Unconditionally Stable Finite-Difference Time-Domain Method Based on the Locally-One-Dimensional Technique

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Abstract— **A new unconditionally stable finite difference time domain (FDTD) technique is discussed. The method employs the locally one dimensional (LOD) operator splitting technique. The resulting LOD-FDTD method is computationally more efficient than the conventional FDTD, and presents a computational cost similar to the ADI-FDTD. The proposed LOD-FDTD is expanded in terms of the Crank-Nicolson scheme that is unconditionally stable and second order accurate. We illustrate the application of this new technique to the modeling of integrated optical waveguides.**

Keywords— **Finite difference time domain, FDTD, locally one-dimensional technique, LOD, Crank-Nicolson, waveguides.**

I. INTRODUCTION

The finite-difference time-domain (FDTD) is a very popular method for the analysis of transient electromagnetic fields in a myriad of communication devices [1,2]. Even though the FDTD method has been successfully employed for communication devices in the microwave and mm-wave range, its computational performance for optical communication devices is seriously hampered by the Courant-Friedrich-Levy (CFL) stability condition [2]. The CFL condition imposes a stringent bound in the time step size, with a direct impact on the simulation of electromagnetic devices which are large with respect to the wavelength of operation.

New approaches have recently appeared in the literature aiming at circumventing this problem, such as the alternating direction implicit FDTD (ADI-FDTD) scheme introduced by Namiki [3] in two dimensions (2-D) based on the formalism first developed by [4,5]. The ADI-FDTD method is not restricted by the Courant stability criterion and therefore it allows much larger time steps compared to the conventional FDTD approach. Zheng [6] and Namiki [7] later extended ADI-FDTD to three-dimensions (3-D). Even though ADI-FDTD presents unconditional stability, numerical dispersion studies have shown that the accuracy of the ADI-FDTD is degraded with the use of large time steps beyond the CFL limit [8]-[10]. This problem was mitigated by Rao [11] with a new ADI-FDTD approach based on the envelope of the field. Since then, FDTD methods that are not restricted by the CFL stability condition have become increasingly popular for the investigation of electromagnetic problems.

This paper introduces a new unconditionally stable FDTD approach based on the locally one dimensional (LOD) operator splitting technique [12], denoted here as LOD-FDTD. Similarly to the ADI-FDTD, the LOD-FDTD reduces considerably the computational effort since much larger time steps can be utilized. Furthermore, the LOD technique greatly simplifies the expansion of the discrete formalism [12].

This paper is organized as follows. Section II presents the derivation of the LOD-FDTD formalism. Section III addresses the validation of the method via a comparison against conventional FDTD and ADI-FDTD results. Finally, some conclusions and perspectives for future work are outlined in Section IV.

II. LOD-FDTD FORMALISM

For simplicity, we consider here TE modes in 2-D. Maxwell's equations in this case can be written as [3]:

$$-\mu\frac{\partial H_z}{\partial t}=\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y} \quad (1)$$

$$\varepsilon\frac{\partial E_x}{\partial t}=\frac{\partial H_z}{\partial y} \quad (2)$$

$$\varepsilon\frac{\partial E_y}{\partial t}=-\frac{\partial H_z}{\partial x} \quad (3)$$

The LOD-FDTD utilizes the same basic principles of the conventional LOD technique introduced in [12], by which a multidimensional equation is split out in successive steps along separate coordinates. In each step, only one *space* dimension is considered.

The LOD-FDTD applied to eqs. (1)-(3) will result in two different propagation sub-steps. The splitting is achieved as follows:

$$\varepsilon\frac{\partial E_y}{\partial t}=-\frac{\partial H_z}{\partial x} \quad (4)$$

$$-\frac{1}{2}\mu\frac{\partial H_z}{\partial t}=\frac{\partial E_y}{\partial x} \quad (5)$$

$$\varepsilon\frac{\partial E_x}{\partial t}=\frac{\partial H_z}{\partial y} \quad (6)$$

