A Simple Approximation to the the κ-μ Phase Probability Density Function

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***Abstract*— In this article, a *new* random variable whose distribution closely follows that of the exact distribution of the κ-μ phase distribution is introduced. Whereas the exact statistics is given in an integral form, the approximate one is obtained in a simple formulation that can be computed in a very efficient manner. More interestingly, whereas the exact phase distribution of the κ-μ model comprises Rice and Nakagami-m as special cases, as designed, the special cases of the approximate phase solution are correspondingly Von Mises and, amazingly, Nakagami-m, obtained in an exact manner.**

***Keywords*— κ-μ model, fading channels, approximation, phase probability density function.**

I. Introduction

In wireless communications, an accurate modeling of the propagating channels is needed for better system design and performance analysis. In particular, the study of phase behavior is useful in the design of optimal carrier recovery schemes needed in the synchronization subsystem of coherent receivers [1].

The κ-μ distribution is a general fading distribution that encompasses several important fading distributions as special cases, namely Rice and Nakagami-*m* [2]. The κ-μ distribution can be used to represent the small-scale variation of the fading signal under line-of-sight (LoS) conditions. Its flexibility renders it suitable to better fit field measurements data in a variety of scenarios, both for low- [2] and high-order statistics [3].

Whereas several statistics of the κ-μ fading channel have been derived in a closed-form fashion, no closed-form has been found for the probability density function (PDF) of the phase. This hinders the practicality of using the κ-μ for studying phase related phenomena. Specifically, finding a set of parameters of the κ-μ distribution that fits an experimental curve is a non linear optimization problem that requires the evaluation of the PDF several times. This calculation can become prohibitive if the exact integral form is used.

In this paper we introduce a simpler and more computationally efficient approximation for the κ-μ phase PDF, which yields excellent results for all of the range of the distribution parameters.

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II. The κ-μ Fading Model

The κ-μ distribution is a general fading distribution that represents the small-scale variations of the fading signal under a LoS condition [2].The κ-μ envelope is given by

$$R^2 = \sum_{i=1}^{\mu}(X_i + p_i)^2 + \sum_{i=1}^{\mu}(Y_i + q_i)^2, \tag{1}$$

in which X_i and Y_i are independent gaussian processes with $E(X_i) = E(Y_i) = 0$ and $E(X_i^2) = E(Y_i^2) = \sigma^2$ and p_i and q_i are the mean values of the in-phase and quadrature components of the multipath cluster i. In [2] it is shown that

$$\kappa = d^2/2\mu\sigma^2 \quad \text{and} \quad \sigma^2 = \frac{\hat{r}^2}{2\mu(1+\kappa)} \tag{2}$$

in which $\hat{r} = \sqrt{E[R^2]}$ and $d^2 = \sum_{i=0}^{\mu} p_i^2 + q_i^2$. Let $p^2 = \sum_{i=0}^{\mu} p_i^2$ and $q^2 = \sum_{i=0}^{\mu} q_i^2$. In [4], a phase displacement ϕ has been established such that $\phi = \arg(p + jq)$. It follows that

$$p = \sqrt{\frac{\kappa}{1+\kappa}}\hat{r}\cos(\phi) \quad \text{and} \quad q = \sqrt{\frac{\kappa}{1+\kappa}}\hat{r}\sin(\phi). \tag{3}$$

Let X be the in-phase and Y be the quadrature part of the signal ($X^2 = \sum_{i=1}^{\mu}(X_i + p_i)^2$ and $Y^2 = \sum_{i=1}^{\mu}(Y_i + q_i)^2$). Let Z denote X or Y and λ denote p or q as required. In [4], the PDF of Z was derived, and it was found to be

$$f_Z(z) = \frac{|z|^{\frac{\mu}{2}} \exp\left(-\frac{(z-\lambda)^2}{2\sigma^2}\right) I_{\frac{\mu}{2}-1}\left(\frac{|\lambda z|}{\sigma^2}\right)}{2\sigma^2|\lambda|^{\frac{\mu}{2}-1}\cosh\left(\frac{\lambda z}{\sigma^2}\right)}. \tag{4}$$

