

Set-Membership Affine-Projection with Estimated Parameters for non-White Inputs

Tadeu N. Ferreira, Markus V. S. Lima, Paulo S. R. Diniz

Abstract— This article proposes versions of the set-membership affine-projection algorithm that estimate an adequate threshold and a proper constraint-vector for a system identification scenario. A novel constraint-vector is proposed for a non-white input, since the Simple-Choice Constraint Vector is asymptotically optimum for white inputs. Non-white inputs are commonplace in communication systems when there is channel encoding or precoding in the transmitter. A fixed and a time-varying threshold set-membership version are used. Simulations show that the proposed algorithm presents lower mean-square error and updates in fewer iterations than the traditional Simple-Choice and Exponential-Decay Constraint Vectors. For the time-varying threshold, the computational complexity of the algorithm is similar to the Simple-Choice Constraint Vector.

Keywords— Adaptive filter, set-membership, constraint vector, parameter estimation.

I. INTRODUCTION

The emergence of the Affine Projection algorithm (APA) [1] provided a data-reuse alternative to Least-Mean Squares (LMS) providing fast convergence. The price for a faster convergence is a higher computational complexity of APA in comparison to LMS. Then, some data selective algorithms have been developed, such as the Set-Membership Affine Projection (SM-AP) [2],[3], in order to avoid the updates in some iterations. The idea of Set-Membership Filtering (SMF) emerged in [4] with the membership set concept used in system modeling [5]. Both the original Optimal Bounding Ellipsoids (OBE) [5] and the SMF algorithms are based on noise modeling, particularly on previous knowledge of noise power.

The use of the constraint vector (CV) and of a threshold are internal characteristics of the SM-AP algorithm. The Simple-Choice CV (SC-CV) has become the hegemonic CV for the implementation of SM-AP [3], [8]. Recently, other CVs have been proposed to exploit specific characteristics of the problems as in [2], [7]. The development of the Optimal CV [6] has lead to a possible break of an SC-CV hegemony. Nevertheless, [6] shows that the SC-CV is asymptotically optimal CV for white input signals. The Optimal CV still relies on a predefined threshold, which requires knowledge of the measurement noise.

The use of time-varying thresholds for the SM-AP has its roots in the development of an SM-AP version with constant rate of updates [9]. A time-varying threshold is particularly

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interesting if the parameters are estimated, since a proper value of the threshold may be calculated adaptively.

The main proposal of this article is to design SM-AP versions based solely on the estimated parameters. One of the proposed algorithms estimates the adequate threshold for a given input and scenario using the Optimal CV in few iterations and designing the proper CV using structures already calculated by the coefficient updates. The other algorithm proposes a time-varying threshold based on the norm of the error vector. A secondary contribution of this article is to develop a CV for non-white input scenarios, such as for channel estimation when a convolutional encoding or a precoding is used in the transmitter.

In this article, Section II describes the SM-AP algorithm and the role of the threshold and the CV. Section III shows the proposed algorithm. In Section IV the main simulation results are presented to validate the proposed algorithm. Section V concludes the article. In this article, minuscule boldface is used for vectors, while capital letter bold symbols are used for matrices.

II. SET-MEMBERSHIP AFFINE PROJECTION ALGORITHM

Consider a tapped-delay line adaptive filter in the system identification configuration [2]. The input of the adaptive filter is $\mathbf{x}(k) \in \mathbb{R}^{N+1}$ at the k^{th} discrete sample time, where $\mathbf{x}(k) = [x(k-N) \ x(k-N+1) \ \dots \ x(k)]^T$. This input $\mathbf{x}(k)$ is also applied to the system whose impulse response $\mathbf{h}(k) \in \mathbb{R}^{N+1}$ is to be identified. The output of the system is $d(k) = \mathbf{x}^T(k)\mathbf{h}(k)$ [2]. The adaptive filter is a tapped-delay line characterized by its parameters, i.e., its coefficients, $\mathbf{w}(k) \in \mathbb{R}^{N+1}$. The output of the filter is $y(k) = \mathbf{x}^T(k)\mathbf{w}(k)$. The error signal is defined as $e(k) = d(k) - y(k) = d(k) - \mathbf{x}^T(k)\mathbf{w}(k)$.

