Fourier Codes and Hartley Codes

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***Abstract*—Real-valued block codes are introduced, which are derived from Discrete Fourier Transforms (DFT) and Discrete Hartley Transforms (DHT). These algebraic structures are built from the eigensequences of the transforms. Generator and parity check matrices were computed for codes up to block length *N*=24. They can be viewed as lattices codes so the main parameters (dimension, minimal norm, area of the Voronoi region, density, and centre density) are computed. Particularly, Hamming-Hartley and Golay-Hartley block codes are presented. These codes may possibly help an efficient computation of a DHT/DFT.**

***Index Terms*—-discrete transforms, DFT, Hartley-DHT, real block codes, eigensequences, lattices.**

***Resumo*—Códigos de bloco sobre os reais construídos a partir de Transformadas discretas de Fourier (DFT) e de Hartley (DHT) são apresentados. Estas estruturas algébricas são construídas usando autoseqüências das transformadas. Matrizes geradoras e de verificação de paridade foram calculadas para códigos até comprimento *N*=24. Estes códigos podem ser encarados como reticulados de modo que seus principais parâmetros foram avaliados (dimensão, norma mínima, área da região de Voronoi e densidade de centros). Em particular, dois códigos ditos de Hamming-Hartley e Golay-Hartley são apresentados. Estes códigos podem vir a auxiliar em um cálculo eficiente de DHT/DFT.**

***Palavras-Chave*—transformadas discretas, DFT, Hartley-DHT, códigos de bloco reais, autoseqüências, reticulados.**

I. INTRODUCTION

Discrete transforms over finite or infinite fields have long been used in the Telecommunication field to achieve different goals. Transforms have significant applications on subjects such as channel coding, cryptography and digital signal and image processing. They were primary conceived to perform an efficient and fast numerical spectral analysis [1], [2]. However, several new applications emerged, which are transformed-based. The better known example is the Discrete Fourier Transform (DFT). Another very rich transform related to the DFT is the Discrete Hartley Transform (DHT) [3], [4]. The DHT has proven over the years to be a powerful tool. A version of the DFT for finite fields was introduced by Pollard in early 70´s [5] and applied as a tool to perform discrete convolutions using integer arithmetic. Some particular transforms, mainly the DHT, have found interesting applications in the optical domain [6], [7]. New digital multiplex systems have been proposed, which are derived from finite field transforms [8]. Recent promising applications of discrete transforms concern the use of transforms to design digital multiplex systems and spread spectrum sequences [9], [10]. This approach exploits orthogonality properties of synchronous multilevel sequences defined over a complex finite field [11] as a tool of spreading sequence design. Additionally, finite field transforms were offered to implement efficient multiple access systems [9], [12].

The uppermost application of discrete transforms is definitely in signal and image compression, where they are the basis of countless algorithms [13]. The decoding of DTMF signals also is always performed on the basis of transforms [14], [15]. Discrete transforms have been successfully applied in error control coding schemes, both in the design of new codes and decoding algorithms [16], [17], [18], [19].

Some cryptographic systems have been devised that exploits discrete transforms [20]. The discrete multitone (DMT) systems are essentially based on DFTs [21]. A related type of modulation, the orthogonal frequency division multiplex (OFDM multicarrier systems), has been effectively applied in digital broadcast and wireless channel communication [22], [23], [24]. They are also a very efficient tool for spectral monitoring, therefore are extensively used in spectral managing [25], [26].

Eigenfunctions of discrete transforms have been one of the focus of several studies [27], [28], [29]. This analysis recently derived new multi-user systems [30]. This paper links the eigenvalues of discrete transforms [31] with the design of block codes defined over the field of real numbers [32], [33]. The structure and parameters of the real codes derived from this approach are presented, and they are put on the lattice codes framework [34], [35], showing that *discrete transforms* have connection, somewhat unexpected, with *lattice theory*.

