_P(k) = \mathbf{W}_P\mathbf{s}_P(k)$ is accordingly defined, and a modeling of the transmit system is generated:

$$\mathbf{y}(k) = \mathbf{E}\mathbf{W}_P\mathbf{H}_P\mathbf{s}_P(k)e^{(j(k-1)\theta(N+N_{\mathrm{CP}}))}, \qquad (4)$$

where $\mathbf{H}_P$ contains the rows and columns of $\mathbf{H}_C$ corresponding to the indices of the non-virtual sub-carriers of $\mathbf{s}(k)$.

The performance of the estimation may be improved by using a Forward-Backward windowing [5]. Then, the following variables are defined:

$$\begin{aligned} \mathbf{y}_F^i(k) &= \begin{bmatrix} y_{i-1}(k) & y_{i-2}(k) & \ldots & y_{i+M-1}(k) \end{bmatrix}^T, && (5) \\ \mathbf{y}_B^i(k) &= \begin{bmatrix} y_{N-i}(k) & y_{N-i-1}(k) & \ldots & y_{N-i-M}(k) \end{bmatrix}^T && (6) \end{aligned}$$

where $M$ is the length of the window. Some other variables are defined based on $\mathbf{s}(k)$:

$$\begin{aligned} \tilde{\mathbf{s}}(k) &= \mathbf{H}_P\mathbf{s}(k)\exp{(j(k-1)\theta(N+N_{CP}))}, && (7) \\ \mathbf{r}(k) &= \exp{(-j\theta(N-1))}\mathrm{diag}\left(\left[1, \exp{(\frac{j2\pi}{N})}, \ldots, \right.\right. \\ & \left.\left. \exp{(\frac{j2\pi(P-1)(N-1)}{N})}\right]\right)\tilde{\mathbf{s}}^*(k). && (8) \end{aligned}$$

Then, the following relations are observed [5]:

$$\begin{aligned} \mathbf{y}_F^i(k) &= \mathbf{E}_{M+1}\mathbf{W}_{M+1}\mathbf{\Delta}^i\tilde{\mathbf{s}}(k), && (9) \\ \mathbf{y}_B^i(k) &= \mathbf{E}_{M+1}\mathbf{W}_{M+1}\mathbf{\Delta}^i\mathbf{r}(k), && (10) \end{aligned}$$

where $\mathbf{\Delta} = \mathrm{diag}(\exp{(j\theta)}, \exp{(j(\theta + 2\pi/N))}, \ldots, \exp{(j(\theta + 2(P-1)\pi/N))})$.

Consider now the correlation $\mathbf{R}$, such that,

$$\mathbf{R} = \frac{1}{K(N-M)} \sum_{i=1}^{N-M}\sum_{k=1}^{K}(\mathbf{y}_F^i(k)+\mathbf{y}_B^i(k))(\mathbf{y}_F^i(k)+\mathbf{y}_B^i(k))^H. \qquad (11)$$

The signal and noise subspaces of $\mathbf{R}$ are extracted by an eigendecomposition. As shown in [5], the noise subspace $\mathbf{U}_n$ of $\mathbf{R}$ is related to $\mathbf{W}^{\perp}$. It is shown in [5] that, in a scenario without CFO,

$$\mathbf{w}_i^H\mathbf{y}(k) = \mathbf{w}_i^H\mathbf{W}_P\mathbf{s}(k) = 0, \qquad i = 1, \ldots, N-P. \quad (12)$$

In the presence of CFO, Eq. (12) is transformed into

$$\mathbf{w}_i^H\mathbf{E}^{-1}\mathbf{y}(k) = 0, \qquad i = 1, \ldots, N-P. \qquad (13)$$

Then, the following cost function is defined [5]:

$$S(\phi) = \sum_{i=1}^{L} \sum_{k=1}^{K} \|\mathbf{w}_i^H \mathbf{Z}^{-1}(\phi)\mathbf{y}(k)\|^2, \tag{14}$$

where $\mathbf{Z}(\phi) = \text{diag}(1, \exp(j\phi), \dots, \exp(j(N-1)\phi))$, such that $S(\phi)$ is minimized when $\phi = \theta$. A grid search may be performed for finding the minimum of $S(\phi)$, which is the estimated $\hat{\theta}$.

### *C. CFO in Sub-Band CAS and Interleaved OFDMA*

In the Orthogonal Frequency-Division Multiple Access (OFDMA) [2], the available sub-carriers are distributed among the users according to a pre-defined algorithm. When a Sub-Band Carrier Assignment Scheme (CAS) is adopted, adjacent sub-carriers are attributed to each of the users. When an interleaved allocation for OFDMA is used [2], the sub-carriers are not distributed in blocks to the users. In general, there is a round-robin allocation.

