Compactly Supported One-cyclic Wavelets Derived from Beta Distributions

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***Resumo*—Novas wavelets contínuas de suporte compacto são construídas, as quais relacionam-se com a distribuição de probabilidade beta. Elas são construídas empregando o conceito de derivada "blur". Estas novas wavelets são unicíclicas e elas podem ser vistas como um tipo de wavelets de Haar suavizadas. Sua relevância decorre do teorema do limite central aplicado para wavelets de suporte compacto.**

***Palavras-Chave*—Wavelets contínuas, wavelets unicíclicas, derivada Blur, distribuição beta, teorema do limite, wavelets de suporte compacto.**

Abstract— **New continuous wavelets of compact support are introduced, which are related to beta distribution. Wavelets can be related to probability distributions using "blur" derivatives. These new wavelets have just one cyclic, so they are termed unicycle wavelets. They can be viewed as a soft variety of Haar wavelets. Their relevance is due to the central limit theory applied for supported compact signals.**

***Index Terms*—Continuous wavelet, one cycle-wavelets, blur derivative, beta distribution, central limit theory, compactly supported wavelets.**

I. PRELIMINARIES AND BACKGROUND

Wavelets are strongly connected with probability distributions. Recently, a new insight into wavelets was presented, which applies Max Born reading for the wave-function [1] and an information theory focus was achieved [2]. Many continuous wavelets are derived from a probability density (e.g. Sombrero). This approach also sets up a link between probability densities, wavelets and "blur derivatives" [3]. Let $P(.)$ be a probability density, $P \in C^\infty$.

If $\lim_{t \to \infty} \frac{d^{n-1}P(t)}{dt^{n-1}} = 0$ then $\psi_n(t) := (-1)^n \frac{d^n P(t)}{dt^n}$ is a wavelet engendered by $P(.)$. Given a mother wavelet ψ that holds the admissibility condition [4–6] then the continuous wavelet transform is defined by

$$CWT(a,b) := \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right) dt \ , \ \forall a \in R - \{0\}, b \in R \ .$$

Continuous wavelets have often unbounded support, such as Morlet, Meyer, Mathieu, de Oliveira wavelets [3,7–8]. In the case where the wavelet was generated by a density, it follows that

$$\frac{1}{\sqrt{|a|}} \psi_n\left(\frac{t-b}{a}\right) = \frac{1}{\sqrt{|a|}} (-1)^n \frac{\partial^n}{\partial t^n} P\left(\frac{t-b}{a}\right).$$

Now $\frac{\partial^n}{\partial b^n} P\left(\frac{t-b}{a}\right) = (-1)^n P^{(n)}\left(\frac{t-b}{a}\right)\frac{1}{a^n}$, so that

$$CWT(a,b) = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{|a|}} \frac{\partial^n}{\partial b^n} P\left(\frac{t-b}{a}\right) dt \ .$$

If the integral and derivative can be commuted, it follows that

$$CWT(a,b) = \frac{\partial^n}{\partial b^n} \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{|a|}} P\left(\frac{t-b}{a}\right) dt \ .$$

Defining the LPFed signal as the "blur" signal

$$\tilde{f}(a,b) := \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{|a|}} P\left(\frac{t-b}{a}\right) dt = \int_{-\infty}^{+\infty} f(t) P_{a,b}(t) dt \ ,$$

an interesting interpretation can be made: Set a scale a and take the average (smoothed) version of the original signal –the blur version $\tilde{f}(a,b)$. The "blur derivative" $\frac{\partial^n \tilde{f}(a,b)}{\partial b^n}$ is the n^{th} derivative regarding the shift b of the blur signal at the scale a. The blur derivative coincide with the wavelet transform CWT(a,b) at the corresponding scale. Details (high-frequency) are provided by the derivative of the low-pass (blur) version of the signal.

Revisiting central limit theorems –

There are essentially three kinds of central limit theorems: for unbounded distributions, for causal distributions and for compactly supported distributions [9]. The random variable corresponding to the sum of N independent and identically distributed (i.i.d.) variables converges to: a Gaussian distribution, a Chi-square distribution or a Beta distribution (see Table I). The Gaussian always plays a very important role and it is associated with Morlet's wavelet, which is known to be of unbounded support. This is the only wavelet that meets the lower bound of Gabor uncertainty inequality [10]. The concept of wavelet entropy was recently introduced and Morlet wavelet also revealed a special wavelet [11]. Among

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all wavelets of compact support, that one linked to the Beta distribution should also play an important practical and theoretical role.