The joint PDF of the normalized envelope P and the phase Θ of the κ-μ model is given as [4]

$$\begin{aligned} f_{P,\Theta}(\rho,\theta) &= \frac{1}{2}\mu^2\kappa^{1-\frac{\mu}{2}}(1+\kappa)^{\frac{\mu+2}{2}}\rho^{\mu+1}|\sin 2\theta|^{\frac{\mu}{2}}|\sin 2\phi|^{1-\frac{\mu}{2}} \\ &\times \exp\left(-\mu(1+\kappa)\rho^2 - \kappa\mu + 2\mu\sqrt{\kappa(1+\kappa)}\rho\cos(\theta-\phi)\right) \\ &\times I_{\frac{\mu}{2}-1}\left(2\mu\sqrt{\kappa(1+\kappa)}\rho|\cos\theta\cos\phi|\right) \\ &\times I_{\frac{\mu}{2}-1}\left(2\mu\sqrt{\kappa(1+\kappa)}\rho|\sin\theta\sin\phi|\right) \\ &\times \operatorname{sech}\left(2\mu\sqrt{\kappa(1+\kappa)}\rho\cos\theta\cos\phi\right) \\ &\times \operatorname{sech}\left(2\mu\sqrt{\kappa(1+\kappa)}\rho\sin\theta\sin\phi\right). \end{aligned} \tag{5}$$

In (5), P denotes the normalized envelope rather than the scaled version found in [4].

III. THE APPROXIMATE κ-μ PHASE PDF

The phase PDF can be calculated by integrating (5) with respect to ρ. Unfortunately, no closed-form is known for this integral.

$$f_\Theta(\theta) = \int_0^\infty f_{P,\Theta}(\rho,\theta)d\rho. \tag{6}$$

It is possible to find a suitable approximation by looking for a candidate function $g^*(\theta)$ with a similar shape of the exact phase PDF. By definition, a continuous probability density function is non-negative and integrates to unitary area over its domain [5]. If $g^*(\theta)$ does not change its sign within its domain and integrates to a finite, non zero number, it can be normalized to have unitary area. In that case, $f^*_\Theta(\theta)$ in (7) defines a probability density function.

$$f^*_\Theta(\theta) = \frac{g^*(\theta)}{\int_{-\pi}^{\pi} g^*(t)dt}. \tag{7}$$

A candidate function is found by approximating the integrand of (6) by a truncated Taylor series and then performing the integration. Unfortunately that integral does not converge. A workaround to address this problem is discussed here. First, the interval of integration is changed,

$$f_\Theta(\theta) = \int_0^1 f_{P,\Theta}(x,\theta)dx + \int_1^\infty f_{P,\Theta}(x,\theta)dx. \tag{8}$$

Next, the variable x of the second integral is changed to $y = 1/x$, so that $y = 0$ when $x = \infty$, $y = 1$ when $x = 1$ and $dx = -\frac{dy}{y^2}$. Accordingly,

$$f_\Theta(\theta) = \int_0^1 f_{P,\Theta}(x,\theta)dx + \int_0^1 \frac{f_{P,\Theta}\left(\frac{1}{y},\theta\right)}{y^2}dy. \tag{9}$$

The dummy variable y is then changed back to x and both integrals are regrouped under a single integral from 0 to 1.

$$f_\Theta(\theta) = \int_0^1 \left(f_{P,\Theta}(x,\theta)dx + \frac{f_{P,\Theta}\left(\frac{1}{x},\theta\right)}{x^2} \right) dx. \tag{10}$$

After fixing θ in (10), the integrand is approximated by a truncated Taylor series around x. The integration of this series converges for the Taylor polynomial of degree 1 around $\rho_0 = 1$, and the resultant expression is shown in (11).

$$\begin{aligned} g(\theta) = {} & \frac{3}{2}\mu^2\kappa^{1-\frac{\mu}{2}}(1+\kappa)^{1+\frac{\mu}{2}}|\sin 2\theta|^{\frac{\mu}{2}}|\sin 2\phi|^{1-\frac{\mu}{2}} \\ & \times \exp\left(-\mu(1+2\kappa) + 2\mu\sqrt{\kappa(1+\kappa)}\cos(\theta-\phi)\right) \\ & \times I_{\frac{\mu}{2}-1}\left(2\mu\sqrt{\kappa(1+\kappa)}|\cos\theta\cos\phi|\right) \\ & \times I_{\frac{\mu}{2}-1}\left(2\mu\sqrt{\kappa(1+\kappa)}|\sin\theta\sin\phi|\right) \\ & \times \operatorname{sech}\left(2\mu\sqrt{\kappa(1+\kappa)}\cos\theta\cos\phi\right) \\ & \times \operatorname{sech}\left(2\mu\sqrt{\kappa(1+\kappa)}\sin\theta\sin\phi\right). \end{aligned} \tag{11}$$