Moose's algorithm is adapted in a straightforward manner to the Sub-band CAS OFDMA and the interleaved OFDMA environment. For the MUSIC algorithm, the unused sub-carriers of each user may be considered virtual.

## IV. Proposed Semi-Blind Two-Stage CFO Estimation

In [6], a two-stage algorithm was proposed, using combinations of ESPRIT (Estimation of Parameters via Rotational Invariance Techniques) and MUSIC. In this paper, we propose the scheme shown in Fig. 2.

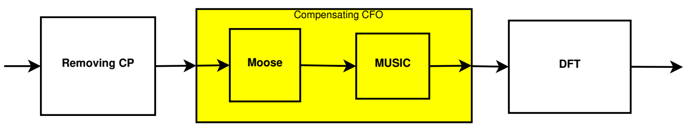


Fig. 2. Scheme of the proposed algorithm.

Since Moose's algorithm is trained, but requires few samples to provide an adequate estimation, its combination with MUSIC is considered a semi-blind two-stage algorithm. Moose provides a low-complexity estimation with few samples, but not very accurate. On the other hand, MUSIC may be used to refine the initial estimation of Moose. A summary of the proposed algorithm is presented in Table I.

## V. Computational Complexity

In this section, the computational complexity of the proposed algorithm is compared to some other algorithms from the literature [3], [6], [10]. The complexity of ESPRIT [10] and E-t-M (ESPRIT-then-MUSIC) are based on the analysis in [6].

In our proposed Moose-MUSIC, ESPRIT is replaced by Moose as the first stage of the algorithm. Since Moose's algorithm presents a much smaller computational complexity than ESPRIT, then our proposed Moose-MUSIC algorithm presents a smaller computational complexity than E-t-M.

TABLE I

Summary of the proposed Moose-MUSIC algorithm for OFDM.

OFDM/OFDMA:
Generate $\mathbf{s}(k), k = 0, 1, 2, \dots, N$,
Perform IDFT $\mathbf{x}(k) = \mathbf{W}^H \mathbf{s}(k)$,
Add CP to $\mathbf{x}(k)$,
Transmit $\mathbf{x}_{\text{CP}}(k)$ through the channel,
Receive $\mathbf{y}(k) = \mathbf{H}_c(k)\mathbf{x}(k) + \mathbf{n}(k)$.
OFDMA: All the sub-carriers belonging to other users are treated as virtual.
Moose (Training):
For each sub-carrier: $\hat{\theta}_{\text{T}} = \dfrac{1}{jN(N_T-1)} \sum \dfrac{\mathbf{y}_i(k)\mathbf{s}_i(k-1)}{\mathbf{y}_i(k-1)\mathbf{s}_i(k)}$.
MUSIC:
Assemble a dense grid centered at $\hat{\theta}_{\text{T}}$,
For each $\theta_i$ of the grid: $S(\phi_i) = \sum \|\mathbf{w}_i^H \mathbf{Z}^{-1}(\phi)\mathbf{y}(k)\|^2$,
Find $\hat{\phi} = \min S(\phi_i)$.

In Table II, there is a summary of the computational complexity of the algorithms, where $T \ll P$ denotes the number of training symbols. FLOPs (Floating Point Operations) are used as defined in [11]. Due to the short training stage, the proposed Moose-MUSIC presents a smaller computational complexity than the previous ESPRIT-MUSIC proposal. Moose's algorithm presents the lowest complexity among the compared algorithms, but it is a trained algorithm, where only trained OFDM are transmitted through the sub-carriers.

TABLE II

Summary of the computational complexity for the proposed and benchmark algorithms.

| **Algor.** | **Operation** | **Complexity** |
|---|---|---|
| ESPRIT | Subs. Separation | $29P^3 + (8/3)M^3 +$ $+15P^2M$ |
| E-t-M | Subs. Separation | $29P^3 + (8/3)M^3 +$ $+15P^2M + 4P^2 +$ $+2(M+1)^2$ |
| $(g < G)$ | Search: Multipl. | $gKP^3(N-P)$ |
| | Search: Addition | $gKP(P-1)(N-P)$ |
| | Search: Comp. | $4P^2 + 2(M+1)^2$ |
| Moose | Division | $KP$ |
| | Arc Tan [12] | $6KP$ |
| | Mean | $2KP$ |
| Proposed | Division | $TP$ |
| | Arc Tan [12] | $6TP$ |
| | Mean | $2TP$ |
| | Subs. Separation | $4P^2 + 2(M+1)^2$ |
| | Search: Multipl. | $G(K-T)P^3(N-P)$ |
| | Search: Addition | $G(K-T)P(P-1)(N-P)$ |
| | Search: Comp. | $4P^2 + 2(M+1)^2$ |