TABLE I. Different versions of the central limit theorem.

Marginal distribution	Central limit distribution as $N\to\infty$
Unbounded support	$G(t\mid m,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-(t-m)^2/2\sigma^2\right)$
Causal distribution	$\chi^2(t\mid\alpha,\beta)=\frac{t^{\alpha}e^{-t/\beta}}{\beta^{\alpha+1}\Gamma(\alpha+1)}u(t)$
Compact support	$beta(t\mid\alpha,\beta)=\begin{cases}Kt^{\alpha}(1-t)^{\beta} & 0<t<1\\ 0 & \text{otherwise}\end{cases}$

Let $p_i(t)$ be a probability density of the random variable t_i, i=1,2,3..N i.e. $p_i(t)\geq 0$, $(\forall t)$ and $\int_{-\infty}^{+\infty}p_i(t)dt=1$. If $p_i(t)\leftrightarrow P_i(w)$, then $P_i(0)=1$ and $(\forall w)$ $|P_i(w)|\leq 1$.

Suppose that all variables are independent. The density $p(t)$ of the random variable corresponding to the sum $t:=\sum_{i=1}^{N}t_i$ is given by the iterate convolution [12]

$p(t)=p_1(t)*p_2(t)*...*p_N(t)$.

If $p_i(t)\leftrightarrow P_i(w)=|P_i(w)|e^{j\Theta_i(w)}$, i=1,2,3..N and $p(t)\leftrightarrow P(w)=|P(w)|e^{j\Theta(w)}$, then $|P(w)|=\prod_{i=1}^{N}|P_i(w)|$ and $\Theta(w)=\sum_{i=1}^{N}\Theta_i(w)$. The mean and the variance of a given random variable t_i are, respectively

$$m_i=\int_{-\infty}^{+\infty}\tau p_i(\tau)d\tau\,,\ \sigma_i^2=\int_{-\infty}^{+\infty}(\tau-m_i)^2p_i(\tau)d\tau\,.$$

The mean and variance of t are therefore $m=\sum_{i=1}^{N}m_i$ and $\sigma^2=\sum_{i=1}^{N}\sigma_i^2$. The following theorems can be proved [9].

Theorem 1. (Central Limit Theorem for distributions of unbounded support). If $\{p_i(t)\}$ are not a lattice (a Dirac comb); $E(t_i^3)<+\infty$ and $\lim_{N\to\infty}\sigma^2=\infty$, then $t:=\sum_{i=1}^{N}t_i$ holds, as $N\to\infty$, $P(w)\sim\exp\left(-\sigma^2w^2/2-jmw\right)$

$$p(t)\sim\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-(t-m)^2/2\sigma^2\right).\ \square$$

According to Gnedenko and Kolmogorov [9], if the probability densities have bounded support, then the corresponding theorem is:

Theorem 2. (Central Limit Theorem for distributions of compact support). Let $\{p_i(t)\}$ be distributions such that $\text{supp}\ p_i(t)=(a_i,b_i)$ $(\forall i)$. Let $a:=\sum_{i=1}^{N}a_i<\infty$ and $b:=\sum_{i=1}^{N}b_i<\infty$. It is assumed without loss of generality that a=0 and b=1. The random variable $t:=\sum_{i=1}^{N}t_i$ holds, as $N\to\infty$,

$$p(t)\sim\begin{cases}K.t^{\alpha}.(1-t)^{\beta} & 0\leq t\leq 1\\ 0 & otherwise\end{cases},$$ where

$\alpha=\frac{m(m-m^2-\sigma^2)}{\sigma^2}$ and $\beta=\frac{(1-m)(\alpha+1)}{m}$ $\square$

II. β-WAVELETS: NEW COMPACTLY SUPPORTED WAVELETS

The Beta distribution is a continuous probability distribution defined in the interval $0\leq t\leq 1$ [13, 14]. It is characterised by two parameters, namely, α and β, according to:

$$P(t):=\frac{1}{B(\alpha,\beta)}t^{\alpha-1}.(1-t)^{\beta-1},\ 1\leq\alpha,\beta\leq+\infty.$$

The normalising factor is $B(\alpha,\beta):=\frac{\Gamma(\alpha).\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, where $\Gamma(.)$ is the generalised factorial function of Euler and $B(.,.)$ is the Beta function [13].

