Investigation on Quantum Entanglement of Pure $C_2 \otimes C_2 \otimes C_2$ Tripartite GHZ States

João Batista Rosa Silva, Paulo Benício Melo de Sousa and Rubens Viana Ramos

Abstract **– The understanding of entanglement measures and separability criteria are crucial problems in quantum information theory. In order to analyse the entanglement of pure tripartite GHZ states we used the Rest matrix. Relations between the maximal eigenvalues of the four possible Rest matrices and the total and bipartite tangles are found.**

Keywords **– Three-way quantum entanglement, tangle, Rest matrix.**

I. INTRODUCTION

Quantum entanglement is, in fact, the most interesting quantum property that, when applied in communication and computation, allows powerful ways of information sending and processing. There are several issues about entanglement that challenges scientists, for example, what is the best way to create [1,2], concentrate [3], send through noisy channels [4], make transformation between classes of entanglement [5] and manipulate entangled states [6], among others. The knowledge of entanglement measures is also an important point. Several works have been done in this direction and there are several measures reliable for pure and mixed states, as for instance those discussed in references [7-10]. Recently the entanglement of pure bipartite $C_2 \otimes C_2$ states have been calculated using the rest matrix (R) [11], $R=\Gamma-\rho_a \otimes \rho_b$, where ρ_a and ρ_b are the individual states of Γ. Relations between the entanglement and some R's properties are expected since a pure state is disentangled only if R is the null matrix. For states $|\Psi\rangle=a|00\rangle+(1-a^2)^{1/2}|11\rangle$, the concurrence, that is also an entanglement measure, of such states is $C(|\Psi\rangle\langle\Psi|) = 2a(1-a^2)^{1/2}$. The eigenvalues of the R matrix for that state are $\{C^2/4+C/2, -C^2/4, -C^2/4, C^2/4-C/2\}$. Since all eigenvalues of R are well related to C, they can be used to measure the entanglement of pure bipartite states. However, if we use the larger eigenvalue, λ_{max}, for example, we have to multiply it by 4/3 in order to have the entanglement measured varying from 0 to 1. In this paper the efforts are concentrated in the generalisation of the use of the R matrix to infer the entanglements of pure tripartite $C_2 \otimes C_2 \otimes C_2$ states pertaining to GHZ-class. In a pure tripartite state there are four possible (pure) entanglements ABC, AB_C, A_BC and AC_B, hence there are also four different R matrices. Using numerical simulations and some analytical work, relations between the maximal eigenvalue of each one of the four possible rest matrices and the tripartite and bipartite tangles are found. Based on these relations, we will see that those eigenvalues can be used to measure or to have a hint about the amount of total entanglement present.

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II. PURE TRIPARTITE GHZ STATES AND THE REST MATRIX

Pure tripartite states pertaining to the GHZ-class can be obtained using the following formula [12]:

$$\begin{gathered}|\psi_{GHZ}\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \\ \lambda_{0,1,2,3,4} \geq 0; \quad \sum_i \lambda_i^2 = 1; \qquad \theta \in [0,\pi]\end{gathered} \cdot (1)$$

The three-way entanglement of such tripartite pure states can be calculated using the 3-tangle, τ_3, an entanglement monotone that can be calculated using the expression [13,14]:

$$\tau_3 = \tau_a - \tau_{ab} - \tau_{ac}, \tag{2}$$

where, for the pure tripartite state Ψ_{abc}, τ_{ab} and τ_{ac} are tangles (entanglement measures) of the bipartite states $\Phi_{ab}=\mathrm{tr}_c(\Psi_{abc})$ and $\Phi_{ac}=\mathrm{tr}_b(\Psi_{abc})$. At last τ_a is given by $\tau_a=4\det|\rho_a|$ where $\rho_a=\mathrm{tr}_{bc}(\Psi_{abc})$. All the possible entanglements present in the state (1) can be easily calculated using the coefficients λ_i i=0,1,2,3 and 4:

$$\tau_3 = 4(\lambda_0\lambda_4)^2 \tag{3}$$

$$\tau_{ab} = 4(\lambda_0\lambda_3)^2 \tag{4}$$

$$\tau_{ac} = 4(\lambda_0\lambda_2)^2 \tag{5}$$

$$\tau_{bc} = 4\left|\lambda_1 e^{i\theta}\lambda_4 - \lambda_2\lambda_3\right|^2 \tag{6}$$

