Wavelet-based Echo Identification in Time Domain Reflectograms

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***Abstract*— The automatic interpretation of reflectometry traces to identify twisted-pair loops can be considered a pattern recognition problem. Thus, any loop make-up algorithm must firstly extract features from the reflectogram before starting a classification process. This paper focuses on feature extraction through the echo identification on time domain reflectometry (TDR) measurements, which are analyzed by a multi-scale edge detection wavelet approach that identifies all singularities in a reflectogram and relate them to theirs respective echoes. From echoes, it is provided an approximation for an auto-regressive TDR trace model, furnishing features for classification.**

***Keywords*— Wavelet transforms, time domain reflectometry, system identification, subscriber loop, loop qualification.**

I. INTRODUCTION

A telephone subscriber line or simply *local loop* is a single-user circuit on a telephone communications system that consists of a metallic twisted-pair network link between a customer and a telephone Central Office (CO). With the growth of high-speed Digital Subscriber Line (DSL) access subscriptions, the demand in the telecommunications industry for equipments and methodologies to accurately predict DSL access performance over local loops increased. In this context, single-ended line tests (SELT) using reflectometry techniques are among the most commonly used methods for locating faults and discontinuities on metallic wires.

SELT measurements require that the test equipment is just in one port of the loop (usually in the CO). Therefore, this type of methodology is less time consuming and expensive than double-ended tests. On the other hand, SELT demands more in relation to noise level at the receiver since the test signals have to propagate a complete round trip of the loop. So, it is desirable to have an automatic single-ended testing technique that could be able to accurately identify any subscriber loops' topology.

The ability to accurately qualify local loops allows telephone companies to improve the performance of its services and offer new ones, or define if these services may or not be available to customers. The system performance analysis requires accurate prediction of the local loop topology, i.e. the estimation of the number of loop sections, the gauge of each section, and the location of gauge changes and bridged taps.

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This system identification problem can be investigated through the analysis of Time Domain Reflectometry (TDR) measurements using an algorithm that is able to extract and analyze information from a reflectogram. This issue can be also considered as a pattern recognition problem and is commonly composed by two stages: the *feature extraction* and *classification* of parameters. The first stage selects the relevant information from the measured data, while the *classification* uses that characteristics extracted by previous stage to choose the best hypothesis for describing the physical system. In [1] and [2], it is proposed a TDR-based solution for loop make-up. In that paper, reflectograms are analyzed by an iterative algorithm based on identification of each loop section through the comparison of intervals of measured and simulated TDR traces using maximum likelihood. However, little accurate information is supplied about how these intervals (features) can be extracted.

In [3], is presented a feature extraction algorithm for a transfer function estimator previously reported in [4]. In that paper, the one-port scattering parameter is firstly measured by means of a network analyzer directly in the frequency domain, like in [4], and after is obtained the time domain counterpart of the scattering parameter. The method relies in the analysis of the first and second derivatives of the one-port scattering parameter in time domain by a set of "maximum-minimum rules" in order to identify attributes of reflected pulses. The method presents good results, but in the presence of noise, the use of derivatives can sometimes be problematic. In [3], this is avoided through the use of average of several measurements.

This paper is also focused on feature extraction stage, but through reflection, or simply *echo*, identification on measured TDR traces. The goal is to obtain the locations of edges of overlapping and non-overlapping echoes in the reflectogram relating each edge with its respective echo. In this context, based in [7], the reflectogram is modeled as a sum of echoes, where, although the resulting signal is smooth, it exhibits a discontinuous change between echo structures. Such changes are in fact irregularities in the trace, which carry key information.

An attractive mathematical tool for analyzing singularities and irregular structures is the wavelet transform, which can characterize the local regularity of signals [5],[6]. Thus, it is well motivated to investigate the wavelet transform approach to echo identification for feature extraction in loop qualification.

The remainder of this paper is organized as follows. In Section II the problem is formulated from modeling of the reflectogram obtained by TDR. In Section III, targeting the echo identification task, it is described a wavelet-based technique for detecting, locating and identifying echo edges in TDR traces. Four examples of application of the developed technique are depicted in Section IV. In this section, it is given special attention to approximation of the reflectogram model resulting of wavelet-based echo identification algorithm. Finally, in Section V, the final considerations about the algorithm are made.

II. Echo Modeling

A TDR trace is composed by a set of echoes which can be classified as *real* and *spurious echoes*. A real echo is the first reflection from a loop discontinuity and, theoretically, contains all information needed to detect, localize and identify the discontinuity that generated it. Spurious echoes are defined as all echoes caused by successive reflections, which do not indicate directly the presence of discontinuities. Spurious echoes will suffer more attenuation as time travel increases into the line until arrives at the source. However, this kind of echo must not be neglected because some of these spurious echoes can reach the source before the real echoes and, sometimes, they can be stronger than real echoes.