The following parameters can be derived:

Supp (P)=(0,1),

mean=m=$\frac{\alpha}{(\alpha+\beta)}$, mode=$\frac{\alpha-1}{(\alpha+\beta-2)}$,

variance=$\sigma^2=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$,

characteristic function=$M(\alpha,\alpha+\beta,j\nu)$, where M(.,.,.) is the Kummer's confluent hypergeometric function [14]. The N^{th} moment of $P(.)$ can be found using

$$moment(N):=\int_0^1t^NP(t)dt=\frac{B(\alpha+\beta,N)}{B(\alpha,N)}.$$

The derivative of the beta distribution can easily be found.

$$P'(t):=\left(\frac{\alpha-1}{t}-\frac{\beta-1}{1-t}\right)P(t).$$

A random variable transform can be made by an afin transform in order to generate a new distribution with zero-mean and unity variance [12], which implies a non-normalised support $T=\frac{1}{\sigma}=T(\alpha,\beta)$.

Let the new random variable is defined by $t':=T.(t-m)$. This variable has zero-mean and unity variance. Its corresponding probability density is given by

$$P(t\,|\,\alpha,\beta) := \frac{1}{B(\alpha,\beta).T(\alpha,\beta)}.$$

$$\left(\frac{t+m(\alpha,\beta).T(\alpha,\beta)}{T(\alpha,\beta)}\right)^{\alpha-1}.\left(1-\frac{t+m(\alpha,\beta).T(\alpha,\beta)}{T(\alpha,\beta)}\right)^{\beta-1}$$

The β-wavelets can now be derived from these adjusted distributions by using the concept of "blur" derivative.
Parameters of the beta-wavelets of parameters α and β are:

$$T := (\alpha+\beta)\sqrt{\frac{\alpha+\beta+1}{\alpha\beta}}$$

$$\text{Supp}\,\psi = \left[-\sqrt{\frac{\alpha+\beta+1}{\beta/\alpha}}, \sqrt{\frac{\beta}{\alpha}}\sqrt{\alpha+\beta+1}\right] := [a,b]$$

$$\text{length Supp}\,\psi = T = (\alpha+\beta)\sqrt{\frac{\alpha+\beta+1}{\alpha\beta}}\,.$$

The parameter $R := b/|a| = \beta/\alpha$ is referred to as "cyclic balance", and is referred to as "cyclic balance", and is defined as the ratio between the lengths of the causal and non-causal piece of the wavelet. The instant of transition t_{zero} between the first and second half cycle is given by

$$t_{zerocross} = \frac{(\alpha-\beta)}{(\alpha+\beta-2)}\sqrt{\frac{\alpha+\beta+1}{\alpha\beta}}\,.$$

The (unimodal) scale function associated with the wavelets is given by

$$\phi_{beta}(t\,|\,\alpha,\beta) := \frac{1}{B(\alpha,\beta)T^{\alpha+\beta-1}}[t-a]^{\alpha-1}.[b-t]^{\beta-1},$$

$a \le t \le b$. Since P(.|α,β) is unimodal, the wavelet generated by $\psi_{beta}(t\,|\,\alpha,\beta) := (-1)\dfrac{dP(t\,|\,\alpha,\beta)}{dt}$ has only one-cycle (a negative half-cycle and a positive half-cycle).

A close expression for first order beta-wavelets can easily be derived. In the support of $\psi(t\,|\,\alpha,\beta)$, $a \le t \le b$,

$$\psi_{beta}(t\,|\,\alpha,\beta) := \frac{-1}{B(\alpha,\beta).T^{\alpha+\beta-1}}.$$

$$\left[\frac{\alpha-1}{t-a}-\frac{\beta-1}{b-t}\right](t-a)^{\alpha-1}.(b-t)^{\beta-1}$$

wavelet function (one-cycle wave), $a \le t \le b$, is

$$\psi_{beta}(t\,|\,\alpha,\beta) := \frac{-1}{B(\alpha,\beta).T^{\alpha+\beta-1}}.$$