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$$-\frac{1}{2}\mu\frac{\partial H_z}{\partial t} = -\frac{\partial E_x}{\partial y} \tag{7}$$

where (4) and (5) are utilized in the first sub-step, while (6) and (7) in the second. Next, the finite difference discretization is carried out via the Crank-Nicolson (CN) scheme [13]. Equation (4) is discretized as follows:

$$\varepsilon\frac{\partial E_y}{\partial t}\Big|^{n+\frac{1}{2}} = -\frac{\partial H_z}{\partial x}\Big|^{n+\frac{1}{2}}$$

$$\varepsilon\frac{E_{y_{i,j+\frac{1}{2}}}^{n+1} - E_{y_{i,j+\frac{1}{2}}}^{n}}{\Delta t} = -\frac{H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}} - H_{z_{i-\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}}}{\Delta x}$$

$$E_{y_{i,j+\frac{1}{2}}}^{n+1} = E_{y_{i,j+\frac{1}{2}}}^{n} - \alpha_1\left(H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}} + H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n} - H_{z_{i-\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}} - H_{z_{i-\frac{1}{2},j+\frac{1}{2}}}^{n}\right) \tag{8}$$

Eq. (5) is discretized as:

$$-\mu\frac{\partial H_z}{\partial t}\Big|^{n+\frac{1}{4}} = \frac{\partial E_y}{\partial x}\Big|^{n+1}$$

$$-\frac{1}{2}\mu\frac{\left(H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}} - H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}\right)}{\frac{\Delta t}{2}} = \frac{E_{y_{i+1,j+\frac{1}{2}}}^{n+1} - E_{y_{i,j+\frac{1}{2}}}^{n+1}}{\Delta x}$$

$$H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}} = H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n} - \alpha_2\left(E_{y_{i+1,j+\frac{1}{2}}}^{n+1} + E_{y_{i+1,j+\frac{1}{2}}}^{n} - E_{y_{i,j+\frac{1}{2}}}^{n+1} - E_{y_{i,j+\frac{1}{2}}}^{n}\right) \tag{9}$$

and

$$H_{z_{i-\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}} = H_{z_{i-\frac{1}{2},j+\frac{1}{2}}}^{n} - \alpha_2\left(E_{y_{i,j+\frac{1}{2}}}^{n+1} + E_{y_{i,j+\frac{1}{2}}}^{n} - E_{y_{i-1,j+\frac{1}{2}}}^{n+1} - E_{y_{i-1,j+\frac{1}{2}}}^{n}\right) \tag{10}$$

Substituting (9) and (10) into (8), results:

$$\begin{aligned}&-\alpha_1\alpha_2 E_{y_{i+1,j+\frac{1}{2}}}^{n+1} + (1+2\alpha_1\alpha_2)E_{y_{i,j+\frac{1}{2}}}^{n+1} - \alpha_1\alpha_2 E_{y_{i-1,j+\frac{1}{2}}}^{n+1} = \\ &(1-2\alpha_1\alpha_2)E_{y_{i,j+\frac{1}{2}}}^{n} + \alpha_1\alpha_2\left(E_{y_{i+1,j+\frac{1}{2}}}^{n} + E_{y_{i-1,j+\frac{1}{2}}}^{n}\right) - 2\alpha_1\left(H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n} - H_{z_{i-\frac{1}{2},j+\frac{1}{2}}}^{n}\right)\end{aligned} \tag{11}$$

where, $\alpha_1 = \frac{\Delta t}{2\varepsilon\Delta x}$ and $\alpha_2 = \frac{\Delta t}{2\mu\Delta x}$.

The second sub-step is obtained by similarly expanding eqs. (6) and (7). This results in:

$$\begin{aligned}&-\beta_1\beta_2 E_{x_{i+\frac{1}{2},j+1}}^{n+1} + (1+2\beta_1\beta_2)E_{x_{i+\frac{1}{2},j}}^{n+1} - \beta_1\beta_2 E_{x_{i+\frac{1}{2},j-1}}^{n+1} = \\ &(1-2\beta_1\beta_2)E_{x_{i+\frac{1}{2},j}}^{n} + \beta_1\beta_2\left(E_{x_{i+\frac{1}{2},j+1}}^{n} + E_{x_{i+\frac{1}{2},j-1}}^{n}\right) + 2\beta_1\left\{H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}} - H_{z_{i+\frac{1}{2},j-\frac{1}{2}}}^{n+\frac{1}{2}}\right\}\end{aligned} \tag{12}$$

where, $\beta_1 = \frac{\Delta t}{2\varepsilon\Delta y}$ and $\beta_2 = \frac{\Delta t}{2\mu\Delta y}$.