$$\tau_a = 4\left[(\lambda_0\lambda_4)^2 + (\lambda_0\lambda_2)^2 + (\lambda_0\lambda_3)^2\right] \tag{7}$$

$$\tau_b = 4\left[(\lambda_0\lambda_4)^2 + (\lambda_0\lambda_3)^2 + \left|\lambda_1 e^{i\theta}\lambda_4 - \lambda_2\lambda_3\right|^2\right]. \tag{8}$$

$$\tau_c = 4\left[(\lambda_0\lambda_4)^2 + (\lambda_0\lambda_2)^2 + \left|\lambda_1 e^{i\theta}\lambda_4 - \lambda_2\lambda_3\right|^2\right] \tag{9}$$

On the other hand, it has been shown in [11] that the rest matrix can be used to calculate the entanglement of pure bipartite $C_2 \otimes C_2$ states. Briefly, if the pure state is entangled $R\neq[0]$ and R has one positive eigenvalue, $\lambda_{max.}$ Its magnitude can be used to measure the entanglement. When considering tripartite states, the definition of the R matrix is not unique. In fact, there are four possible R matrices that can be useful for tripartite systems. If Ψ_{abc} is the total tripartite state having partial states Φ_{ab}, Φ_{ac}, Φ_{bc}, ρ_a, ρ_b, and ρ_c, then the four R matrices are:

$$R_{abc} = \Psi_{abc} - \rho_a \otimes \rho_b \otimes \rho_c \quad (10)$$

$$R_{a\_bc} = \Psi_{abc} - \rho_a \otimes \Phi_{bc} \quad (11)$$

$$R_{ab\_c} = \Psi_{abc} - \Phi_{ab} \otimes \rho_c \quad (12)$$

$$R_{ac\_b} = \Psi_{abc} - \left(I^{(2x2)} \otimes U\right)\left(\Phi_{ac} \otimes \rho_b\right)\left(I^{(2x2)} \otimes U\right)^{+} \quad (13)$$

where I is the identity matrix and the unitary matrix U corresponds to the swap operation between qubits b and c:

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (14)$$

For a disentangled pure state, the eigenvalue of maximal absolute value is $\lambda_{max}=0$ and for a maximally entangled state, $\lambda_{max}=0.875$ for R_{abc}, and $\lambda_{max}=0.75$ for R_{a_bc}, R_{ab_c} and R_{ac_b},. Hence, the measures used will be the normalized versions $E_{abc}=(8/7)\lambda_{max}(R_{abc})$, $E_{a\_bc}=(4/3)\lambda_{max}(R_{a\_bc})$, $E_{ab\_c}=(4/3)\lambda_{max}(R_{ab\_c})$ and $E_{ac\_b}=(4/3)\lambda_{max}(R_{ac\_b})$. In order to show the usefulness of the R matrices in the study of pure tripartite GHZ states, given by (1), some analytical and numerical work is realized. Initially, we can obtain analytically the following equations relating the coefficients of (1), τ_a, τ_b and τ_c and one of the eigenvalues of the R matrices:

$$\lambda_{a\_bc} = \lambda_0^2\left(\lambda_2^2 + \lambda_3^2 + \lambda_4^2\right) = 0.25\tau_a \quad (15)$$

$$\lambda_{ab\_c} = \lambda_0^2\left(\lambda_4^2 + \lambda_2^2\right) + \Delta - 2\cos(\theta)\prod_{i=1}^{4}\lambda_i = 0.25\tau_c \quad (16)$$

$$\lambda_{ac\_b} = \lambda_0^2\left(\lambda_4^2 + \lambda_3^2\right) + \Delta - 2\cos(\theta)\prod_{i=1}^{4}\lambda_i = 0.25\tau_b \quad (17)$$

where $\Delta = \lambda_1^2\lambda_4^2 + \lambda_2^2\lambda_3^2$. Hence, there is a very clear relation between one of the eigenvalues of the R matrices and the bipartite entanglement between a single qubit and the bipartite state. However, if we consider the eigenvalues of largest absolute value and the entanglement measured by τ_a, τ_b and τ_c the following relations are found:

$$E_{a\_bc} = \left(\tau_a + 2\sqrt{\tau_a}\right)/3 \quad (18)$$

$$E_{ab\_c} = \left(\tau_c + 2\sqrt{\tau_c}\right)/3\,. \quad (19)$$

$$E_{ac\_b} = \left(\tau_b + 2\sqrt{\tau_b}\right)/3 \quad (20)$$

Hence, the eigenvalue of maximal absolute value of the R matrices given in (4)-(6) are measures of the bipartite entanglement between the bipartite and single-qubit states of GHZ-class pure tripartite states $C_2 \otimes C_2 \otimes C_2$. Equations (11)-(13) are exactly equal to those shown in the Introduction and reference [11], valid to pure two-qubit states.

