Second-Order Statistics of Maximal-Ratio and Equal-Gain Combining in Hoyt Fading

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***Abstract*— Exact expressions for the level crossing rate and average fade duration of M-branch equal-gain and maximal-ratio combining systems in a Hoyt fading environment are presented. The expressions apply to unbalanced, independent diversity channels and have been validated by specializing the general results to some particular cases whose solutions are known and, more generally, by means of simulation.**

***Index Terms*— Average fade duration, equal-gain combining, Hoyt fading channels, level crossing rate, maximal-ratio combining.**

I. INTRODUCTION

THE performance of wireless communication systems is considerably affected by the multipath propagation phenomena. Diversity-combining techniques are effective means used for mitigating the deleterious effects of fading. The rate of occurrence of fades, or level crossing rate (LCR), and the average fade duration (AFD) provide a dynamic characterization of the communication channel. As second-order statistical quantities, they complement the static probabilistic description of the fading signal (the first-order statistics), and have found several applications in the modelling and design of practical systems.

LCR and AFD expressions of a single channel have been derived for Rayleigh [1], Rice [2], Nakagami-*m* [3] and, more recently, Hoyt [4] fading environments. Several works have addressed the second-order statistics of diversity-combining systems, including the following. In [5], LCR and AFD expressions of selection combining (SC), maximal-ratio combining (MRC), and equal-gain combining (EGC) for dual-branch diversity in correlated Rayleigh channels were presented. The case of M independent identically distributed (iid) Nakagami-*m* channels was solved in [6] for SC, MRC, and EGC. Some results involving the independent but non-identical Nakagami-*m* diversity case were presented in [7].

The Hoyt (Nakagami-q) distribution spans the range of the fading figure from the one-sided Gaussian to the Rayleigh distributions, and has found applications in mobile satellite propagation channels [4]. Despite its practical interest very little attention has been paid to this type of fading. This paper provides *exact* LCR and AFD expressions for MRC and EGC in Hoyt channels. The formulas apply to M unbalanced, independent branches and have been validated by specializing the general results to some particular cases whose solutions are known and, more generally, by means of simulation.

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II. BRANCH STATISTICS

The Hoyt fading envelope R_i and phase Θ_i at the *i*th branch, $i = 1, \ldots, M$, is modelled as

$$R_i = \sqrt{X_i^2 + Y_i^2} \quad (1)$$

$$\Theta_i = \arctan \frac{Y_i}{X_i} \quad (2)$$

where X_i and Y_i are independent zero-mean Gaussian random variables (RVs) with variances $\sigma_{X_i}^2$ and $\sigma_{Y_i}^2$, respectively. The joint probability density function (JPDF) $p_{R_i,\Theta_i}(\cdot,\cdot)$ of R_i and Θ_i is given by

$$p_{R_i,\Theta_i}(r_i,\theta_i) = \frac{r_i}{\Omega_i \pi \sqrt{1-b_i^2}} \exp\left(-\frac{1-b_i\cos 2\theta_i}{\Omega_i(1-b_i^2)} r_i^2\right) \quad (3)$$

where $\Omega_i = \mathrm{E}[R_i^2]$ and $b_i \triangleq (\sigma_{X_i}^2 - \sigma_{Y_i}^2)/(\sigma_{X_i}^2 + \sigma_{Y_i}^2)$, $-1 \leq b_i \leq 1$ is the Hoyt fading parameter. The probability density function (PDF) $p_{R_i}(\cdot)$ of R_i is obtained as

$$p_{R_i}(r_i) = \frac{2r_i}{\Omega_i\sqrt{1-b_i^2}} \exp\left(-\frac{r_i^2}{\Omega_i(1-b_i^2)}\right) I_0\left(\frac{b_i r_i^2}{\Omega_i(1-b_i^2)}\right) \quad (4)$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind and zeroth order.

In order to derive the LCR and AFD of EGC and MRC, we shall make use of the conditional PDF (CPDF) $p_{\dot{R}_i|R_i,\Theta_i}(\cdot|\cdot,\cdot)$ of the envelope time derivative $\dot{R}_i$ at the *i*th branch given R_i and Θ_i. From (1), $R_i\dot{R}_i = X_i\dot{X}_i + Y_i\dot{Y}_i$, where $\dot{X}_i$ and $\dot{Y}_i$ denote, respectively, the time derivatives of X_i and Y_i. Knowing that $X_i = R_i \cos\Theta_i$ and $Y_i = R_i \sin\Theta_i$, then