Since (11) will be multiplied by a scaling constant, the terms that are not function of θ can be dropped. The simplified version of $g(\theta)$ is given by (12).

$$\begin{aligned} g^*(\theta) = {} & \kappa^{1-\frac{\mu}{2}}|\sin 2\theta|^{\frac{\mu}{2}} \exp\left(2\mu\sqrt{\kappa(1+\kappa)}\cos(\theta-\phi)\right) \\ & \times I_{\frac{\mu}{2}-1}\left(2\mu\sqrt{\kappa(1+\kappa)}|\cos\theta\cos\phi|\right) \\ & \times I_{\frac{\mu}{2}-1}\left(2\mu\sqrt{\kappa(1+\kappa)}|\sin\theta\sin\phi|\right) \\ & \times \operatorname{sech}\left(2\mu\sqrt{\kappa(1+\kappa)}\cos\theta\cos\phi\right) \\ & \times \operatorname{sech}\left(2\mu\sqrt{\kappa(1+\kappa)}\sin\theta\sin\phi\right). \end{aligned} \tag{12}$$

The term $\kappa^{1-\frac{\mu}{2}}$ was maintained because otherwise the inderterminacy obtained by setting $\kappa \to 0$ cannot be resolved. For $\mu \geq 0$, $g^*(\theta)$ is always positive. Hence, a probability density function can be found by the means of (7). Denote by $\mathbb{O}$ the random variable that has the resulting PDF. Let $S(\kappa,\mu,\phi) = \left(\int_{-\pi}^{\pi} g^*(\theta)d\theta\right)^{-1}$. The PDF of $\mathbb{O}$ is given by

$$\begin{aligned} f_{\mathbb{O}}(\theta) = {} & S(\kappa,\mu,\phi)\cdot|\sin 2\theta|^{\frac{\mu}{2}} \exp\left(2\mu\sqrt{\kappa(1+\kappa)}\cos(\theta-\phi)\right) \\ & \times I_{\frac{\mu}{2}-1}\left(2\mu\sqrt{\kappa(1+\kappa)}|\cos\theta\cos\phi|\right) \\ & \times I_{\frac{\mu}{2}-1}\left(2\mu\sqrt{\kappa(1+\kappa)}|\sin\theta\sin\phi|\right) \\ & \times \operatorname{sech}\left(2\mu\sqrt{\kappa(1+\kappa)}\cos\theta\cos\phi\right) \\ & \times \operatorname{sech}\left(2\mu\sqrt{\kappa(1+\kappa)}\sin\theta\sin\phi\right). \end{aligned} \tag{13}$$

Numerical evaluations have shown that $f_{\mathbb{O}}(\theta)$ follows very closely the exact PDF of the phase of the κ-μ channel. This approximation has an almost closed-form expression as it demands only the calculation of a single integral, instead of one for each point. As a result, it can be computed substantially faster than the exact phase PDF. In Figures 1 - 4 the approximation of the phase distribution is compared to its exact value. This comparison shows a close fit between both formulations.

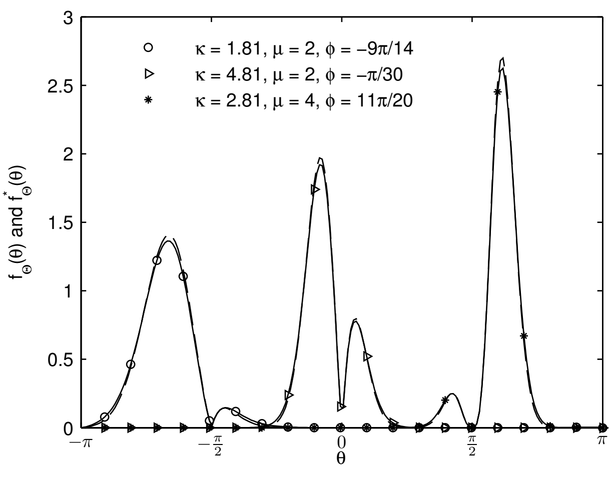

Fig. 1. Phase PDF of the κ-μ model. Comparison between exact (solid line) and approximate (dotted line) solutions.

IV. SPECIAL CASES

As it was discussed, the κ-μ distribution encompasses several other distributions, including Nakagami-*m* and Rice.