The codebook of these codes is simply a list of all eigensequences associated with a particular real-valued eigenvalue of the corresponding discrete transform [31]. The generator (G) and parity check (H) matrices are introduced for such codes, which play a role somewhat similar to the standard G and H matrices of block codes [36].

The paper is organized as follows. Section II presents the major concepts and makeup of the transformed-based real block codes, with focus on the so-called Fourier codes. Section III shows several codes derived from discrete Hartley transforms (named Hartley codes). Concluding remarks are presented in Section IV.

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II. ARE DISCRETE TRANSFORMS RELATED TO REAL BLOCK CODES?

Let $[W]_N := \left(\exp(-j\frac{2\pi}{N}k.n) \right)$ be the $N \times N$ DFT matrix, where k,n=0,1,2,…,N-1. If $x[n]$ is an eigensequence of the linear transform W, then its spectrum (Fourier spectrum) is given by:

$$[W]_N\, x[n] = \lambda x[n]\,. \qquad (1)$$

The four possible eigenvalues of a DFT with blocklength N, λ, are only $\pm\sqrt{N}, \pm j\sqrt{N}$ [29],[30]. Each one of the four eigenvalues engender a linear subspace of eigensequences of blocklength N, denoted by $V_N^+, V_N^-, V_N^j, V_N^{-j}$, respectively [30].

Since we intend to deal solely with "real codes", just the real-valued eigenvalues are of interest. Accordingly, the eigensequences $x[n]$ must hold the relationship

$$[W]_N\, x[n] = \pm\sqrt{N}.x[n]\,, \qquad (2)$$

so that $x[n]\left([W]_N \mp \sqrt{N} I_N\right)^T = 0$, where the sign depends on if the eigenvalue is positive or negative, respectively. As a result, the matrix $H^T := [W]_N \mp \sqrt{N} I_N$ plays a role to some extent analogous to the parity check matrix of standard block codes.

$$\text{Here } N\text{-}K = rank([W]_N \mp \sqrt{N} I_N). \qquad (3)$$

It seems appropriate to adopt a notation corresponding to the standard approach of block codes, that is,

$$H^T = [I_{N-K} \,\vdots\, P_\lambda] \qquad (4)$$

in such a way that the generator matrix of the real code can be put under the format

$$G = [-P_\lambda^T \,\vdots\, I_K]\,. \qquad (5)$$

Two block codes of parameters $[N,K^+]$ and $[N,K^-]$ may be generated, where

$$K^{\mathrm{sgn}(\lambda)} = N - rank\left([W]_N - \lambda I_N\right),\ \lambda = \pm\sqrt{N}\,, \qquad (6)$$

where $sgn(.)$ is the usual signal function.

Example 1. As a naïve and clarifying example, let us consider the Fourier codes of length N=4. We start with the matrix

$$\left([W]_N - \lambda I_N\right) = \begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & -(2+j) & -1 & j \\ 1 & -1 & -1 & -1 \\ 1 & j & -1 & -(2+j) \end{vmatrix} \qquad (7)$$

Thus, $K^+ = 2$. In standard echelon form, the "parity check" matrix, is $\begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & -1 \end{vmatrix}$, so we have the following two Fourier codes:

$$\text{F: [4,2]} \quad G_4^+ = \begin{vmatrix} 1 & 0 & \vdots & 1 & 0 \\ 2 & 1 & & 0 & 1 \end{vmatrix} \qquad (8)$$

$$\text{F: [4,1]} \quad G_4^- = \begin{vmatrix} -1 & 1 & 1 & \vdots & 1 \end{vmatrix}. \qquad (9)$$

Codes conceived by this approach can be interpreted as lattice codes [37], [35].