Since E_{a_bc}, E_{ab_c} and E_{ac_b} are (pure) bipartite entanglement measures, one could expect E_{abc} being an entanglement measure for the pure tripartite state. However this is not true. In fact, E_{abc} measures the presence of any type of entanglement in the state and not a unique type. This can see in Fig. 1 in which some relation between E_{abc} and τ, $\tau=\tau_3+\tau_a+\tau_b+\tau_c+\tau_{ab}+\tau_{ac}+\tau_{bc}$, exist. Such relation is well approximated by:

$$\lambda_{\max}\left(R_{abc}\right) \approx \frac{\tau}{16} + \frac{5\sqrt{\tau}}{16}\,. \quad (21)$$

$$E_{abc} \approx \left(\tau + 5\sqrt{\tau}\right)/14 \quad (22)$$

Looking at Fig. 1, we can roughly say the larger the value of E_{abc} the larger the value of τ. The relation is not precise because the order induced by E_{abc} and τ is not the same. The error of (21)-(22) is lower than 0.03 for one million of randomly chosen states.

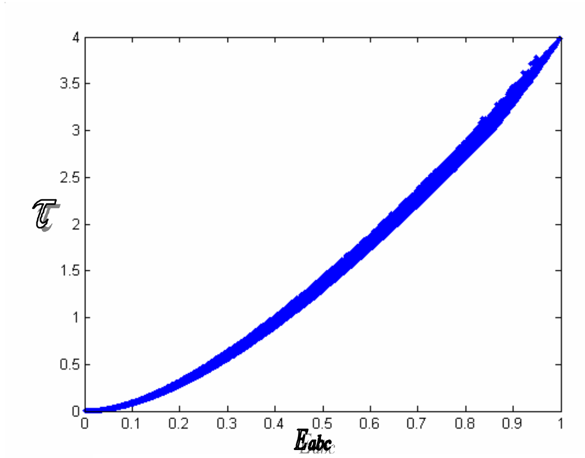

Fig. 1 - E_{abc} versus F(τ), $\tau=\tau_3+\tau_a+\tau_b+\tau_c+\tau_{ab}+\tau_{ac}+\tau_{bc}$.

Hence, in general, E_{abc} cannot be used to measure the 3-way entanglement of pure tripartite ($C_2 \otimes C_2 \otimes C_2$) states but, based on Fig. 1, we can say that E_{abc} give us a good hint of the sum of all entanglements present in the tripartite state and, certainly, is criterion to find the presence of any entanglement, since E_{abc} is null only for completely disentangled states. Moreover, there is a particular class of pure tripartite states whose entanglement can be measured by E_{abc}. In order to check this, let us analyze the behavior of E_{abc} and τ_3 for the following states:

$$\left|\psi_{GHZ}\right\rangle = \sqrt{p}\left|000\right\rangle + \sqrt{1-p}e^{i\varphi}\left|111\right\rangle. \quad (23)$$

The calculation of E_{abc} and τ_3 for states (21) having φ chosen randomly for each value of p, can be seen in Fig. 2. As can be seen in this figure, there is a clear relation between E_{abc} and τ_3. In fact, for any pure tripartite state LU (local unitary operation) equivalent to (23), E_{abc} can be used to measure the entanglement. This happens because for these states the three partial bipartite states are disentangled, hence τ_{ab}, τ_{bc} and τ_{ac} vanish and (2) reduces to $\tau_3 = \tau_a=\tau_b=\tau_c$. The relation between E_{abc} and τ_3 is given by (24) and (25).