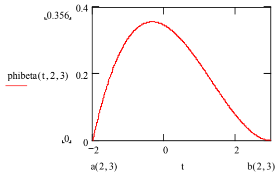
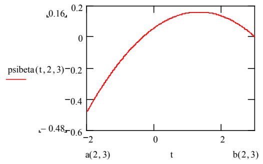
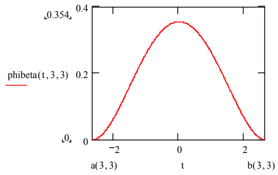
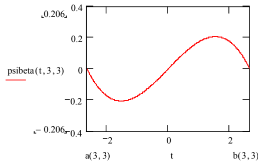
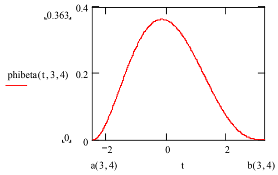
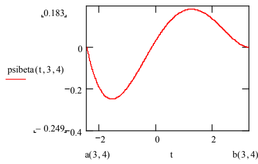
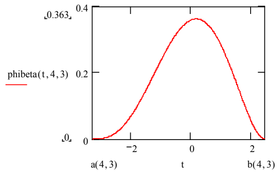
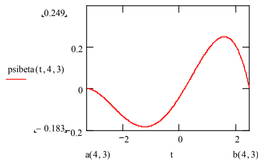
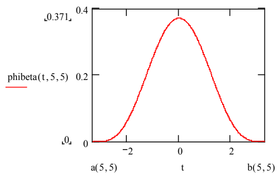
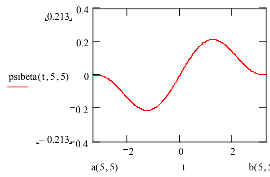
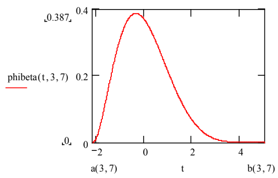
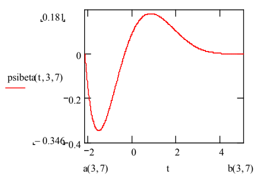
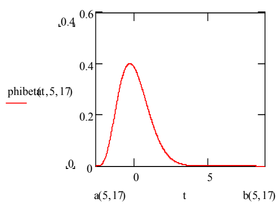
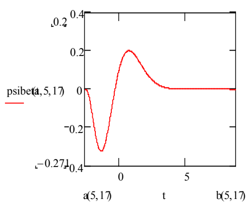
$$\left[(\beta-1).(t-a)^{\alpha-1}.(b-t)^{\beta-2}-(\alpha-1).(t-a)^{\alpha-2}.(b-t)^{\beta-1}\right]$$

Although scale and wavelets can be found for any α,β>1, the behaviour of the wavelet in the extreme points of the support can be discontinuous. However, it is a simple matter to guarantee the continuity of the wavelet according to:

Proposition 1. Beta one-cycle wavelets of parameters α>2 and β>2 are smooth continuous wavelets of compact support.

Proof. Clearly, $\psi_{beta}(t\,|\,\alpha,\beta) = 0\ \forall t<a$, and $\forall t>b$. The only concerns are therefore the extreme points of the support, but $\psi_{beta}(a\,|\,\alpha,\beta) = \psi_{beta}(b\,|\,\alpha,\beta) = 0$ provided that α>2 and β>2. □

TABLE II. Unicycle-beta scale function and wavelet for different parameters: a) α=2, β=3 b) α=β=3 c) α=3, β=4 d) α=4, β=3 e) α=β=5 f) α=3, β=7 g) α=5, β=17.

Scale function $\phi(t\,|\,\alpha,\beta)$	Wavelet function $\psi(t\,|\,\alpha,\beta)$	
		(a)
		(b)
		(c)
		(d)
		(e)
		(f)
		(g)

The beta wavelet spectrum can be derived in terms of the Kummer hypergeometric function [14]. Let the Fourier transform pair $\psi_{beta}(t\,|\,\alpha,\beta) \leftrightarrow \Psi_{BETA}(w\,|\,\alpha,\beta)$. The spectrum is also denoted by $\Psi_{BETA}(w)$ for short. It can be shown that

$$\Psi_{BETA}(w) =$$
$$-jw.M\left(\alpha, \alpha+\beta, -jw(\alpha+\beta)\sqrt{\frac{\alpha+\beta+1}{\alpha.\beta}}\right).e^{-jw\sqrt{\frac{\alpha}{\beta}(\alpha+\beta+1)}}.$$

The spectrum of a number of unicyclic beta wavelets is presented in Figure 1. Only symmetrical ($\alpha=\beta$) cases have zeroes in the spectrum. The evaluation was carried out using the relationship:

$$M(\alpha, \alpha+\beta, j\nu) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}.\int_0^1 e^{j\nu t} t^{\alpha-1}(1-t)^{\beta-1}\,dt.$$