The corresponding codes engendered by the unitary Hartley transform of length four have the following matrices:

$$\text{H: [4,3]} \quad G_4^+ = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} \qquad (10)$$

$$\text{H: [4,1]} \quad G_4^- = \begin{vmatrix} -1 & 1 & 1 & \vdots & 1 \end{vmatrix}. \qquad (11)$$

Incidentally, these generator matrices corresponds exactly to the lattices D_3 (checkerboard lattice) and A_1, the best packing in dimension 3 and 1, respectively [34], [35]. It seems to be appealing to make use of the unitary form of the DFT/DHT operator. However, lattice codes thus derived are tantamount to those obtained from the standard DHT. Fourier codes parameters were computed for blocklengths up to 24 and the results are summarized in Table I. The density achieved by the corresponding sphere packing is indicated.

TABLE I. Fourier lattice Codes on an N-dimensional Euclidean space. K is the dimension, μ the minimal norm, and Δ the density of the associated lattice packing.

N	K^+	μ^+	Δ^+	K^-	μ^-	Δ^-
3	1	9.4641	1	1	2.5359	1
4	2	2	0.55536	1	4	1
5	2	4	0.82582	1	5.5279	1
6	2	3.5505	0.56921	2	3.5505	0.56921
7	2	2.8931	0.37722	2	2.4577	0.44763
8	3	3	0.26029	2	2.3431	0.42505
9	3	2.7852	0.26073	2	2.5014	0.43446
10	3	2.7906	0.25213	3	6.8377	0.40055
11	3	2.9203	0.2483	3	5.4021	0.42541
12	4	4	0.11577	3	4.906	0.44661
13	4	3.4628	0.12822	3	4.2503	0.39718
14	4	3.5376	0.15731	3	3.8717	0.30403
15	4	3.4413	0.15837	4	7.418	0.21238
16	4	4.2625	0.1412	3	3.2543	0.27218
17	5	5.3824	0.089659	4	2.9451	0.063149
18	5	4.8172	0.096808	4	3.0745	0.069407
19	5	4.7151	0.11403	4	3.304	0.088451
20	5	5.0325	0.13599	4	4.1055	0.15164
21	5	4.9701	0.10052	4	3.6455	0.14034
22	6	7.2991	0.071307	5	8.387	0.10898
23	6	5.0715	0.034921	5	8.6131	0.15042
24	6	4.0909	0.021066	5	5.777	0.1134

Further parameters for the Fourier Codes defined over the Euclidean space are shown in Table II, including the determinant of the Gram matrix [35], and their centre densities. In the next section, codes derived from the discrete Hartley transform (DHT) are introduced. Long codes (at high dimension) can easily be designed by using long blocklength discrete transforms.

TABLE II. Fourier Codes on an N-dimensional Euclidean space. Further parameters: δ is the centre density of the associated lattice packing, and det(Λ) is the volume of the Voronoi region.

N	K^+	δ^+	det Λ^+	K^-	δ^-	det Λ^-
3	1	0.5	3.0764	1	0.5	1.5925
4	2	0.17678	2.8284	1	0.5	2
5	2	0.26287	3.8042	1	0.5	2.3511
6	2	0.18119	4.899	2	0.18119	4.899
7	2	0.12007	6.0237	2	0.14249	4.3122
8	3	0.06214	10.453	2	0.1353	4.3296
9	3	0.062245	9.3343	2	0.13829	4.5221
10	3	0.060192	9.6812	3	0.095625	23.373
11	3	0.059278	10.524	3	0.10156	15.454
12	4	0.023459	42.628	3	0.10662	12.74
13	4	0.025982	28.844	3	0.094819	11.552
14	4	0.031878	24.536	3	0.072582	13.12
15	4	0.032093	23.062	4	0.043038	79.91
16	4	0.028614	39.685	3	0.064979	11.294
17	5	0.017033	123.31	4	0.012797	42.364
18	5	0.018391	86.543	4	0.014065	42.005
19	5	0.021663	69.642	4	0.017924	38.066
20	5	0.025835	68.723	4	0.030728	34.283
21	5	0.019096	90.122	4	0.028438	29.207
22	6	0.013799	440.35	5	0.020704	307.47
23	6	0.0067576	301.61	5	0.028577	238.09
24	6	0.0040764	262.42	5	0.021544	116.35