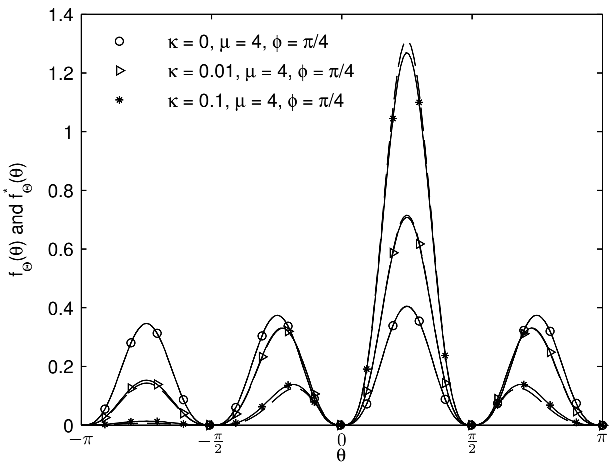

Fig. 2. Phase PDF of the κ-μ model. Comparison between exact (solid line) and approximate (dotted line) solutions.

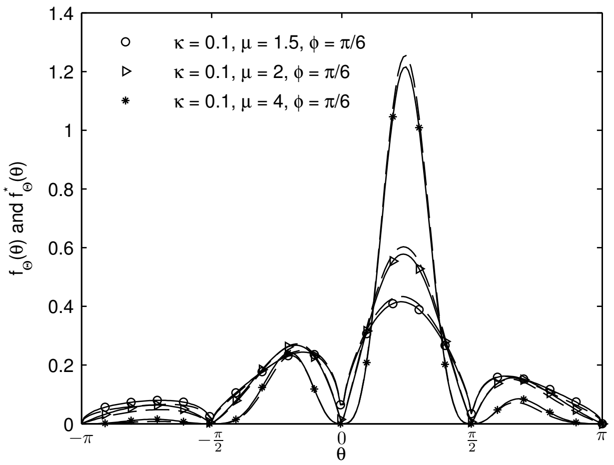

Fig. 3. Phase PDF of the κ-μ model. Comparison between exact (solid line) and approximate (dotted line) solutions.

In this section, the behavior of the approximate expression for those two special cases is explored.

A. Rice

The Rice distribution is obtained from the κ-μ distribution by setting $\mu = 1$. It can be obtained by substituting $\mu = 1$ in (5) and integrating the resulting expression with respect to ρ from $-\infty$ to ∞, yielding

$$\begin{aligned} f_\Theta(\theta)_{Rice} &= \\ \tfrac{\exp(-\kappa)}{2\pi}&(1+\sqrt{\kappa\pi}\exp(\kappa\cos^2(\theta-\phi))\cos(\theta-\phi) \\ &\times[1+\mathrm{erf}(\sqrt{\kappa}\cos(\theta-\phi))]). \end{aligned} \quad (14)$$

The approximate phase PDF is obtained in the same fashion, by substituting $\mu = 1$ in (13). Surprisingly, this leads to a rather simple and closed-form expression,

$$f^*_\Theta(\theta)_{Rice} = \frac{\exp\left(2\sqrt{\kappa(1+\kappa)}\cos(\theta-\phi)\right)}{2\pi I_0(2\sqrt{\kappa(1+\kappa)})}. \quad (15)$$

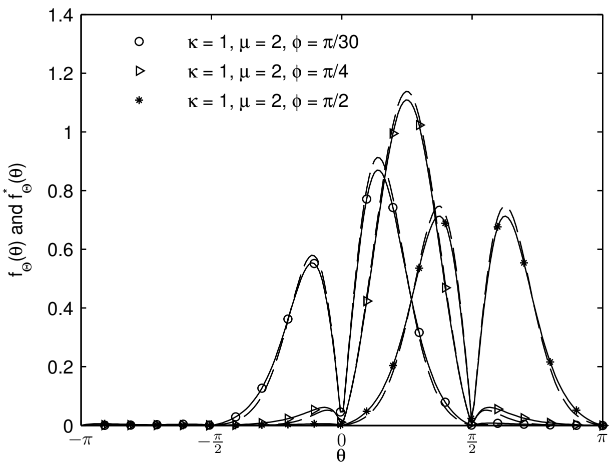

Fig. 4. Phase PDF of the κ-μ model. Comparison between exact (solid line) and approximate (dotted line) solutions.

The expression (15) is the PDF of the von Mises distribution, which can be viewed as a circular analog of the normal distribution [6]. In Figure 5, exact and approximate solutions are compared for several values of κ. The parameter ϕ only shifts the distributions horizontally. The approximate and exact solutions become closer as κ approaches both zero and infinity.