$$\dot{R}_i = \dot{X}_i \cos\Theta_i + \dot{Y}_i \sin\Theta_i \quad (5)$$

For isotropic scattering, $\dot{X}_i$ and $\dot{Y}_i$ are known to be zero-mean Gaussian RVs with variances $\dot{\sigma}_{X_i}^2 = (\sqrt{2}\pi f_m)^2 \sigma_{X_i}^2$ and $\dot{\sigma}_{Y_i}^2 = (\sqrt{2}\pi f_m)^2 \sigma_{Y_i}^2$, respectively, where f_m is the maximum Doppler shift in Hertz [1]. Thus, from (5), $\dot{R}_i$, given R_i and Θ_i, is also zero-mean Gaussian distributed, so that

$$p_{\dot{R}_i|R_i,\Theta_i}(\dot{r}_i|r_i,\theta_i) = \frac{1}{\sqrt{2\pi}\dot{\sigma}_{R_i}} \exp\left(-\frac{1}{2}\left(\frac{\dot{r}_i}{\dot{\sigma}_{R_i}}\right)^2\right) \quad (6)$$

with $\dot{\sigma}_{R_i}^2 = \Omega_i(\pi f_m)^2(1 + b_i\cos 2\theta_i)$.

The LCR and AFD of a random signal are defined, respectively, as the average number of upward (or downward) crossings per second at a given level and as the mean time the signal remains below this level after crossing it in the downward direction. The LCR $n_R(r)$ and AFD $T_R(r)$ of the combiner output R at level r are, respectively, given by [2]

$$n_R(r) = \int_0^\infty \dot{r} p_{R,\dot{R}}(r,\dot{r}) d\dot{r} \quad (7)$$

$$T_R(r) = \frac{P_R(r)}{n_R(r)} \quad (8)$$

where $p_{R,\dot{R}}(\cdot,\cdot)$ is the JPDF of R and its time derivative $\dot{R}$, and $P_R(\cdot)$ is the cumulative distribution function (CDF) of R. In the following, (7) and (8) shall be calculated for M-branch EGC and MRC in a Hoyt fading environment.

III. Equal-Gain Combining

In EGC, the received signals are cophased and added so that the combiner output envelope R, already taking into account the resultant output noise power, is written as $R = \frac{1}{\sqrt{M}} \sum_{i=1}^M R_i$. Thus

$$\dot{R} = \frac{1}{\sqrt{M}} \sum_{i=1}^M \dot{R}_i \quad (9)$$

The CDF of R can be calculated as the integration of the JPDF of R_i, $i = 1, \ldots, M$, over the M-dimensional volume bounded by the hyperplane $\sqrt{M} r = \sum_{i=1}^M r_i$ and the coordinate hyperplanes [8]

$$P_R(r) = \int_0^{\sqrt{M}r} \int_0^{\sqrt{M}r - r_M} \cdots \int_0^{\sqrt{M}r - \sum_{i=3}^M r_i} \int_0^{\sqrt{M}r - \sum_{i=2}^M r_i} \times p_{R_1,\ldots,R_M}(r_1,\ldots,r_M) dr_1 dr_2 \cdots dr_{M-1} dr_M \quad (10)$$

where $p_{R_1,\ldots,R_M}(r_1,\ldots,r_M) = \prod_{i=1}^M p_{R_i}(r_i)$ is the JPDF of R_i, $i = 1, \ldots, M$, since the branches are independent, and $p_{R_i}(\cdot)$ is given by (4).

Note, from (6) and (9), that $\dot{R}$, given R_i and Θ_i, $i = 1, \ldots, M$, is a zero-mean Gaussian variate with CPDF

$$p_{\dot{R}|R_1,\ldots,R_M,\Theta_1,\ldots,\Theta_M}(\dot{r}|r_1,\ldots,r_M,\theta_1,\ldots,\theta_M) = \frac{1}{\sqrt{2\pi}\dot{\sigma}_R} \exp\left(-\frac{1}{2}\left(\frac{\dot{r}}{\dot{\sigma}_R}\right)^2\right) \quad (11)$$

and variance $\dot{\sigma}_R^2 = \frac{(\pi f_m)^2}{M} \sum_{i=1}^M \Omega_i (1 + b_i \cos 2\theta_i)$. Next, we shall exploit this fact by including the variates Θ_is in the formulation of $p_{R,\dot{R}}(\cdot,\cdot)$. As shall be seen, this will greatly simplify the calculations. Derivating (10) with respect to r to obtain $p_R(r)$ as in [6] and then using the Bayes' rule, $p_{R,\dot{R}}(\cdot,\cdot)$ can be found as (12), where $p_{R_1,\ldots,R_M,\Theta_1,\ldots,\Theta_M,\dot{R}}(\cdot,\ldots,\cdot,\cdot,\ldots,\cdot,\cdot)$ is the JPDF of R_i, Θ_i, $i = 1, \ldots, M$, and $\dot{R}$. Of course,