The proposed Moose-MUSIC presents an asymptotic computational complexity in the order of $\mathcal{O}(P^4)$, whereas Moose's algorithm has an asymptotic complexity of $\mathcal{KP}$. The ESPRIT algorithm has a computational complexity in the order of $\mathcal{O}(P^3 + M^3)$, while E-t-M presents a complexity of $\mathcal{O}(P^4)$. In our simulations, $P = 36$, $M = 35$ and $K$ varies with values 20 and 50. Then, for the used configuration, the asymptotic computational complexity of ESPRIT is in the order of $10^5$ FLOPs, the complexity of Moose is in the order of $10^3$ FLOPs, whereas E-t-M and the proposed Moose-MUSIC present computational complexity in the order of $10^6$ FLOPs.

## VI. Simulation Results

The performance of the proposed algorithms was evaluated by computer simulations. The figure of merit is the CFO normalized by inter-subcarrier spacing (NRMSE):

$$\mathrm{NRMSE} = \sqrt{\frac{1}{Q}\sum_{i=1}^{Q}\frac{(\hat{\theta}-\theta)^2}{2\pi/N}}, \tag{15}$$

where $Q$ is the number of Monte Carlo (MC) runs. The algorithms have been simulated for $Q = 200$ MC runs in the ensemble in a transmission through a random channel with 10 taps. The signal is transmitted using $N = 64$ subcarriers, where $N - P = 28$ of them are virtual. The Cramer-Rao lower bound (CRLB) for the CFO estimation scenario is plotted according to the Eq. (30) of [13].

In the first setup, we are addressing the well-known characteristics of good performance with few OFDM symbols of both Moose algorithm and of the semi-blind class of algorithms. In this scenario, 20 OFDM symbols are transmitted with 3 training symbols. In the proposed algorithm, the Moose algorithm works on the first two OFDM symbols. The results are shown in Fig. 3. As depicted in Fig. 3, the proposed algorithm presents the best performance in comparison to the benchmark algorithms when a small number of OFDM symbols are used.

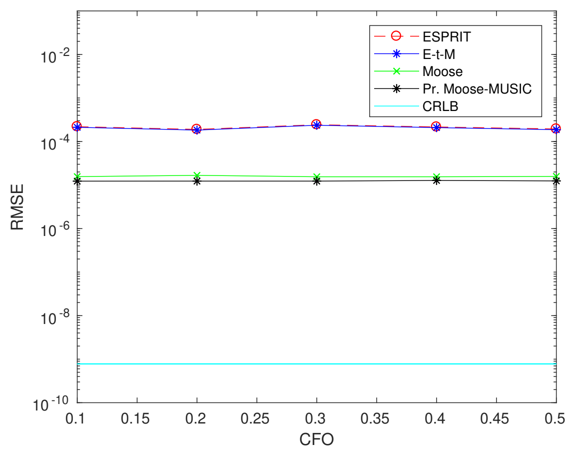


Fig. 3. Results for different values of CFO and 20 OFDM symbols.

In the second scenario, the number of OFDM symbols is increased to 50, while maintaining 3 training OFDM symbols. The results are shown in Fig. 4. The proposed algorithm presents a slightly better perfor- mance than the benchmark algorithms when 50 symbols are used for the whole range of CFOs.

The sub-band CAS OFDMA scenario was also simulated, where the target user transmits in 12 sub-carriers. The CFO detection is assumed to be performed separately for each user, then the other users' sub-carriers are treated as virtual sub-carriers. Fig. 5 shows the results for this scenario when 20 symbols are transmitted. Note that the performance advantage of Moose-MUSIC reduces for a short transmitted sequence, due to the use of few sub-carriers per user.

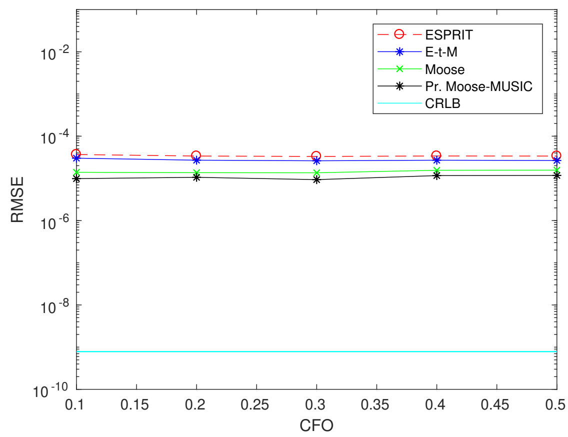


Fig. 4. Results for different values of CFO and 50 OFDM symbols.