III. NUMERICAL RESULTS

This section presents a performance analysis of the present formalism against the conventional FDTD and the ADI-FDTD. For this purpose, we use the parameter CFLN (Courant-Friedrich-Levy number) to denote the ratio between the actual time step and the maximum time step allowed by the CFL stability condition in conventional FDTD, i.e.,

$$CFLN = \frac{\Delta t}{\Delta t_{\max}^{FDTD}} \tag{13}$$

where in 2-D.

$$\Delta t_{\max}^{FDTD} = \frac{1}{c\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}}} \tag{14}$$

The simulations were first performed in free space, where c is speed of light in vacuum. The discretization parameters in this case are listed in Table I.

TABLE I: FREE SPACE SIMULATION PARAMETERS.

L_x	L_y	Number of points in *x*	Number of points in *y*
687 cm	687 cm	100 to 400 *	100 to 400 *

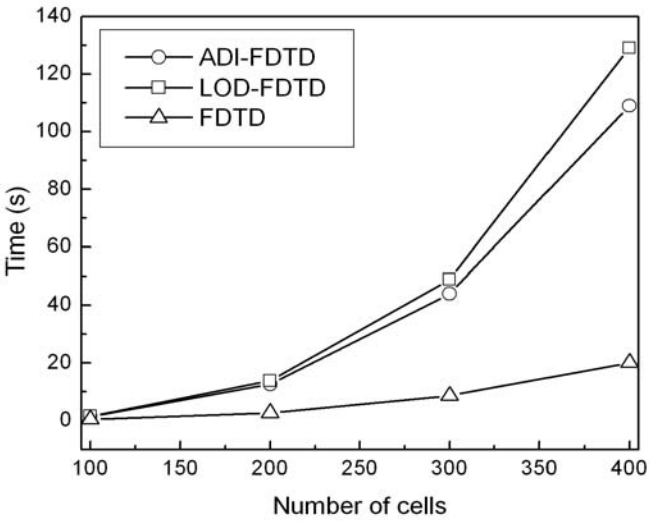
* The number of points varied from 100 to 400 in the simulations.

A soft-source is introduced at the center of the computational domain as follows:

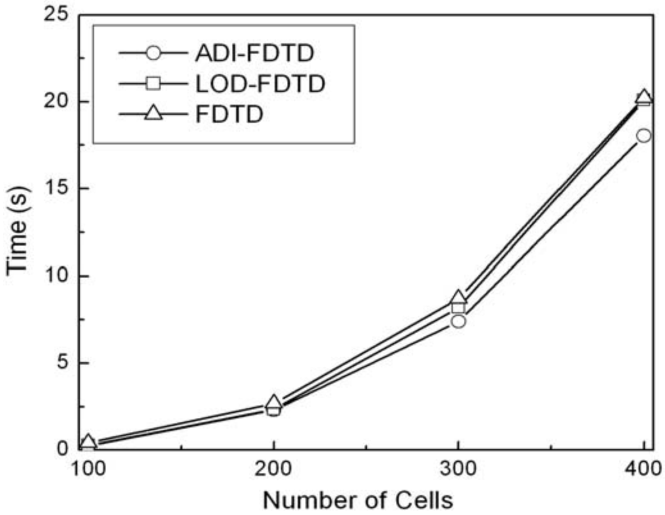
$$H_z = H_z + \sin^2\left(\frac{\pi t}{T}\right) \tag{15}$$

where T=9.4ns.