$$E_{abc} = -0.5a\begin{bmatrix} -3a + 3a^3 - \\ \left(4 - 3a^2 - 6a^4 + 13a^6 - 12a^8 + 4a^{10}\right)^{1/2} \end{bmatrix} \quad (24)$$

$$a=\sqrt{p}=\sqrt{\frac{1\pm\sqrt{1-\tau_3}}{2}}\ . \qquad (25)$$

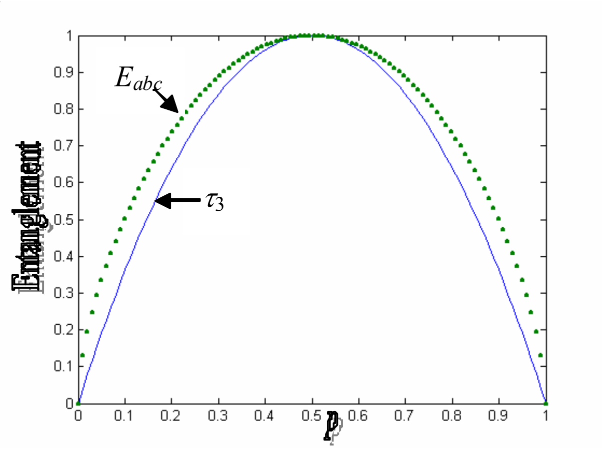

Fig.2 – Entanglement (E_{abc} – dotted line; τ_3 - continuous line) versus p for states (23).

III. R Matrix for Three-Way Disentangled Pure GHZ States

The investigation of the R matrices showed us clear relations between the maximal eigenvalues of the R matrices and the bipartite and total entanglements. However, a generally valid relation between the R matrices and τ_3 was not found. In this section the aim is to present, in some particular cases, the relations between the coefficients of the three-way disentangled pure GHZ states (τ_3=0) and the maximal eigenvalue of R_{abc}.

From (3) we see that a pure tripartite GHZ state is three-way disentangled if $\lambda_0\lambda_4=0$, hence, we have the following situations:

1. $\lambda_0=0$ or $\lambda_4=0$, $\lambda_1\neq0$ and $\lambda_0\lambda_2\lambda_3=0$

$$\lambda_{\max}(R_{abc})=\frac{(\tau_{ab}+\tau_{ac}+\tau_{bc})}{4}+\frac{\sqrt{(\tau_{ab}+\tau_{ac}+\tau_{bc})}}{2} \qquad (26)$$

2. $\lambda_4=0$ and $\lambda_1=0$

$$\lambda_{\max}(R_{abc})=\frac{1}{6}\left(12\sqrt{x}+y\right)^{1/3}-\frac{6z}{\left(12\sqrt{x}+y\right)^{1/3}}+w \qquad (27)$$

$$\begin{aligned}x = {}&-3((\lambda_2)^2-1)((\lambda_0)^2-1)((\lambda_0)^2+(\lambda_2)^2)(-48(\lambda_0)^4(\lambda_2)^2+4(\lambda_0)^4+\\&4(\lambda_2)^4-48(\lambda_0)^2(\lambda_2)^4-16(\lambda_0)^6+32(\lambda_0)^8+104(\lambda_0)^4(\lambda_2)^4-16(\lambda_2)^6-\\&12(\lambda_0)^{12}(\lambda_2)^2+32(\lambda_2)^8+32(\lambda_0)^{12}+20(\lambda_0)^{10}(\lambda_2)^2-40(\lambda_0)^{10}+\\&16(\lambda_0)^4(\lambda_2)^{14}-16(\lambda_2)^{14}+51(\lambda_0)^{10}(\lambda_2)^4-2(\lambda_0)^{10}(\lambda_2)^6+\\&54(\lambda_0)^8(\lambda_2)^8-40(\lambda_2)^{10}-2(\lambda_0)^6(\lambda_2)^{10}+32(\lambda_2)^{12}-29(\lambda_2)^{12}(\lambda_0)^4-\\&29(\lambda_0)^{12}(\lambda_2)^4+4(\lambda_2)^{16}+84(\lambda_2)^6(\lambda_0)^2+84(\lambda_0)^6(\lambda_2)^2-\\&39(\lambda_0)^6(\lambda_2)^4-39(\lambda_0)^4(\lambda_2)^6-56(\lambda_2)^8(\lambda_0)^2-56(\lambda_0)^8(\lambda_2)^2-\\&59(\lambda_0)^8(\lambda_2)^4-94(\lambda_0)^6(\lambda_2)^6-59(\lambda_0)^4(\lambda_2)^8+20(\lambda_2)^{10}(\lambda_0)^2+\\&58(\lambda_0)^8(\lambda_2)^6+58(\lambda_0)^6(\lambda_2)^8+51(\lambda_2)^{10}(\lambda_0)^4-12(\lambda_2)^{12}(\lambda_0)^2+\\&9(\lambda_0)^{12}(\lambda_2)^6-29(\lambda_0)^{10}(\lambda_2)^8-29(\lambda_0)^8(\lambda_2)^{10}+9(\lambda_0)^6(\lambda_2)^{12}-\\&16(\lambda_0)^{14}+16(\lambda_0)^{14}(\lambda_2)^4+4(\lambda_0)^{16}+8(\lambda_0)^2(\lambda_2)^2+4(\lambda_0)^{16}(\lambda_2)^2+\\&4(\lambda_0)^2(\lambda_2)^{16})\end{aligned} \qquad (28)$$