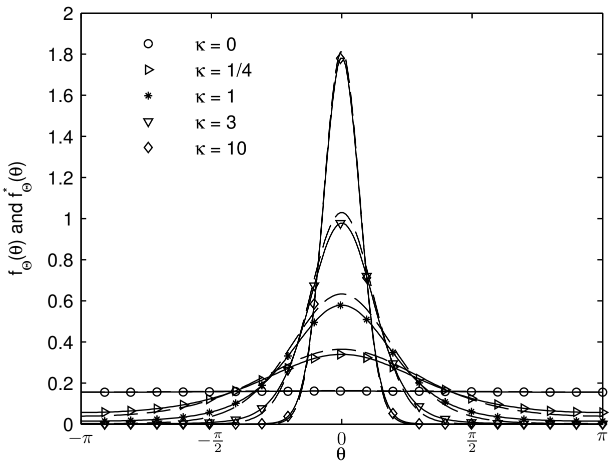

Fig. 5. Phase PDF of the κ-μ model. Comparison between exact (solid line) and approximate (dotted line) solutions for the Rice distribution ($\mu = 1$, $\phi = 0$). In this case, the approximate solution is the Von Mises distribution.

B. Nakagami-*m*

Nakagami-*m* is a particular case of the κ-μ distribution with $\kappa \to 0$. Its phase distribution was derived in [7] and is given by

$$f_\Theta(\theta)_{nak-m} = \frac{|\sin 2\theta|^{m-1}\Gamma(\mu)}{2^\mu\Gamma^2(\mu/2)}. \quad (16)$$

Obtaining the approximate expression for the Nakagami-*m* case is more complicated, since setting $\kappa = 0$ leads to an indeterminacy. It can be shown that

$$\lim_{x\to 0}\left(x^{\frac{1}{2}-\frac{m}{4}} I_{\frac{\mu}{2}-1}(ax^{1/2})\right) = \frac{(a/2)^{\frac{m}{2}-1}}{\Gamma\left(\frac{m}{2}\right)}. \qquad (17)$$

To solve the indeterminacy, the limit (17) is used in (11) and then the resulting expression is scaled to unitary area. The approximation of the phase PDF of the Nakagami-*m* model obtained this way reduces to the exact Nakagami-*m* phase PDF. This is an interesting result that shows one particular case in which the proposed approximate phase PDF reduces to the exact phase PDF.

V. Comparison between approximate and exact solutions

This section shows two measures of the efficiency of the approximation: the error between exact and approximate solutions and the time needed to compute both formulations. These two aspects are important to quantify the usefullness of the approximation. Whereas the latter shows the main advantage of using an approximation, the former reveals the quality of the fit between the curves.

The error was measured as the energy of the difference between the two PDFs,

$$\text{error} = \int_{-\pi}^{\pi} \left[f_{\kappa-\mu}(\theta) - f_{\mathbb{P}}(\theta)\right]^2 d\theta. \qquad (18)$$

Figures 6 - 8 show the behavior of the error as a function of the parameters. It reaches its maximum value at $\kappa = 1$ and $\mu = 1$, and it goes down as κ and μ increase. The error is periodic in relation to ϕ, with period $\pi/2$, reaching its lowest value at odd multiples of $\pi/4$.

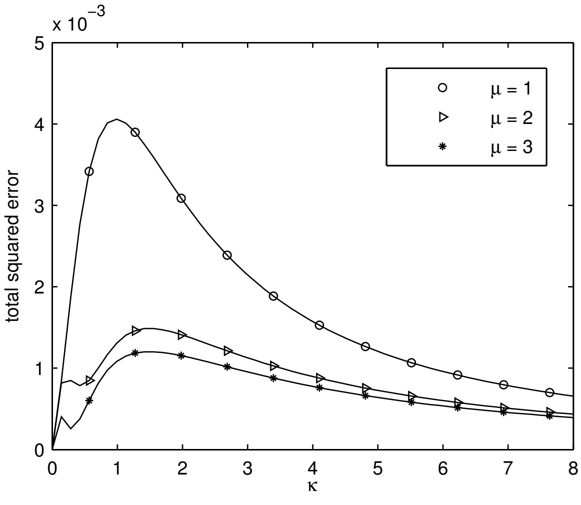

Fig. 6. Total squared error between exact and approximate solutions for the PDF of the phase of the κ-μ process as a function of κ ($\phi = 0$).