$$p_{R_1,\ldots,R_M,\Theta_1,\ldots,\Theta_M,\dot{R}}(r_1,\ldots,r_M,\theta_1,\ldots,\theta_M,\dot{r}) = p_{\dot{R}|R_1,\ldots,R_M,\Theta_1,\ldots,\Theta_M}(\dot{r}|r_1,\ldots,r_M,\theta_1,\ldots,\theta_M) \times p_{R_1,\ldots,R_M,\Theta_1,\ldots,\Theta_M}(r_1,\ldots,r_M,\theta_1,\ldots,\theta_M) \quad (13)$$

where $p_{\dot{R}|R_1,\ldots,R_M,\Theta_1,\ldots,\Theta_M}(\cdot|\cdot,\ldots,\cdot,\cdot,\ldots,\cdot)$ is given by (11) and $p_{R_1,\ldots,R_M,\Theta_1,\ldots,\Theta_M}(r_1,\ldots,r_M,\theta_1,\ldots,\theta_M) = \prod_{i=1}^M p_{R_i,\Theta_i}(r_i,\theta_i)$, since the branches are independent. Using (13) into (12) and (7), the output LCR of an M-branch EGC system in a Hoyt fading environment can be finally written as (14), where $p_{R_i,\Theta_i}(\cdot,\cdot)$ is given by (3). (Observe that the inclusion of Θ_is in the formulations led to a closed-form integration over $\dot{r}$ in (7).) From (8), (10), and (14), the output AFD of EGC in a multi-branch Hoyt fading is obtained.

IV. Maximal-Ratio Combining

In MRC, the received signals are cophased, each signal is amplified appropriately for an optimal combining, and the resultant signals are added so that the combiner output envelope R is given by $R = \sqrt{\sum_{i=1}^M R_i^2}$. Thus

$$\dot{R} = \sum_{i=1}^M \frac{R_i}{R} \dot{R}_i \quad (15)$$

The MRC analysis follows exactly the same steps detailed for EGC in the previous section, considering that, for MRC, the hyperplane used to compute $P_R(\cdot)$ is $r^2 = \sum_{i=1}^M r_i^2$ and that $\dot{\sigma}_R^2 = (\pi f_m)^2 (\sum_{i=1}^M R_i^2)^{-1} \sum_{i=1}^M \Omega_i R_i^2 (1 + b_i \cos 2\theta_i)$. The resulting $P_R(\cdot)$, $p_{R,\dot{R}}(\cdot,\cdot)$ and $n_R(\cdot)$ are given by (16), (17), and (18), respectively.

$$P_R(r) = \int_0^r \int_0^{\sqrt{r^2 - r_M^2}} \cdots \int_0^{\sqrt{r^2 - \sum_{i=3}^M r_i^2}} \int_0^{\sqrt{r^2 - \sum_{i=2}^M r_i^2}} \times p_{R_1,\ldots,R_M}(r_1,\ldots,r_M) dr_1 dr_2 \ldots dr_{M-1} dr_M \quad (16)$$

As before, $p_{R_1,\ldots,R_M}(r_1,\ldots,r_M) = \prod_{i=1}^M p_{R_i}(r_i)$, since the branches are independent. From (8), (16), and (18), the output AFD of MRC in multi-branch Hoyt fading is obtained.

V. Results

The formulations obtained in this paper can be specialized into those already found in the literature. In particular, for balanced diversity channels and $b_i = 0$, $i = 1, \ldots, M$, they reduce to the M-branch EGC and MRC of the iid Rayleigh case, given by [6, Eqs. 23 and 24] for $m = 1$ and [6, Eqs. 38 and 39], respectively. In the same way, for balanced channels and $b_i \to \pm 1$, $i = 1, \ldots, M$, they reduce to the iid one-sided Gaussian case, given by [6, Eqs. 23, 24, 36 and 37] with $m = 0.5$. For the more general cases, including identical and non-identical fading branches, *exhaustive* simulations have been carried out and compared with the analytical expressions obtained here. All the cases investigated revealed an excellent agreement between analytical and simulation results. Figs. 1 and 2 show the LCR and the AFD of EGC and MRC, respectively, for $M = 1, 2, 4$ and $b_i = 0, 0.9, 0.999$, considering identical Hoyt-fading channels. For the sake of clarity, the simulation data have been omitted in the figures. In fact, they are practically coincident with the theoretical curves.

