The General Solution for Crossing Rates and Fade Durations of Selection Combining

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***Abstract*— General and compact expressions for level crossing rate and average fade duration of selection combining are derived. The expressions can be directly applied to any multi-branch, correlated, non-identical, and unbalanced fading environment for which the joint statistics of the envelopes and each of their time derivatives are known. The general solutions specialize to simpler calculations for some particular cases. In the derivation process, novel, generalized, joint crossing rate concepts, consistent with the statistical theory, are introduced. Simple bounds for the quantities investigated are also attained, and they are written in terms of the individual branch measures.**

***Keywords*— Arbitrary fading channels, average fade duration, level crossing rate, selection combining.**

I. Introduction

DIVERSITY-COMBINING techniques are widely known to improve the performance of wireless communication systems. Among these techniques, selection combining (SC) is particularly attractive for its implementation simplicity. The performance of combining algorithms can be evaluated by several means including the level crossing rate (LCR) and average fade duration (AFD) statistics [1], [2], [3], [4]. In [1], exact closed-form expressions for the LCR and AFD of SC over an arbitrary number of independent, identically distributed (iid) Nakagami-*m* branches were derived. These results have been extended to independent, but non-identical Nakagami-*m* channels in [2], [3]. The Rice fading condition was also investigated in [3]. However, LCR and AFD expressions for SC in a correlated fading environment are known only for dual-branch diversity over Rayleigh channels having the same mean power [4]. A recent work [5] announced a general approach for evaluating the impact of fading correlation on the LCR and AFD of SC, but the assumptions in it, in fact, lead to a special case of the general result obtained here (see section IV-A). In this letter, general and compact expressions for LCR and AFD of SC are derived. The expressions can be directly applied to any multi-branch, correlated, non-identical, and unbalanced fading environment for which the joint statistics of the envelopes and each of their time derivatives are known. Novel, generalized, joint crossing rate concepts, consistent with the statistical theory, are also introduced. Simple bounds for the output LCR and AFD are attained, and they are written in terms of the individual branch LCRs and AFDs.

II. System Model and Preliminaries

Let $R_i(t)$ be the signal envelope at the *i*th branch, $i = 1, 2, \ldots, M$. At any instant t the SC scheme picks the branch with the best signal, so that the combiner output envelope $R(t)$ can be written as[1]

$$R(t) = \max_{i\in\{1,\ldots,M\}} R_i(t) \tag{1}$$

(For the sake of clarity, from this point on the time variable shall be omitted.)

The LCR of a random signal is defined as the average number of upward (or downward) crossings per second at a given level. The LCR $n_R(r)$ of the combiner output envelope R at level r is given by

$$n_R(r) = \int_0^\infty \dot{r} p_{R,\dot{R}}(r,\dot{r}) d\dot{r} \tag{2}$$

where $p_{R,\dot{R}}(\cdot,\cdot)$ is the joint probability density function (JPDF) of R and its time derivative $\dot{R}$. The AFD is defined as the mean time a random signal remains below a given level after crossing it in the downward direction. In this case, the AFD $T_R(r)$ of R at level r is given by

$$T_R(r) = \frac{P_R(r)}{n_R(r)} \tag{3}$$

where $P_R(\cdot)$ is the cumulative distribution function (CDF) of R.

III. The General Solution

From (1), the JPDF of R and $\dot{R}$ can be expressed as follows

$$\begin{aligned} p_{R,\dot{R}}(r,\dot{r}) &= \sum_{i=1}^{M} p_{R_i,\dot{R}_i}(r,\dot{r}) \Pr\left[R_i - \text{chosen} | R_i = r, \dot{R}_i = \dot{r}\right] \\ &= \sum_{i=1}^{M} p_{R_i,\dot{R}_i}(r,\dot{r}) \Pr\left[R_j \le r, j \ne i | R_i = r, \dot{R}_i = \dot{r}\right] \\ &= \sum_{i=1}^{M} p_{R_i,\dot{R}_i}(r,\dot{r}) P_{\mathbf{R}_i|R_i,\dot{R}_i}(\mathbf{r}|r,\dot{r}) \quad (4) \\ &= \sum_{i=1}^{M} \int_0^r \!\!\cdots\! \int_0^r p_{R_i,\dot{R}_i,\mathbf{R}_i}(r,\dot{r},\mathbf{r}_i) \mathbf{dr}_i \quad (5) \end{aligned}$$