In the APA [2], L previous scalar inputs are used together with the current scalar input $x(k)$, where L is known as data reuse factor. Then the matrix data is $\mathbf{X}(k) \in \mathbb{R}^{(N+1) \times (L+1)}$, where $\mathbf{X}(k) = [\mathbf{x}(k) \ \mathbf{x}(k-1) \ \dots \ \mathbf{x}(k-L)]$.

In the SMF approach, the parameter vector $\mathbf{w}(k)$ is estimated with an error which is bounded by a pre-defined threshold $\bar{\gamma}$. The Set-Membership Adaptive Recursive Techniques (SMART) [4] allowed the use of the SMF approach in online applications [6]. At the k^{th} iteration, a new input vector $\mathbf{x}(k)$ and a sample of the desired signal $d(k)$ are received by the adaptive filter. The pair $(\mathbf{x}(k), d(k))$ generates the constraint set $\mathbb{H}(k) = \{\mathbf{w} \in \mathbb{R}^{N+1} : |d - \mathbf{w}^T \mathbf{x}(k)| \leq \bar{\gamma}\}$. SM-AP uses the intersection of the previous L constraint sets and the current constraint set, that is, $\Psi_{k-L}^k \triangleq \bigcap_{i=k-L}^k \mathbb{H}(i)$. At each iteration, SM-AP chooses a new parameter vector $\mathbf{w}(k+1)$

belonging to Ψ_{k-L}^k , which is the closest one to $\mathbf{w}(k)$ for the SC-CV. In summary, SM-AP minimizes $\|\mathbf{w}(k+1) - \mathbf{w}(k)\|^2$ subjected to $\mathbf{w}(k+1) \in \Psi_{k-L}^k$ [3],[6].

A. Threshold and the Constraint-Vector

An important definition to the SM-AP algorithm is the constraint set, which is defined as $\mathbb{H}(k) = \{\mathbf{w} \in \mathbb{R}^{N+1} : |d(k) - \mathbf{w}^T(k)\mathbf{x}(k)| \leq \bar{\gamma}\}$ [6], where $\bar{\gamma}$ is the threshold. Therefore, if $|d(k) - \mathbf{w}^T(k)\mathbf{x}(k)| \leq \bar{\gamma}$ already holds for the current coefficient vector, then it is not going to be updated, i.e. $\mathbf{w}(k+1) = \mathbf{w}(k)$.

A cost function often used is $\|\mathbf{w}(k+1) - \mathbf{w}(k)\|^2$, which is minimized subjected to $\mathbf{d}(k) - \mathbf{X}^T(k)\mathbf{w}(k) = \gamma$, where γ is the CV. The CV defines the subset of Ψ_{k-L}^k where the projection is performed [7]. The coefficient updates in SM-AP is given by:

$$\mathbf{w}(k+1) = \begin{cases} \mathbf{w}(k) + \mathbf{X}(\delta\mathbf{I} + \mathbf{X}^T\mathbf{X})^{-1}(\mathbf{e}(k) - \gamma(k)) & |e(k)| > \bar{\gamma}, \\ \mathbf{w}(k) & |e(k)| \leq \bar{\gamma} \end{cases} \quad (1)$$

where $\mathbf{e}(k) = \mathbf{d}(k) - \mathbf{X}^T(k)\mathbf{w}(k)$ and δ is a regularization parameter. Parameter $\bar{\gamma}$ is typically set to $\sqrt{\tau\sigma_n^2}$ [2], where τ is usually chosen as 4 or 5 and σ_n^2 is the measurement noise variance, which is assumed to be known. In this paper, an estimation of the measurement noise is used.

Several CV $\gamma(k)$ exists [2]. The most popular CV is the Simple Choice CV (SC-CV), defined as $\gamma_{SC}(k) = [\bar{\gamma}\text{sign}(e(k)) \quad e(k-1) \quad \dots \quad e(k-N)]^T$ [2]. In [6], the SC-CV is proved to be asymptotically optimal for the white input case.