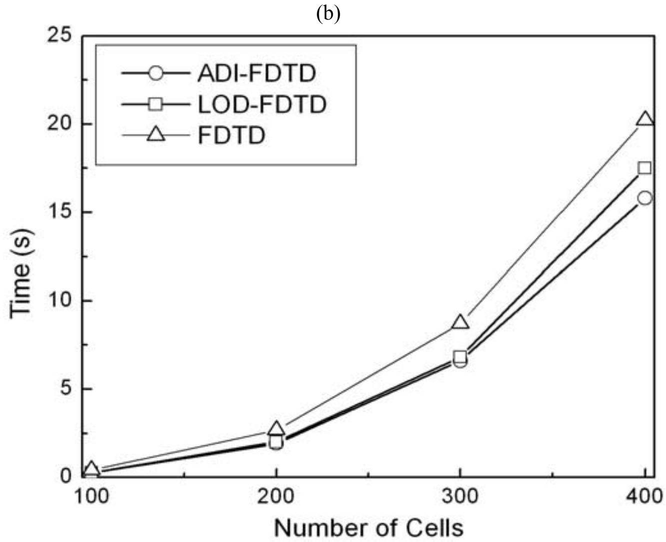
Numerical results for the CPU time required for the conventional FDTD, the ADI-FDTD from [3], and the present LOD-FDTD are shown in Fig. 1(a)-(c). These results show the overall CPU time required to simulate the same structure for different grid cell sizes. These results show that, in this case, the LOD-FDTD CPU time cost with CFLN=6 is equivalent to the conventional FDTD using the same discretization parameters. For a CFLN=7, the LOD-FDTD presents a smaller time cost than the conventional FDTD. When compared to the ADI-FDTD, the time costs are similar, with a slight disadvantage for the LOD-FDTD. However, it is important to notice that the ADI-FDTD scheme utilized here [3] exhibits increasing numerical dissipation for larger CFLN, which degrades the simulation results of long waveguide structures. This problem, however, is not present in the proposed LOD-FDTD, as illustrated next. It is worth mentioning that operator splitting techniques, such as ADI and LOD, cause the method to become implicit. Consequently, the solution of two tri-diagonal linear systems is required in each iteration [4]. In explicit methods, such as FDTD, this is not necessary, resulting in lower computational cost for same CFLN as shown in Fig. 1(a).



(a)



(b)



(c)

Fig. 1: Comparison of the computational effort for the conventional FDTD and the ADI-FDTD methods with the present LOD-FDTD. (a) CFLN=1, (b) CFLN=6, and (c) CFLN=7.

Next, field propagation along the *z*-varying three-layer integrated optical waveguide illustrated in Fig. 2 is simulated in order to demonstrate the efficiency and versatility of the proposed LOD-FDTD. The pertinent parameters relative to the computational domain and waveguide structure are listed in Tables II and III, respectively. We compare the simulation results obtained with LOD-FDTD against ADI-FDTD and standard FDTD.

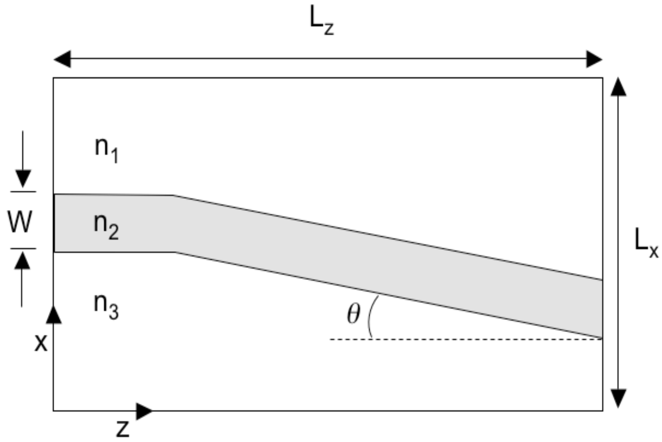


Fig. 2: Longitudinally varying three layer waveguide.