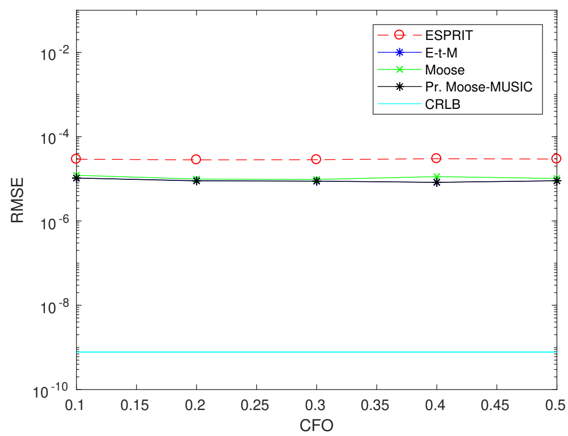


Fig. 5. Results for sub-band CAS OFDMA scenario with 20 OFDM symbols.

A simulation was performed with a noisy environment, with a signal-to-noise ratio (SNR) of 30 dB, and 20 OFDM blocks, with 3 training bits. Results are shown in Fig. 6. The performance of Moose-based algorithm are affected by the presence of noise. Nevertheless, Moose-MUSIC still presents a superior performance in comparison to the benchmark algorithms.

Interleaved OFDMA was simulated, with the results presented in Fig. 7. ESPRIT was slightly changed in order to use non-continguous sub-carriers, similarly to [14]. RMSE for all the algorithms present similar results. Table III shows the measured running time for different parts of the algorithm. It is noticeable that Moose-MUSIC presents a shorter running time than M-t-M and E-t-M.

TABLE III

Measured running time per MC run for parts of algorithms.

| Part | Running Time (s) |
|---|---|
| Subspace Separation | 0.0115 |
| ESPRIT | 0.00768 |
| MUSIC | 19.5 |
| Moose | 0.000416 |

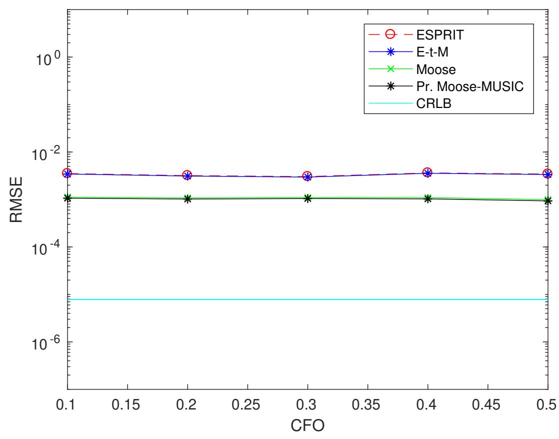


Fig. 6. Results for the sub-band CAS OFDMA scenario with 20 transmitted OFDM symbols and SNR = 30 dB.

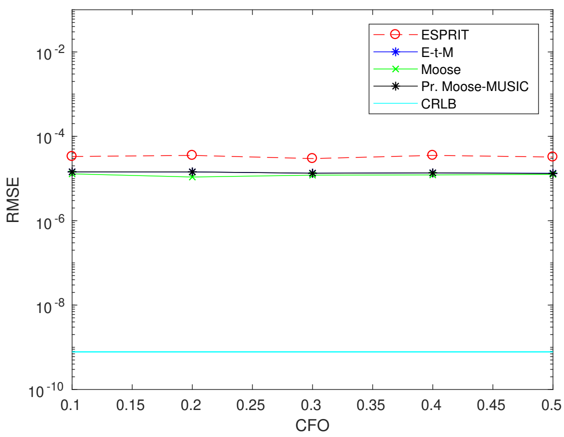


Fig. 7. Results for the interleaved OFDMA scenario with 20 OFDM symbols.

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## VII. Conclusion

This article presents the semi-blind Moose-MUSIC algorithm for CFO estimation in OFDM and OFDMA systems. Proposed Moose-MUSIC shows a superior performance in terms of NRMSE when a small number of samples is used, without increasing the computational complexity when compared to the blind counterpart MUSIC. Moose-MUSIC still presents an advantage over the benchmark algorithms when a larger number of symbols are transmitted, as well as in an OFDMA scenarios of transmission. Moreover, Moose-MUSIC needs a shorter running time than both E-t-M and M-t-M.

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