B. Optimal Constraint-Vector

Optimal CV [6] finds the CV that minimizes the SM-AP problem formulated as:

$$\min_{\gamma, \mathbf{w}(k+1)} \|\mathbf{w}(k+1) - \mathbf{w}(k)\|^2 + \frac{1}{\delta} \|\mathbf{d}(k) - \mathbf{X}^T(k)\mathbf{w}(k)\|, \quad (2)$$

s. t. $\|\gamma(k)\|_\infty \leq \bar{\gamma}.$

Eq. (2) allows as possible results the CVs whose elements are less than the threshold $\bar{\gamma}$, which leads to the use of the original constraint $\mathbf{w}(k+1) \in \Psi_{k-L}^k$ of the SM-AP.

In general, the choice of $\bar{\gamma}$ and $\gamma(k)$ in SM-AP is empirical. The Optimal CV [6] leads to a more mathematical approach, but it presents a high computational burden.

The use of the Optimal CV generates CVs whose largest element may be smaller than the pre-defined $\bar{\gamma}$. It indicates that the modeling for $\bar{\gamma}$ may be changed based on the Optimal CV. The choice of an Optimal CV may not lead to a projection which has a distance of $\bar{\gamma}$ from the a posteriori null error which is the solution to the SM-APA cost function. On the other hand, if a $\hat{\gamma}$ may be selected as the largest element of the Optimal CV, restoring the property of the SM-APA solution. Then, the Optimal CV indirectly provides an estimation of the threshold $\bar{\gamma}$.

III. PROPOSED ALGORITHMS

Our objective is to estimate $\bar{\gamma}$ and then build a CV. The CV generation also uses the estimation of how far from the white distribution the input is. This is performed by using a time-average estimate of the correlation of the input signal. A rough instantaneous estimation of the correlation is already performed in the APA by calculating $\mathbf{X}^T(k)\mathbf{X}(k)$.

Using the Optimal CV is a problem due to an inherent high computational complexity and to the nature of SM-AP, which calculates Optimal CV if the $\bar{\gamma}$ is high enough. Then, an underestimated $\bar{\gamma}$ would not be detected. Our strategy aims to circumvent both issues. The algorithm is divided in two stages: a short training stage and a large execution stage. During training, the update is performed by APA, but the Optimal CV is also calculated, and the input correlation is estimated. After training, $\bar{\gamma}$ is estimated by $\|\gamma_m\|_\infty$, where γ_m is the median of each element along the training, obtained by the Optimal CV. The CV building is a more complicated issue.

A. Building a Constraint-Vector

One of the first SM algorithms was the SM-NLMS algorithm [4]. Defining SM-NLMS does not lead to creating a CV. SM-NLMS uses a scalar error, which is compared to $\bar{\gamma}$ at each iteration. Moreover, the coefficient vector is updated by using $\bar{\gamma}\text{sign}(e(k))$.

The emergence of SM-AP [3] and SM Binormalized NLMS (SM-BNLMS) [8] leads to the creation of the CV for the SM algorithms. The solution, which is still placed at $\bar{\gamma}\text{sign}(e(k))$ instead of $e(k)$, is advantageous since the $\mathbf{w}(k+1)$ is updated inside the current constraint set while generating the minimum distortion to the original error, because it is placed on the edge of Ψ_{k-L}^k . The other entries of the CV are chosen not to change the current a posteriori error. Then,

$$\gamma_{SC}(k) = \begin{bmatrix} \bar{\gamma}\text{sign}(e(k)) \\ e(k-1) \\ \vdots \\ e(k-N) \end{bmatrix}. \quad (3)$$

Eq. (3) leads to the update based on the vector $\mathbf{e}(k) - \gamma(k) = [\bar{\gamma}\text{sign}(e(k)) \quad 0 \quad \dots \quad 0]^T$. Due to the SM-AP's good performance with SC-CV, it is the most popular version of the SM-AP nowadays. Recently, the optimality of SC-CV for orthogonal inputs and for white signals (in the mean) has been shown [6]. Then, some design guidelines may be developed:

- The CV should converge to the SC-CV for white inputs;
- The first component of the CV should be $e(k) - \bar{\gamma}\text{sign}(e(k))$, which is the choice of the large majority of the existing CV [2].