TABLE II: COMPUTATIONAL DOMAIN PARAMETERS

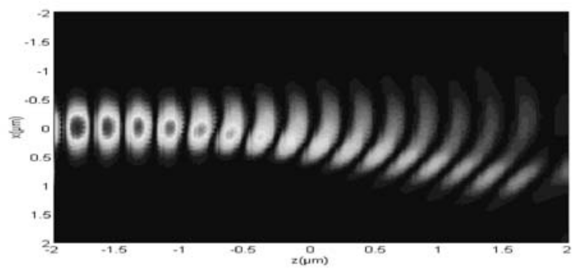
$N_x \times N_z$	$\Delta x\,(\mu m)$	$\Delta z\,(\mu m)$
400×4000	0.01	0.01

TABLE III: PHYSICAL AND GEOMETRICAL PARAMETERS RELATIVE TO SLAB WAVEGUIDE STRUTURE. ($\lambda = 1.55\mu\, m$).

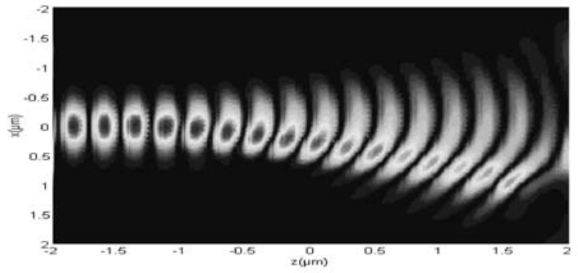
$\lambda\,(\mu m)$	$w\,(\mu m)$	n_1	n_2
1.55	0.5	3.4	3.1

The simulation results obtained for bending angle θ=16^o are shown in Fig. 3(a), (b), and (c) for the ADI-FDTD, LOD-FDTD, and standard FDTD respectively. The initial pulse profile E_0 is assumed as the fundamental mode of the planar waveguide multiplied by a time sinusoid given by: $H_z = E_0 \cdot \sin(\omega T)$, where $\lambda = 1.55 \mu m$.

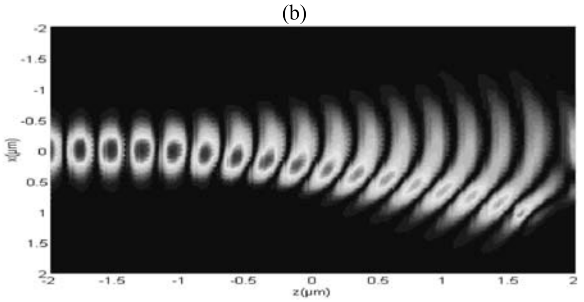
We note that the ADI-FDTD result presents a strong numerical dissipation for CFLN=5. On the other hand, the LOD-FDTD result with CFLN=7 does not show numerical dissipation, and yet it agrees quite well with the conventional FDTD. As a result, the proposed LOD-FDTD scheme appears to be as a good alternative for the simulation of electromagnetic structures where obeying the CFL criterion would otherwise make the simulation too costly.



(a)



(b)



(c)

Fig. 3: Evolution of the fundamental mode along a longitudinally varying three layer waveguide. The tilting angle $\theta=16^o$. (a) ADI-FDTD with CFLN=5, (b) present LOD-FDTD with CFLN=7, (c) Standard FDTD CFLN=1. The tilting occurs at Lz/8.

IV. CONCLUSIONS

This paper introduced a novel FDTD formalism for the time-domain simulation of electrically large electromagnetic structures. This method employs the locally one-dimensional (LOD) technique in conjunction wit the Crank-Nicolson (CN) discretization. The resulting LOD-FDTD is able to produce more efficient simulations than the conventional FDTD in structures requiring stringent time steps from the CFL criterion. For the same discretization parameters, the CPU time cost of the present LOD-FDTD formalism was slightly worse than that obtained with a conventional ADI-FDTD. However, this latter scheme exhibits numerical dissipation that can degrade the results for long integration times.

The authors are currently working on the extension of the LOD-FDTD to investigate electromagnetic metamaterials exhibiting anisotropic and frequency dispersive behavior. Higher-order approaches in terms of the envelope approximation and extension to 3-D are also under consideration.

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