where $\mathbf{R}_i$, $\mathbf{r}$, $\mathbf{r_i}$, $\mathbf{dr_i}$, $p_{R_i,\dot{R}_i}(\cdot,\cdot)$, $P_{\mathbf{R}_i|R_i,\dot{R}_i}(\cdots|\cdot,\cdot)$, and $p_{R_i,\dot{R}_i,\mathbf{R}_i}(\cdot,\cdot,\cdots)$ denote, respectively, the sets $\{R_j\}_{j=1,j\ne i}^{M}$, $\{r\}_{j=1}^{M-1}$, $\{r_j\}_{j=1,j\ne i}^{M}$, and $\{dr_j\}_{j=1,j\ne i}^{M}$, the JPDF of R_i and its time derivative $\dot{R}_i$, the conditional joint CDF (CJCDF) of $\mathbf{R}_i$ given R_i and $\dot{R}_i$, and the JPDF of R_i, $\dot{R}_i$, and $\mathbf{R}_i$. In order to obtain (5) from (4), the CJCDF in (4) has been expressed as the integral of the corresponding conditional JPDF (CJPDF) from which the Bayes' rule has been applied.

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[1]The branches are assumed to have identical noise power.

Inserting (5) into (2) and rearranging the order of the operations, we obtain the general, compact expression for the LCR of SC as

$$n_R(r) = \sum_{i=1}^{M} N_{R_i,\mathbf{R}_i}(r,\mathbf{r}) \tag{6}$$

where

$$N_{R_i,\mathbf{R}_i}(r,\mathbf{r}) \triangleq \int_0^r \cdots \int_0^r n_{R_i,\mathbf{R}_i}(r,\mathbf{r}_i)\, \mathbf{dr}_i \tag{7}$$

can be understood as a LCR joint cumulative function (LCRJCF) of R_i and $\mathbf{R}_i$; and

$$n_{R_i,\mathbf{R}_i}(r,\mathbf{r}_i) \triangleq \int_0^\infty \dot{r} p_{R_i,\dot{R}_i,\mathbf{R}_i}(r,\dot{r},\mathbf{r}_i)\, d\dot{r} \tag{8}$$

as the corresponding LCR joint density function (LCRJDF), so that the marginal LCR $n_{R_i}(\cdot)$ (LCRM) of R_i, i.e. each individual branch LCR, can be calculated by

$$\begin{aligned} n_{R_i}(r) &= \int_0^\infty \cdots \int_0^\infty n_{R_i,\mathbf{R}_i}(r,\mathbf{r}_i)\, \mathbf{dr}_i \\ &= N_{R_i,\mathbf{R}_i}(r,\boldsymbol{\infty}) \end{aligned} \tag{9}$$

where $\boldsymbol{\infty}$ denotes the set $\{\infty\}_{j=1}^{M-1}$. The LCRJDF and the LCRJCF constitute a generalization of the usual definition for LCR as given in (2). The new concepts introduced, namely LCRJDF, LCRJCF, and LCRM are consistent with the well-established statistical theory.

The CDF of SC is known to be given by

$$P_R(r) = P_{R_1,\ldots,R_M}(r,\ldots,r) \tag{10}$$

where $R_1,\ldots,R_M$ and $P_{R_1,\ldots,R_M}(\cdot,\ldots,\cdot)$ denote the branch envelopes and their joint CDF (JCDF), respectively. From (3), (6), and (10) the general, compact expression for the AFD of SC can be written as

$$T_R(r) = \frac{P_{R_1,\ldots,R_M}(r,\ldots,r)}{\sum_{i=1}^{M} N_{R_i,\mathbf{R}_i}(r,\mathbf{r})} \tag{11}$$

Note, from (6) and (11), that the general expressions for the LCR and AFD presented here depend on the joint statistics of $R_1,\ldots,R_M$ and $\dot{R}_i$, $i=1,2,\ldots,M$. These M joint statistics are the only input information required for evaluating the LCR and AFD of SC over generalized fading channels.

IV. Special Cases

In this section, (6) and (11) are specialized to some particular cases for which simplifications are accomplished.