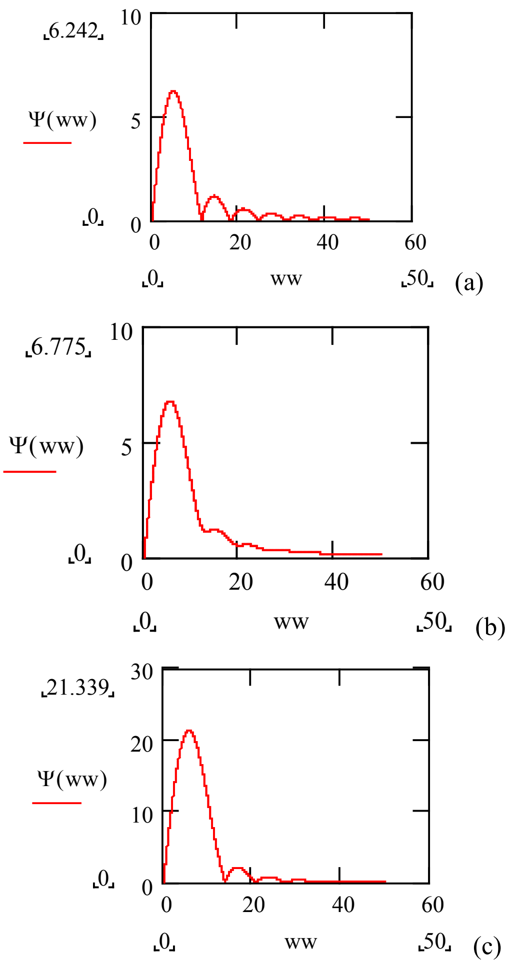


Figure 1. Magnitude of the spectrum $\Psi_{BETA}(w)$ of a few beta wavelets, $|\Psi_{BETA}(w|\alpha,\beta)| \times w$ for: a) symmetric beta wavelets $\alpha=\beta=3$; b) asymmetric beta wavelets $\alpha=3$, $\beta=4$; c) $\alpha=\beta=4$.

III. HIGH-ORDER WAVELETS

Since the beta distribution is unimodal, the first derivative has just one cycle. Higher derivatives may also generate beta wavelets. Higher order beta wavelets are defined by

$$\psi_{beta(N)}(t|\alpha,\beta) := (-1)^N \frac{d^N}{dt^N} P(t|\alpha,\beta).$$

This is henceforth referred to as an N-order beta wavelet. They exist for order $N \leq MIN(\alpha,\beta)-1$. After some algebraic handling, their close expression can be found:

$$\psi_{beta(N)}(t|\alpha,\beta) = \frac{(-1)^N}{B(\alpha,\beta).T^{\alpha+\beta-1}}.\sum_{n=0}^{N} \text{sgn}(2n-N).$$
$$\frac{\Gamma(\alpha)}{\Gamma(\alpha-(N-n))}.(t-a)^{\alpha-1-(N-n)}.\frac{\Gamma(\beta)}{\Gamma(\beta-n)}.(b-t)^{\beta-1-n}.$$

A couple of high beta wavelets are shown in Figure 2.

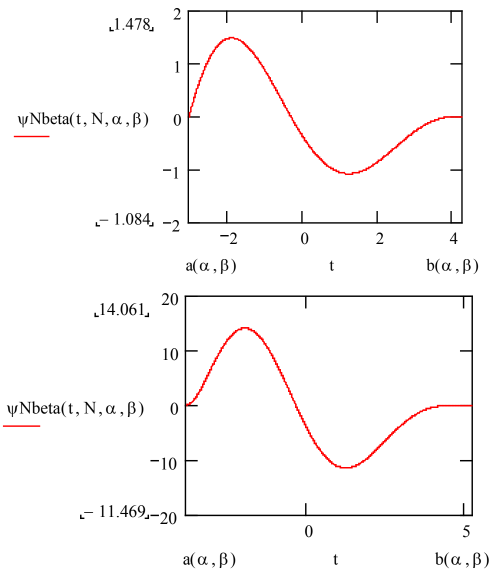


Figure 2. High–order beta wavelets for different parameters: a) N=3, α=5, β=7; b) N=5, α=8, β=11.

With the aim of allowing the investigation of potential applications of such wavelets, software to compute them should be written. Nowadays one of the most powerful software supporting wavelet analysis is the Matlab™ [16], especially when the wavelet graphic interface is available. In the Matlab™ wavelet toolbox, there exist five kinds of wavelets (type the command *waveinfo* on the prompt): (*i*) crude wavelets (*ii*) Infinitely regular wavelets (*iii*) Orthogonal and compactly supported wavelets (*iv*) biorthogonal and compactly supported wavelet pairs. (*v*) complex wavelets. Figure 3 illustrates the beta implementation over Matlab. The m-files to allow the computation of beta wavelet transform are currently (*freeware*) available at the URL: http://www2.ee.ufpe.br/codec/WEBLET.html (new wavelets).