III. BLOCK-CODES LIKE STRUCTURES DERIVED FROM HARTLEY TRANSFORMS: A TABLE OF HARTLEY CODES

Let $[H]_N := \left(cas(\frac{2\pi}{N} k.n) \right)$ n,k=0,1,…,N-1 be the $N\times N$ DHT matrix. The (real) casoidal Hartley kernel is $cas(x):=cos(x)+sin(x)$ as usual. Then the eigensequences $x[n]$ of the DHT must hold $[H]_N\, x[n] = \pm\sqrt{N}.x[n]$, so that $x[n]\left([H]_N \mp \sqrt{N} I_N\right)^T = 0$, where the sign depends on if the eigenvalue is positive or negative, respectively. Again, the matrix $H^T := [H]_N \mp \sqrt{N} I_N$ plays a role somewhat similar to the parity check matrix of eigensequences. Here again

$$N\text{-}K = rank([W]_N \mp \sqrt{N} I_N). \qquad (12)$$

Hartley codes have dimensions given by

$$K^+ = \left\lceil \frac{N}{2} \right\rceil + \delta_{N \bmod 4, 0} \text{ and } K^- = \left\lfloor \frac{N}{2} \right\rfloor - \delta_{N \bmod 4, 0}, \qquad (13)$$

where $\delta_{k,l}$ denotes the Kronecker symbol. It is worthwhile to remark that the "rate" $K^{\pm}/N$ of these codes is always about a half.

It is worthwhile to mention that the lattices Λ and $\Lambda^{\perp}$ corresponding to the codes [N,K^+] and [N,K^-] are dual lattices. Furthermore, the computational complexity required to compute the parameters of Hartley codes is much higher than that required by Fourier codes of the same length.

Corollary. The matrix of the DHT of blocklength N has only eigenvalues +1 and -1 and they are repeated $\left\lceil \frac{N}{2} \right\rceil + \delta_{N \bmod 4, 0}$ times and $\left\lfloor \frac{N}{2} \right\rfloor - \delta_{N \bmod 4, 0}$ times, respectively.

The generator and the parity check matrices of the real block code of length N associated with the eigenvalue λ of the DFT (DHT) will be denoted by ${}_s G_N^{\mathrm{sgn}(\lambda)}$ and ${}_s H_N^{\mathrm{sgn}(\lambda)}$, where s denote Fourier or Hartley code $s \in \{F, H\}$, and $\mathrm{sgn}(\lambda) \in \{+,-\}$.

Example 2. As a naïve and illustrative example, let us consider the Hartley codes of block length 4.

$${}_H G_4^- = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{vmatrix} \text{ and } ({}_H H_4^-)^T = \begin{vmatrix} -1 & 1 & 1 \vdots 1 \end{vmatrix}.$$

Let also build a Hartley code of block length N=7 associated with the eigenvalue $-\sqrt{7}$. The generator matrix is:

$${}_H G_7^- = \begin{bmatrix} 1 & 0 & 0 & 0 & 0.2540 & 0.4240 & 0.9678 \\ 0 & 1 & 0 & 0 & -0.6697 & -0.2831 & -0.0472 \\ 0 & 0 & 1 & 0 & 1.2068 & -0.4899 & -1.7169 \\ 0 & 0 & 0 & 1 & -0.4629 & 0.2270 & -0.7641 \end{bmatrix}$$

A particular and remarkable case occurs for blocklengths for which there exists eigensequences with integer components. We have shown that this happens when $N=m^2$ and this is illustrated in the example in the sequel.