There is a wide variety of non-white signals, which makes it harder to design proper CVs. The nullity of almost all of the elements in $\mathbf{e}(k) - \gamma(k)$ for the SC-CV removes the effects of $e(k-1), \dots, e(k-L)$ in the coefficient updates. The inclusion of the effects of all elements, as in the Fixed-Modulus Error Bound (FMEB) CV [2], in the Trivial Choice (TC) CV [2] or in the Exponentially-Decay (ED) CV [7] resulted in worse performance when compared to the SC-CV in some scenarios.

The inclusion of the effect of some of the elements of $\mathbf{e}(k) - \gamma(k)$ may be a possible solution. Our proposal is to include only the effects of $e(k)$ and $e(k-1)$. The proposed CV is then configured as:

$$\gamma(k) = \begin{bmatrix} e(k) - \bar{\gamma} \text{sign}(e(k)) \\ e(k-1) - \alpha(k) \text{sign}(e(k-1)) \\ e(k-2) \\ \vdots \\ e(k-L) \end{bmatrix}. \quad (4)$$

The value of $\alpha(k)$ is used based on the (2, 1) entry of the estimated correlation matrix, since $e(k-1) = d(k-1) - \mathbf{w}^T(k-1)\mathbf{x}(k-1)$, and the (2, 1) estimates the correlation between $\mathbf{x}(k-1)$ and $\mathbf{x}(k)$.

After some experimental tests, α was chosen as $|r_{2,1}|/5$, where $r_{2,1}$ is the (2, 1) entry of the estimated correlation matrix.

Keep in mind that the threshold $\bar{\gamma}$ used in our proposed algorithm is also estimated from the training part, using $\|\gamma_m\|_\infty$, with γ_m as the median of each element along the training. Median eliminates possible outliers in the optimal CV estimation. This idea is inspired by the time-varying threshold used in [10]. The whole proposed algorithm is summarized in Table I.

TABLE I
PROPOSED ALGORITHM WITH FIXED THRESHOLD.

Stage 1: Training
Coefficient updates using APA;
Correlation matrix estimate updates using $\mathbf{X}^T(k)\mathbf{X}(k)$;
$\bar{\gamma}$ estimate updates using $\ \gamma_{opt}(k)\ _\infty$
Stage 2: Running
Coefficient updates using the proposed SM-AP;
$\bar{\gamma}$ defined by the estimate of the training stage;
CV defined in (4), based on the training stage.

B. Time-Varying Threshold

Another approach for performing the estimation of the SM threshold is to use a time-varying threshold. The proposed estimation is inspired by the time-varying threshold [11], by avoiding the term containing σ_n^2 , which requires previous knowledge of the measurement noise variance. The estimated threshold $\hat{\gamma}(k)$ at iteration k is given by:

$$\hat{\gamma}(k) = \lambda \hat{\gamma}(k-1) + (1-\lambda)\epsilon \|\mathbf{e}(k)\|, \quad (5)$$

where λ is a forgetting factor and ϵ is a scaling factor for adjusting the proportionality between $\hat{\gamma}(k)$ and $\|\mathbf{e}(k)\|$. In our simulations, $\hat{\gamma}(0) = 0$.

A summary of the proposed algorithm with a time-varying threshold is presented in Table II.

C. Remarks on Convergence

The proposed CV defined in Eq. (4) shows an intermediary behavior between the SC-CV and the Fixed Modulus Error-Bound (FMEB) CV [2] [7], since the first element of the proposed CV is identical to both SC-CV and FMEB CV, the

TABLE II
PROPOSED ALGORITHM WITH TIME-VARYING THRESHOLD.

Stage 1: Training
Coefficient updates using SM-AP;
Correlation matrix estimate updates using $\mathbf{X}^T(k)\mathbf{X}(k)$;
$\bar{\gamma}$ estimate updates using Eq. (5), starting with $\hat{\gamma}(0) = 0$
Stage 2: Running
Coefficient updates using the SM-AP with the proposed CV;
$\bar{\gamma}$ estimate updates using Eq. (5).
CV defined in (4), based on the training stage.

second element combines the weighted FMEB CV and the SC-CV, while the others are identical to the SC-CV corresponding element. Then, the same reasoning applied to the convergence of ED (Exponentially Decay) CV in [7] may also be applied here to the proposed CV.