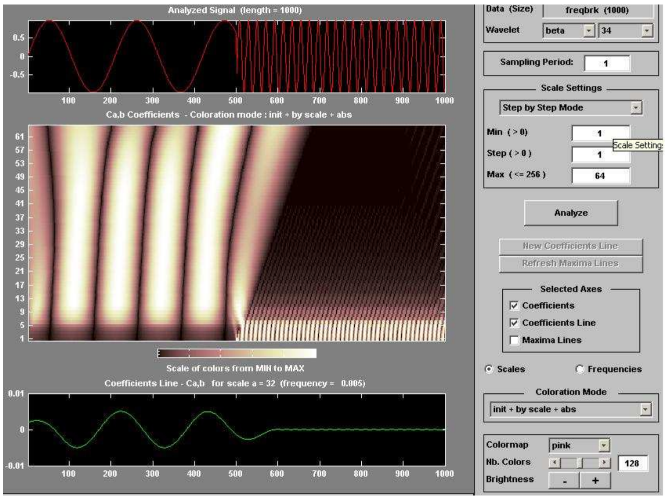


Figure 3. Beta wavelets displayed over Matlab™ using the *wavemenu* command.

IV. CONCLUDING REMARKS

Compactly supported wavelets are among the most functional and useful wavelets. This correspondence introduces a new family of wavelets of this class. These wavelets can be viewed as some kind of soft Haar wavelets. It remains to be investigated how beta wavelets can be approximated using FIR or IIR filters. In comparison with other wavelets of compact support (e.g. dBN, coiflets etc.), the beta wavelets derived in this work have some idiosyncrasies and advantages: *i*) They are regular and smooth, *ii*) have only one cycle, *iii*) have an analytical formulation (close formulae), *iv*) They importance relay on the Central Limit Theorem. This behaviour can be useful for analysing signals from certain modulation schemes or from power systems disturbances.

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APPENDIX

Let $D^{\beta eta} := \{\phi_{beta}(t\,|\,\alpha,\beta)\}_{\alpha\beta \in R}$ be the set of all possible signals of the kind beta probability density.

Lemma 1. The square of a normalised beta density $P(t) = \phi(t\,|\,\alpha,\beta) = \dfrac{1}{B(\alpha,\beta)} t^{\alpha-1}.(1-t)^{\beta-1}$ is proportional to another beta density.

Proof. A straightforward algebraic handling yields $\phi^2(t\,|\,\alpha,\beta) = \lambda_0.\phi(t\,|\,2\alpha-1, 2\beta-1)$, where $\lambda_0 = \lambda_0(\alpha,\beta) := \dfrac{B(2\alpha-1, 2\beta-1)}{B^2(\alpha,\beta)}$. □

Corollary. $D^{\beta eta}$ is a close class regarding the rising to a power (pair exponent) and repeated convolution (a number pair of times), i.e. $\phi_{beta(t|\alpha,\beta)} \in D^{\beta eta} \Rightarrow \phi^2_{beta}(t\,|\,\alpha,\beta) \in D^{\beta eta}$ and $\phi_{beta(t|\alpha,\beta)} * \phi_{beta(t|\alpha,\beta)} \in D^{\beta eta}$. □

A similar property is shared with the other densities concerning the Central Limit Theorem.

Lemma 2. $\dfrac{1}{2\pi}\int_{-\infty}^{+\infty} |M(\alpha, \alpha+\beta, j\nu)|^2 d\nu = \lambda_0(\alpha,\beta)$.

Proof. Parseval's identity furnishes $\dfrac{1}{2\pi}\int_{-\infty}^{+\infty} |M(\alpha, \alpha+\beta, j\nu)|^2 d\nu = \int_0^1 \phi^2(t\,|\,\alpha,\beta)dt$ and the proof follows by applying lemma 1 □

It can be also proved the following proposition:

Proposition 2. Let $\alpha > 1$ and $\beta > 1$. The admissibility constant c_ψ of a unicyclic beta wavelet is

$$c_\psi(\alpha,\beta) = \frac{2\pi.\lambda_0(\alpha,\beta)}{T(\alpha,\beta)} < +\infty .$$