Example 3. For instance, take the case of the DHT N=9 and select the eigenvalue λ=1. The Hartley code [9,5] has generator matrix ${}_H G_9^+$ given by

$$\begin{matrix} 6.2150 & 6.1346 & 2.7832 & 2.5123 & 1 & 0 & 0 & 0 & 0 \\ -2.0501 & -2.7832 & -0.3139 & -2.0030 & 0 & 1 & 0 & 0 & 0 \\ 1.0000 & 0.0000 & 0.0000 & 1.0000 & 0 & 0 & 1 & 0 & 0 \\ -3.8490 & -5.1346 & -2.7832 & -0.7802 & 0 & 0 & 0 & 1 & 0 \\ 2.6840 & 2.7832 & 1.3139 & 0.2710 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

so there are sequences with all-integer components such as:

[1 0 0 1 0 0 1 0 0], [10 1 1 7 1 1 7 1 1], [10 1 1 1 1 1 1 1 1].

Example 4. The Hamming-Hartley and Golay-Hartley codes. The generator matrix of the real Hamming-Hartley code [N,K^+]=[7,4] is

$${}_H G_7^+ = \begin{vmatrix} 3.9372 & 3.81064 & 1.66971 & 1 & 0 & 0 & 0 \\ 1.82300 & 1 & 1 & 0 & 1 & 0 & 0 \\ -4.75175 & -6.31546 & -2.50481 & 0 & 0 & 1 & 0 \\ 2.63705 & 2.50481 & 0.83511 & 0 & 0 & 0 & 1 \end{vmatrix}$$

The generator matrix of the real Golay-Hartley code [N,K^+]=[23,12] is given in Appendix.

TABLE III. Hartley Codes on an N-dimensional Euclidean space. K is the dimension, μ the minimal norm, and Δ the density of the associated lattice packing.

N	K^+	μ^+	Δ^+	K^-	μ^-	Δ^-
3	2	2	0.7221	1	2.5359	1
4	3	2	0.74048	1	4	1
5	3	4	0.42552	2	1.5814	0.4614
6	3	4.9148	0.21497	3	1.3924	0.24623
7	**4**	3.3937	0.19977	**3**	1.5515	0.22407
8	5	2	0.062949	3	2.3431	0.30672
9	5	3	0.054289	4	1.7306	0.12155
10	5	5.2063	0.075916	5	1.8957	0.077169
11	6	3.7751	0.052394	5	2.6115	0.14192
12	7	2.5744	0.014886	5	2.2205	0.066449
13	7	3.0709	0.0099451	6	1.8702	0.024582
14	6	2	0.010263	7	2.2839	0.02135
15	8	4.1987	0.015084	7	2.1276	0.014352
16	9	2.7463	0.0022665	7	2.2606	0.012029
17	9	3.3358	0.0021189	8	1.9687	0.00391907
18	9	4.7288	0.0033659	9	2.5477	0.0047677
23	12	-	-	11	-	-

TABLE IV. Hartley Codes on an N-dimensional Euclidean space. Further parameters: δ is the centre density of the associated lattice packing, and det(Λ) is the volume of the Voronoi region.

N	K^+	δ^+	det Λ^+	K^-	δ^-	det Λ^-
3	2	0.22985	2.1753	1	0.5	1.5925
4	3	0.17678	2	1	0.5	2
5	3	0.10159	9.8438	2	0.14687	2.6919
6	3	0.051319	26.54	3	0.058782	3.494
7	**4**	0.040483	17.781	**3**	0.053494	4.5157
8	5	0.011959	14.782	3	0.073223	6.1229
9	5	0.010314	47.233	4	0.024631	7.5998
10	5	0.014422	134.01	5	0.01466	10.547
11	6	0.010139	82.915	5	0.026962	12.774
12	7	0.0031505	67.882	5	0.012624	18.189
13	7	0.0021049	188.36	6	0.0047569	21.488
14	6	0.001986	62.94	7	0.0045188	31.127
15	8	0.003716	326.64	7	0.030375	36.13
16	9	0.00068728	267.95	7	0.0025442	53.298
17	9	0.00064239	687.59	8	0.00006449	60.769
18	9	0.00000161	2081.2	9	0.00005246	90.866
23	12	-	-	11	-	-

A glimpse on the properties of these lattices may be a bit disappointing, because they do not provide dense packings. Nonetheless, it is significant to stress at this point, that the aim of this paper is not to find dense packings or thin coverages. After all, this task is recognized as an exceptionally hard one.