The proposed CV is also a possible solution to Eq. (2) of the Optimal CV. Then, the reasoning used in [6] to address the convergence of the Optimal CV may also be applied here to the special case of a fixed CV inside the feasible set of solutions to Eq. (2).

The use of a time-varying threshold may present some concerns over convergence, since a small rate of updates in the beginning may slow the convergence down. Nevertheless, our algorithm starts with a zero threshold, that is, every iteration presents an update, with a slow increase in its value until a proper estimate for the threshold is reached.

D. Computational Complexity Remarks

As presented in [12], the asymptotic computational complexity of the Optimal CV lies between $\mathcal{O}(L^{3.5})$ and $\mathcal{O}(L^4)$, during the training stage. On the other hand, the computational complexity of a pre-defined CV is $\mathcal{O}(L)$ after the training stage. In our proposed algorithm with fixed CV, the number of training samples is much smaller than the number of samples. In Section IV, the training part comprises 20 samples out of 3,500. Besides that, the proposed fixed CV requires fewer updates than the SC-CV.

For the proposed algorithm with a time-varying threshold, the Optimal CV is not used in the training part. The estimated threshold is updated in every iteration using Eq. (5). A summary of the asymptotic computational complexity of the proposed CVs for the SM-AP with SC-CV is presented in Table III.

TABLE III
COMPARISON FOR THE ASYMPTOTIC COMPUTATIONAL COMPLEXITY OF THE PROPOSED CVs AND THE SM-AP WITH SC-CV.

Version of the SM-AP	Complexity
SC-CV	$\mathcal{O}(L)$
Proposed Fixed CV	$\mathcal{O}(L_T^{3.5})$ then $+$ $\mathcal{O}(L)$,
Proposed Time-Varying CV	$\mathcal{O}(L)$

IV. RESULTS

This section presents the results of computer simulations for the performance comparison of the proposed CVs, presented in

Section III, compared to SM-AP with SC-CV [2] and ED-CV. Besides that, the fixed threshold algorithm is also simulated in a situation where the parameters were known. Simulations have been performed on Matlab software in a Ubuntu Linux distribution.

The algorithms are compared in terms of mean-squared error (MSE), that is, $\sum_{m=0}^{M-1} \sum_{k=0}^{K-1} |e_m(k)|^2$, where $M = 600$ is the number of Monte-Carlo runs and $K = 3,500$ is the amount of transmitted symbols, and $e_m(k)$ is the error at the m -th Monte-Carlo run. Only 20 symbols are used in the training phase. Another comparative figure-of-merit used in the simulations is the Misalignment $\phi(k)$, defined in Eq. (6).

$$\phi(k) = \frac{1}{M} \sum_{m=0}^{M-1} \sum_{k=0}^{K-1} \frac{\|\mathbf{w}_m(k) - \mathbf{h}(k)\|_2^2}{\|\mathbf{w}_m(k)\|_2^2}, \quad (6)$$

where $\mathbf{w}_m(k)$ is the coefficient vector in the m -th Monte-Carlos run, and h is the impulse response of the channel, which has all the $N = 16$ taps equal to one during the first 200 iterations and then all of them change to 2.

As presented in [6], the SC-CV is the Optimal CV in the mean for the white input. Our focus in the simulations are the colored-input scenarios.

In the first experiment, a 4th order auto-regressive (AR) is used to generate the adaptive filter input signal as $\mathbf{x}(n) = -0.95\mathbf{x}(n-1) - 0.19\mathbf{x}(n-2) - 0.09\mathbf{x}(n-3) + 0.5\mathbf{x}(n-4) + \mathbf{n}(k)$. The MSE results for the first experiment are shown in Fig. 1. The proposed algorithm with fixed threshold presents a lower final MSE. Note also that the training part comprises only 20 samples. The performance is not degraded even when there is a change to the impulse response at the 200-th iteration. For the time-varying threshold, the forgetting factor is set as $\lambda = 0.999$ and $\epsilon = 0.75$.