VI. Conclusions

Exact formulas for level crossing rate and average fade duration of the M-branch EGC and MRC techniques in a Hoyt fading environment were presented. These formulas have been validated by specializing the general results to some particular cases whose solutions are known and, more generally, by means of simulation.

$$p_{R,\dot{R}}(r,\dot{r}) = \sqrt{M}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_0^{\sqrt{M}r}\int_0^{\sqrt{M}r-r_M}\cdots\int_0^{\sqrt{M}r-\sum_{i=3}^{M}r_i} \times p_{R_1,R_2,\ldots,R_M,\Theta_1,\ldots,\Theta_M,\dot{R}}\left(\left(\sqrt{M}r-\sum_{i=2}^{M}r_i\right),r_2,\ldots,r_M,\theta_1,\ldots,\theta_M,\dot{r}\right)dr_2\cdots dr_{M-1}dr_M d\theta_1\cdots d\theta_M \quad (12)$$

$$n_R(r) = \sqrt{\pi/2}f_m\int_0^{2\pi}\cdots\int_0^{2\pi}\int_0^{\sqrt{M}r}\int_0^{\sqrt{M}r-r_M}\cdots\int_0^{\sqrt{M}r-\sum_{i=3}^{M}r_i}\sqrt{\sum_{i=1}^{M}\Omega_i(1+b_i\cos 2\theta_i)} \times p_{R_1,\Theta_1}\left(\left(\sqrt{M}r-\sum_{i=2}^{M}r_i\right),\theta_1\right)\prod_{i=2}^{M}p_{R_i,\Theta_i}(r_i,\theta_i)dr_2\cdots dr_{M-1}dr_M d\theta 1\cdots d\theta_M \quad (14)$$

$$p_{R,\dot{R}}(r,\dot{r}) = \int_0^{2\pi}\cdots\int_0^{2\pi}\int_0^{r}\int_0^{\sqrt{r^2-r_M^2}}\cdots\int_0^{\sqrt{r^2-\sum_{i=2}^{M}r_i^2}} \times\frac{r}{\sqrt{r^2-\sum_{i=2}^{M}r_i^2}}p_{R_1,R_2,\ldots,R_M,\Theta_1,\ldots,\Theta_M,\dot{R}}\left(\left(\sqrt{r^2-\sum_{i=2}^{M}r_i^2}\right),r_2,\ldots,r_M,\theta_1,\ldots,\theta_M,\dot{r}\right)dr_2\cdots dr_{M-1}dr_M d\theta 1\cdots d\theta_M \quad (17)$$

$$n_R(r) = \sqrt{\pi/2}f_m\int_0^{2\pi}\cdots\int_0^{2\pi}\int_0^{r}\int_0^{\sqrt{r^2-r_M^2}}\cdots\int_0^{\sqrt{r^2-\sum_{i=3}^{M}r_i^2}}\sqrt{\frac{\Omega_1\left(r^2-\sum_{i=2}^{M}r_i^2\right)(1+b_1\cos 2\theta_1)+\sum_{i=2}^{M}\Omega_i r_i^2(1+b_i\cos 2\theta_i)}{r^2-\sum_{i=2}^{M}r_i^2}} \times p_{R_1,\Theta_1}\left(\left(\sqrt{r^2-\sum_{i=2}^{M}r_i^2}\right),\theta_1\right)\prod_{i=2}^{M}p_{R_i,\Theta_i}(r_i,\theta_i)dr_2\cdots dr_{M-1}dr_M d\theta 1\cdots d\theta_M \quad (18)$$

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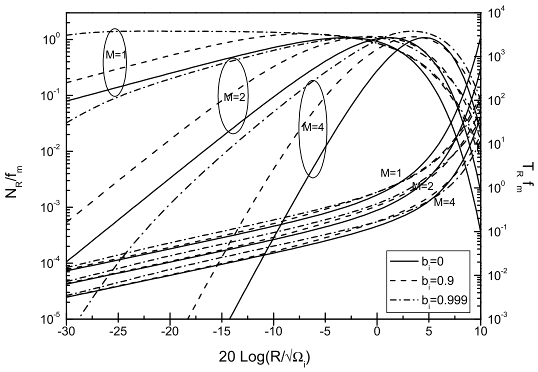

Fig. 1. LCR and AFD of EGC for identical Hoyt-fading channels ($M = 1,\ 2,\ 4$ and $b_i = 0,\ 0.9,\ 0.999$).

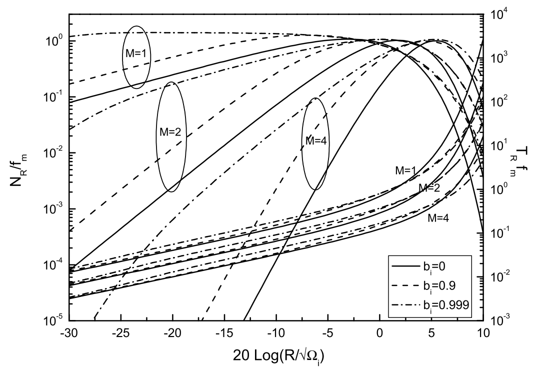

Fig. 2. LCR and AFD of MRC for identical Hoyt-fading channels ($M = 1,\ 2,\ 4$ and $b_i = 0,\ 0.9,\ 0.999$).