$$\begin{aligned}y = {}&8(\lambda_0)^{12}+24((\lambda_2)^2-1)(\lambda_0)^{10}+(96-36(\lambda_2)^2-60(\lambda_2)^4)(\lambda_0)^8+\\&(144(\lambda_2)^2+168(\lambda_2)^4-152-160(\lambda_2)^6)(\lambda_0)^6+(-60(\lambda_2)^8+96(\lambda_2)^4\\&+168(\lambda_2)^6-276(\lambda_2)^2+72)(\lambda_0)^4+(144(\lambda_2)^6-36(\lambda_2)^8+24(\lambda_2)^{10}-\\&276(\lambda_2)^4+144(\lambda_2)^2)(\lambda_0)^2+8(\lambda_2)^{12}+72(\lambda_2)^4-152(\lambda_2)^6+96(\lambda_2)^8-\\&24(\lambda_2)^{10}\end{aligned} \qquad (29)$$

$$\begin{aligned}z = {}&-1/9(\lambda_0)^8+2/9(1-(\lambda_2)^2)(\lambda_0)^6+(2/9-1/3(\lambda_2)^4+1/9(\lambda_2)^2)(\lambda_0)^4\\&+(4/9(\lambda_2)^2-1/3+1/9(\lambda_2)^4-2/9(\lambda_2)^6)(\lambda_0)^2+2/9((\lambda_2)^4+(\lambda_2)^6)-\\&1/3(\lambda_2)^2-1/9(\lambda_2)^8\end{aligned} \qquad (30)$$

$$w = ((\lambda_2)^2-2/3)(\lambda_0)^4+((\lambda_2)^4-5/3(\lambda_2)^2+2/3)(\lambda_0)^2+2/3(\lambda_2)^2-2/3(\lambda_2)^4 \qquad (31)$$

This last equations gives us a good idea about how hard is the problem to find the exact formula for $\lambda_{max}(R_{abc})$. However, (27)-(31) can be very well approximated by:

$$\lambda_{\max}(R_{abc})\approx\frac{(\tau_{ab}+\tau_{ac}+\tau_{bc})}{4}+\frac{\sqrt{(\tau_{ab}\tau_{ac}\tau_{bc})}}{8} \qquad (32)$$

Unfortunately we have not been able to find even an approximate formula for $\lambda_{max}(R_{abc})$ when $\lambda_4=0$ and $\lambda_0\lambda_1\lambda_2\lambda_3\neq0$, however, from (26) and (32) we can observe how $\lambda_{max}(R_{abc})$ measures the total bipartite entanglement of the three-way tripartite disentangled pure state for the particular cases considered.

IV. Conclusions

Pure tripartite $C_2\otimes C_2\otimes C_2$ states pertaining to GHZ-class were analysed using analytical and numerical procedures. It was shown analytically the relations between τ_a and $(4/3)\lambda_{\max}(R_{a\_bc})$, τ_c and $(4/3)\lambda_{\max}(R_{ab\_c})$, and τ_b and $(4/3)\lambda_{\max}(R_{ac\_b})$. Hence, the maximal eigenvalues of R_{a_bc}, R_{ab_c} and R_{ac_b} matrices can be used to measure the bipartite entanglement between a bipartite state and a single qubit state, since both belong to a pure tripartite state. On the other hand, $(8/7)\lambda_{\max}(R_{abc})$ cannot be used, in general, to measure the tripartite entanglement, since it is not compatible to τ_3. However, in the special case of states LU equivalent to GHZ states giving in (23), it can be used. Further, $(8/7)\lambda_{\max}(R_{abc})$ can give us a hint of the total amount of entanglement ($\tau_3+\tau_a+\tau_b+\tau_c+\tau_{ab}+\tau_{bc}+\tau_{ac}$) of the tripartite GHZ state. In fact, the maximal eigenvalue of R_{abc} is an entanglement witness, since it will be zero only when none entanglement is present.

Acknowledgments

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