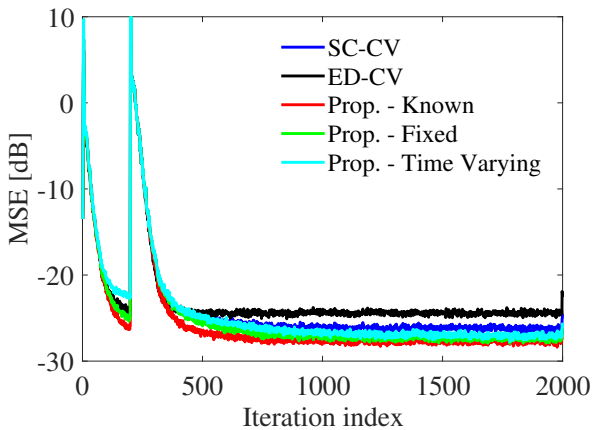


Fig. 1. MSE for an AR-4 process input.

In the same simulation, the misalignment was also measured. The results are shown in Fig. 2. The proposed algorithm with a fixed threshold presents a lower final misalignment. The parameter ϵ of the proposed time-varying threshold was reset as $\epsilon = 1$.

Another simulation was performed using an AR-1 input, generated by $\mathbf{x}(n) = -0.95\mathbf{x}(n-1)\mathbf{n}(k)$. Such a scenario

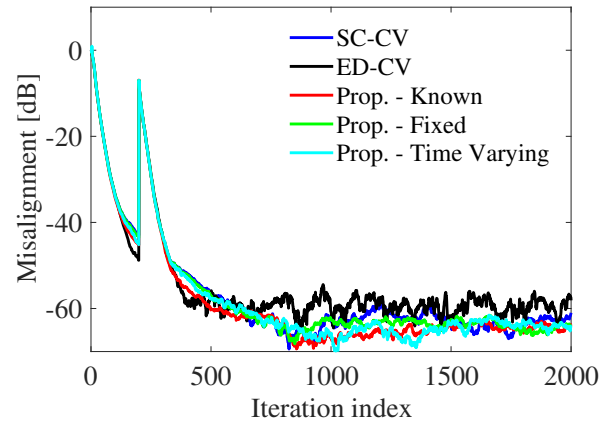


Fig. 2. Misalignment for an AR-4 process input.

tends to be more favorable to the SC-CV, as shown in [6]. The results are presented in Fig. 3.

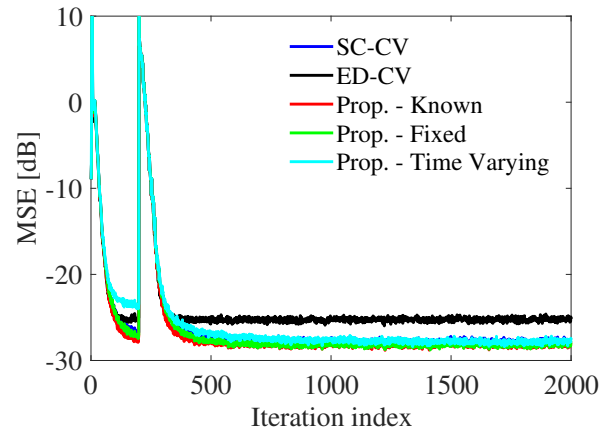


Fig. 3. MSE for an AR-1 process input.

One may realize in Fig. 3 that the proposed algorithm with time-varying threshold presents a slower convergence than the proposed fixed threshold, as well as the SC-CV and ED-CV.

Fig. 4 shows the results for the misalignment in the AR-1 simulated environment. One observes that the results are equivalent for all the CVs, except for the ED-CV, which has the worst performance.

Since SM algorithms do not update in every iteration, Table IV shows the update rate for each algorithm, defined as the ratio of the number of iterations with updates to the number of updates where the algorithm runs. For an AR-4 input, the proposed algorithm with a fixed threshold, then the algorithm's training part is not considered in the ratio since an affine projection update is used.