A. *Second-Order Independence (SOI)*

The SOI assumption concerns the case in which each envelope time derivative $\dot{R}_i$, $i=1,2,\ldots,M$ is independent of all of the remaining fading envelopes $R_j, j \neq i$. In this case, (8), (6), and (11) specialize to

$$n_{R_i,\mathbf{R}_i}(r,\mathbf{r}_i) = p_{\mathbf{R}_i|R_i}(\mathbf{r}_i|r)\, n_{R_i}(r) \tag{12}$$

$$n_R(r) = \sum_{i=1}^{M} P_{\mathbf{R}_i|R_i}(\mathbf{r}|r) n_{R_i}(r) \tag{13}$$

$$T_R(r) = \frac{P_{R_1,\ldots,R_M}(r,\ldots,r)}{\sum_{i=1}^{M} P_{\mathbf{R}_i|R_i}(\mathbf{r}|r) n_{R_i}(r)} \tag{14}$$

where $p_{\mathbf{R}_i|R_i}(\cdots|\cdot)$ and $P_{\mathbf{R}_i|R_i}(\cdots|\cdot)$ are the CJPDF and the CJCDF of $\mathbf{R}_i$ given R_i, respectively. Although for some cases such as Rayleigh, Rician, and Nakagami-*m* R_i and $\dot{R}_i$ are independent random variables, dependence between $\dot{R}_i$ and R_j, $j \neq i$ may occur in a correlated environment as demonstrated in [4, Eqn. (21)] for the bivariate Rayleigh fading. Therefore, the case explored in [5], although claimed to be the general case, is, in fact, the special SOI case.

B. *First-Order Independence (FOI)*

The FOI assumption concerns the case in which the envelopes are independent random variables. This is a subset of the previous case. In this scenario, further simplifications can be attained such that

$$n_{R_i,\mathbf{R}_i}(r,\mathbf{r}_i) = n_{R_i}(r) \prod_{\substack{j=1 \\ j\neq i}}^{M} p_{R_j}(r_j) \tag{15}$$

$$n_R(r) = \sum_{i=1}^{M} n_{R_i}(r) \prod_{\substack{j=1 \\ j\neq i}}^{M} P_{R_j}(r) \tag{16}$$

$$T_R^{-1}(r) = \sum_{i=1}^{M} T_{R_i}^{-1}(r) \tag{17}$$

where $P_{R_i}(\cdot)$ and $T_{R_i}(\cdot)$ correspond to the CDF and the AFD of R_i, respectively. Results (16) and (17) are particularly interesting for they express the LCR and AFD of the combiner output as functions of the LCRs and AFDs of the individual input branches. The result in (16) coincides with [6, eq. (2)], and if applied to Nakagami-*m* or Rice fading branches — whose individual PDFs, LCRs, and AFDs are known — directly leads to the LCR expressions presented in [2], [3]. On the other hand, although [2], [3] have also obtained AFD expressions of SC in the aforementioned fading conditions, their results are for particular fading environments. The result of (17) is, in fact, the general solution, and to the best of the authors' knowledge, this result is new.

C. *Commutative Joint Statistics (CJS)*

The CJS assumption concerns the case in which any joint statistical function involving a subset of $\{R_i,\dot{R}_i\}$ and a subset of $\{R_j,\dot{R}_j\}$, $j \neq i$, does not alter if the subscripts are interchanged. This is the case of identically distributed, identically correlated fading processes. Note that the required interchangeability comprises not only the fading envelopes themselves but also their time derivatives, since the latter is an integral part of the general formulae (6) and (11). In this case, these formulae specialize to

$$n_R(r) = M N_{R_i,\mathbf{R}_i}(r,\mathbf{r}) \tag{18}$$

$$T_R(r) = \frac{P_{R_1,\ldots,R_M}(r,\ldots,r)}{M N_{R_i,\mathbf{R}_i}(r,\mathbf{r})} \tag{19}$$

for any $i \in \{1,2,\ldots,M\}$. In [4, eq. (8)], the LCR of SC for the identical dual-branch Rayleigh fading scenario is provided as the sum of two double integrals. Our result (18) states that only one of these integrals must be actually evaluated, because they are identical.