IV. CONCLUSIONS

The aim of this paper is to launch an alternative approach for the designing of block codes over the field of real numbers. The most relevant issue here is to establish a structure in terms of eigensequences, which may possibly help an efficient computation of a DHT/DFT by partitioning the transform into "sub-transforms" defined over the invariant spaces V_N^+, V_N^-. Long codes can easily be designed by using long blocklength discrete transforms. In spite of the fact that this paper is only a preliminary investigation on the link between discrete transforms and lattices, it opens a path for unusual applications of the lattice theory. In particular, Discrete Hartley transform matrices seem to have a strong connection with lattice construction. The structure of these codes was briefly examined, but their performance over noisy channels was not addressed [38]. For instance, for the additive white Gaussian channel not only the lattice density plays a role on the error performance, but also the number of neighbours (contact number). At any rate, it must be said that these are not codes intended for the Gaussian additive noise channel. Further transforms close related to the DHT such as the Generalized-DHT should also be scrutinized to find further lattice codes. The particular mathematical arrangement behind such codes may potentially assist the implementation of fast transforms using trellis, a topic which is currently under investigation. Finally, the search for finite groups that lie hidden in these lattices is also another subject for upcoming researches.

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APPENDIX Real parity check matrix of the Golay-Hartley code, ${}_{H}H_{23}^{+}$.

1	0	0	0	0	0	0	0	0	0	0	0	0.0707	0.1328	0.1806	0.2052	0.2576	0.2725	0.3446	0.3546	0.4865	0.4931	0.9975
0	1	0	0	0	0	0	0	0	0	0	0	-0.2096	-0.3738	-0.4600	-0.4311	-0.4377	-0.2763	-0.2355	0.0703	0.0802	0.8307	0.4427
0	0	1	0	0	0	0	0	0	0	0	0	0.3139	0.4952	0.4639	0.1856	-0.0295	-0.4240	-0.4346	-0.6661	0.0749	-0.3157	-0.6636
0	0	0	1	0	0	0	0	0	0	0	0	-0.3631	-0.4411	-0.1470	0.3930	0.5700	0.6670	0.0734	0.2235	-0.6462	-1.1716	-0.1579
0	0	0	0	1	0	0	0	0	0	0	0	0.4208	0.3418	-0.1762	-0.6303	-0.1794	0.2343	1.1547	0.1737	-1.0834	-0.7214	-0.5346
0	0	0	0	0	1	0	0	0	0	0	0	-0.4200	-0.0859	0.5313	0.5639	-0.3692	0.0609	0.1053	-0.1063	0.0355	-0.8583	-0.4572
0	0	0	0	0	0	1	0	0	0	0	0	0.4704	-0.0758	-0.4440	0.1100	1.1463	-0.2617	-1.2212	0.2180	0.2166	-0.4859	-0.6726
0	0	0	0	0	0	0	1	0	0	0	0	-0.4108	0.4409	0.4098	0.0483	-0.0264	-0.1150	-0.3806	-0.5899	0.5268	-0.2818	-0.6212
0	0	0	0	0	0	0	0	1	0	0	0	0.4964	-0.4527	0.5468	0.4790	-1.2225	0.1046	0.1456	-0.6493	0.1306	0.1316	-0.7100
0	0	0	0	0	0	0	0	0	1	0	0	-0.3338	1.1216	-0.3070	-0.7009	0.4065	-0.8335	0.5190	-0.3977	-0.1826	0.3034	-0.5947
0	0	0	0	0	0	0	0	0	0	1	0	0.7645	-0.5050	-0.1304	-0.4541	0.0619	-0.0811	-0.2977	0.3274	-0.5944	0.4580	-0.5491
0	0	0	0	0	0	0	0	0	0	0	1	-0.1386	-0.2349	-0.3342	0.2470	-0.4127	0.3455	-0.4255	0.3412	-0.3778	0.2531	-0.2632