The proposed algorithm with fixed threshold updates in a smaller number of iterations compared to the SC-CV. The proposed algorithm with time-varying threshold presents an intermediary rate of iterations when compared to SC-CV and ED-CV for an AR-4 input. On the other hand, the version with a time-varying threshold presents a lower complexity for the update than the fixed threshold version, as shown

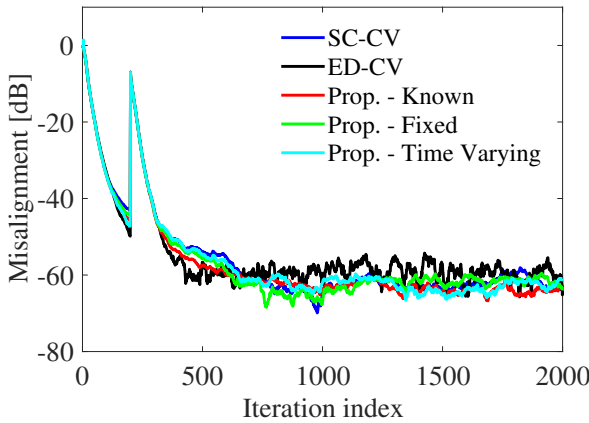


Fig. 4. Misalignment for an AR-1 process input.

TABLE IV

SM UPDATE RATES FOR EACH ALGORITHM IN EACH OF THE SCENARIOS.

CV	Update Rate			
	AR4	AR1	White	Rep. Code
SC-CV	0.271	0.187	0.156	0.193
ED-CV	0.335	0.281	0.203	0.181
Proposed: Fixed	0.215	0.165	0.151	0.258
Proposed: Time-Varying	0.304	0.168	0.150	0.165

in Section III-D. Then, the proposed fixed threshold version presents lower MSE, a slightly lower misalignment, and fewer updates than the SC-CV by using only 20 samples of training. The proposed time-varying threshold version presents lower MSE and misalignment, with an update rate greater than SC-CV, but smaller than ED-CV.

In the case of AR-1 input, both proposed versions of SM-AP present a smaller update rate than SC-CV and ED-CV.

In our simulations, we also plotted the proposed algorithm with a fixed threshold considering the situation where the parameters were all known. As can be seen in Figs. 1 and 2, the version with known parameters presents a lower MSE and misalignment. Then, we conclude that the performance of the proposed algorithm presents a sub-optimal estimation.

It is remarkable that the training of the proposed fixed CV is performed only once, even if the impulse response is changed, as depicted in Figs. 1-4. On the other hand, if the input signal changes its input statistics, the adaptive filter should be retrained.

The proposed algorithm was also tested in a scenario where a simple channel encoding known as repetition code [13] with length 4 was used. The Misalignment for the proposed algorithm as well as the SC-CV and ED-CV is shown in Fig. 5. The update rates are also shown in Table IV.

V. CONCLUSION

This article describes two versions of the SM-AP algorithm solely based on estimation. One of them estimates the CV parameters based on the optimal CV. The other one is based on a time-varying estimation of the noise power for the SM threshold. Simulation results show that the proposed algorithm outperforms the benchmarks SC-CV and ED-CV in terms of

mean-squared error with less update, with the exception of the scenario with repetition code, where the fixed version presented more updates. Another contribution of this article is that the proposed CV is developed mainly for non-white inputs, which is common when the transmitter has an encoder.

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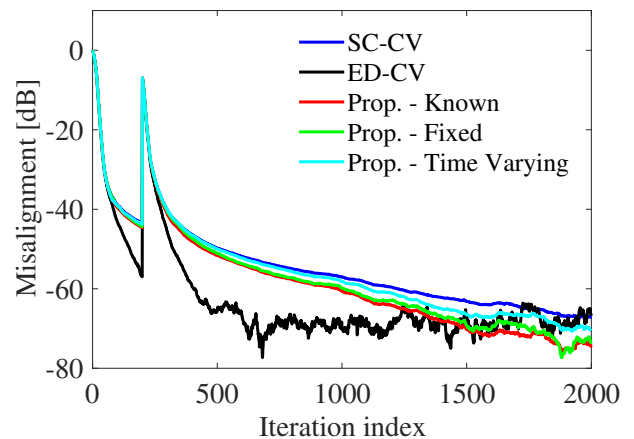


Fig. 5. Misalignment for the scenario with Repetition Code [13].