The CPU time was measured by the Mathematica software [8]. In Figure 9, the time needed to map the exact and approximate solutions is given as a function of the number of points used in the mapping. All the times were measured on the same computational system. The time needed to calculate the exact expression is around two orders of magnitude greater than the time needed to calculate the approximation.

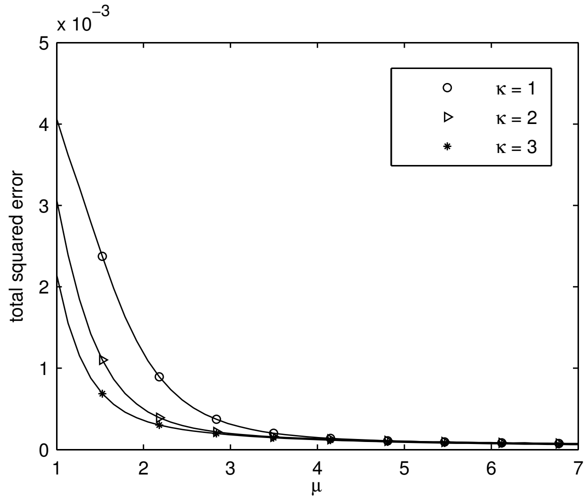

Fig. 7. Total squared error between exact and approximate solutions for the PDF of the phase of the κ-μ process as a function of μ ($\phi = \pi/5$).

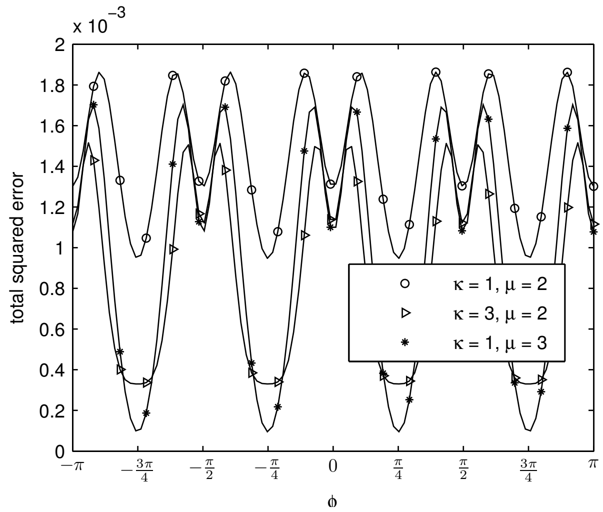

Fig. 8. Total squared error between exact and approximate solutions for the PDF of the phase of the κ-μ process as a function of ϕ.

VI. Conclusion

In this paper, a new random variable $\mathbb{O}$ is described. This new RV has an almost closed-form expression and can be computed efficiently. The $\mathbb{O}$ random variable was derived as a simpler approximate solution of the κ-μ phase PDF. Sample curves comparing both formulations were provided. It is shown that the proposed approximation of the phase distribution reduces to the von Mises distribution in the Ricean case. It is also shown that, in the particular case when $\kappa \to 0$, the approximate solution coincides with the Nakagami-*m* phase distribution. Finally, a comparison is made between the exact and approximate expressions. This comparison shows the overall quality of the fit and the superiority of the approximation in terms of computational performance.

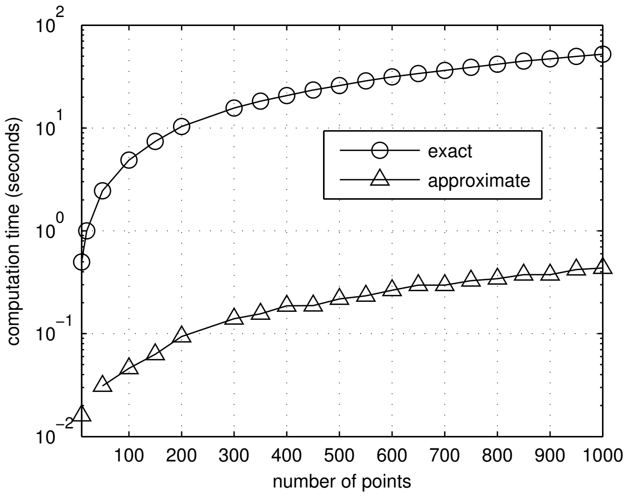

Fig. 9. CPU time needed to map the approximate and exact solutions. The κ-μ parameters used were $\kappa = 2$, $\mu = 2$ and $\phi = \pi/4$.

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