D. Independent, Identically Distributed (IID)

The IID assumption comprises the intersection of the cases in sections IV-B and IV-C, leading to the well-known simple solution

$$n_R(r) = M n_{R_i}(r) P_{R_i}^{M-1}(r) \tag{20}$$

$$T_R(r) = \frac{T_{R_i}(r)}{M} \tag{21}$$

for any $i \in \{1, 2, \ldots, M\}$. Note clearly that the output AFD is reduced by a factor of M if compared to the AFDs of the input branches.

V. Performance Bounds

Since the event $\{R_i = r, \dot{R}_i = \dot{r}\}$ is a subset of the event $\{R_i \leq r\}$, it follows that

$$P_{\mathbf{R}_i|R_i}(\mathbf{r}|r) \leq F_{\mathbf{R}_i|R_i}(\mathbf{r}|r) \tag{22}$$

where $F_{\mathbf{R}_i|R_i}(\cdots|r_i)$ is the CJCDF of $\mathbf{R_i}$ given that $R_i \leq r_i$. Therefore, from (4) and (22)

$$p_{R,\dot{R}}(r,\dot{r}) \leq \sum_{i=1}^{M} p_{R_i,\dot{R}_i}(r,\dot{r}) F_{\mathbf{R}_i|R_i}(\mathbf{r}|r) \tag{23}$$

Substituting the right-hand side of (23) into (2), we obtain

$$n_R(r) \leq \sum_{i=1}^{M} F_{\mathbf{R_i}|R_i}(\mathbf{r}|r) n_{R_i}(r) \tag{24}$$

as the upper bound for the LCR of R. Noting that $F_{\mathbf{R}_i|R_i} \leq 1$, a less tight bound can be written as

$$n_R(r) \leq \sum_{i=1}^{M} n_{R_i}(r) \tag{25}$$

Combining (24) with (3) and (10), the lower bound for the AFD of R is found as

$$\begin{aligned} T_R^{-1}(r) &\leq \frac{\sum_{i=1}^{M} F_{\mathbf{R_i}|R_i}(\mathbf{r}|r) n_{R_i}(r)}{P_{R_1,\ldots,R_M}(r,\ldots,r)} \\ &\leq \sum_{i=1}^{M} \frac{P_{R_1,\ldots,R_M}(r,\ldots,r)\, n_{R_i}(r)}{P_{R_i}(r) P_{R_1,\ldots,R_M}(r,\ldots,r)} \\ &\leq \sum_{i=1}^{M} \frac{n_{R_i}(r)}{P_{R_i}(r)} \\ &\leq \sum_{i=1}^{M} T_{R_i}^{-1}(r) \end{aligned} \tag{26}$$

The equality in (26) corresponds to (17), i.e., it is attained in case the branches are independent from each other. Furthermore, note that the lower bound of the AFD for SC is a function of the individual branch AFDs only.

An upper bound can also be found for the output AFD. This is obtained for the case in which the branch envelopes are weighted versions of a same signal (fully correlated branches). In this case, the branch with the highest root mean square value — thus with the smallest AFD — is always selected as the combiner output, i.e.

$$T_R^{-1}(r) \geq \max_{i \in \{1,2,\ldots,M\}} T_{R_i}^{-1}(r) \tag{27}$$

Combining (26) with (27), the absolute bounds of the output AFD for SC are obtained as

$$\max_{i \in \{1,2,\ldots,M\}} T_{R_i}^{-1}(r) \leq T_R^{-1}(r) \leq \sum_{i=1}^{M} T_{R_i}^{-1}(r) \tag{28}$$

VI. Conclusions

All the results presented in this letter arises from the formulation of the output JPDF of R and $\dot{R}$ as a weighted sum of the input JPDFs of R_i and $\dot{R}_i$, $i = 1, 2, \ldots, M$. This paradigm leads to a general unified treatment for calculating the LCR and AFD of multi-branch SC over arbitrarily correlated, non-identical fading channels. For some particular cases, the general solutions are specialized to simpler expressions. Simple and straightforward bounds of the output LCR and AFD are derived in terms of the individual branch LCRs and AFDs. A well-accepted result which can be directly verified from the formulations is that the best performance is achieved with independent diversity branches, deteriorating as the correlation among them increases.

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