Thus, if there are N discontinuities in a given loop, N real echoes will be generated for an input signal $s(t)$, while a theoretically infinity number of spurious echoes will be generated [7]. In this way, the TDR trace can be expressed as an infinite-order autoregressive model, which is given by

$$\begin{aligned} r\left(t\right) = e^{(0)}\left(t-\xi^{(0)}\right) + \sum_{i=1}^{N} e_r^{(i)}\left(t-\xi_r^{(i)}\right) \\ + \sum_{j=1}^{\infty} e_s^{(j)}\left(t-\xi_s^{(j)}\right) + n(t), \end{aligned} \tag{1}$$

where $e^{(0)}(t)$ is the echo generated by impedance mismatch in the connection between source and loop that reaches the source in $\xi^{(0)}$; $e_r^{(i)}(t)$, with $i = 1, 2, ..., N$, is the ith real echo generated by ith discontinuity that reaches the source in $\xi_r^{(i)}$; $e_s^{(j)}(t)$, with $j = 1, 2, ..., \infty$, is the jth spurious echo and $\xi_s^{(j)}$ is the time of arrival of the jth spurious echo at the source. $n(t)$ represents the effect of noise that, in this paper, will be considered like an Additive White Gaussian Noise (AWGN). In [7], j is represented as the value of i from $N+1$. However, in spite of this representation try to introduce the concept of dependency between spurious and real echoes, it can also induce to the false idea that the spurious echoes always occur after all real echoes reach the source.

A real echo $e_r^{(i)}(t)$ can be expressed by [7]

$$e_r^{(i)}\left(t\right) = s(t) * h_r^{(i)}(t) \tag{2}$$

where $h_r^{(i)}$ is the echo path impulse response. In the frequency domain, the correspondent transfer function is given by

$$H_r^{(i)}\left(f\right) = F\left[h_r^{(i)}\left(t\right)\right] = K\tau_r^{(i)}(f)K\rho_r^{(i)}(f)E_r^{(i)}(f) \tag{3}$$

where $E_r^{(i)}(f)$ is the transfer function of the echo path from the source to the ith discontinuity and back to the source, $K\tau_r^{(i)}(f)$ is the term that takes into account all the transmission coefficients preceding the ith discontinuity, and $K\rho_r^{(i)}(f)$ takes into account the reflection coefficients and, for real echoes, is equal to reflection coefficient of the ith discontinuity $\rho^{(i)}(f)$.

The spurious echoes can be expressed in the same way by

$$e_s^{(j)} = s(t) * h_s^{(j)}(t) \tag{4}$$

and

$$H_s^{(j)}\left(f\right) = F\left[h_s^{(j)}\left(t\right)\right] = K\tau_s^{(i)}(f)K\rho_s^{(i)}(f)E_s^{(i)}(f). \tag{5}$$

The distance d from the source to the ith discontinuity that generates a real echo is estimated from the computation of the round-trip delay time $T_r^{(i)}$ of the ith real echo by using

$$d^{(i)} = \frac{T_r^{(i)} \cdot v_p}{2} \tag{6}$$

where $T_r^{(i)}$ is obtained by the difference between time of arrival of the echo on the input port and the reference time (time of arrival of $e^{(0)}(t)$, $\xi^{(0)}$) and v_p is the velocity of propagation of the signal in the line.

In addition, for subsequent development of this work, the following basic assumptions are adopted [8]:

1) The input signal $s(t)$ is a wideband rectangular pulse;
2) The propagation channel, the copper loop twisted pair, is approximately a time-invariant linear system [9];
3) The number of real echoes N in the reflectogram is unknown *a priori* and remains unchanged at least the twisted pair is modified; and
4) Each echo structure in the reflectogram is smooth, but is also irregular and exhibits discontinuities at its edges.

As a result of those assumptions, the signal crossing the loop will conserve some characteristics of the input signal. Each echo, as well as the input signal, will be composed by a rise edge and a fall edge whose interval of time defined between them is approximately the same to the width of the input pulse. Thus, the "echo interval" is defined as the interval between time of arrival of a given echo on the input port of the loop (rise edge) and the time that the echo starts to dissipate (fall edge).

In this way, since the reflectogram is defined as an analytic function, given by (1), and the echoes can be considered irregular structures, the rise and fall edges are characterized by isolated singularities. Hence, since the objective of the *feature extraction* step in the subscriber loop make up from TDR measurements is to detect and estimate the location of the echoes in reflectogram, it can be reached through characterization of the singularities in the TDR trace.

III. Wavelet Approach to Echo Identification

Singularities are points at which a given mathematical object, as a signal, is not defined, or it fails to be well-behaved in some particular way, such as differentiability or continuity. Singularities and irregular structures are extremely important in signal processing, where they often carry essential information about the function. The feature extraction problem presented here can be viewed as an edge detection problem in a signal. This section shows that the singularities of a reflectogram can be detected and characterized through the analysis of the modulus of its Continuous Wavelet Transform (CWT).

A. Multi-scale Edge Detection from Wavelet Transform

In reflectometry, since the reflectograms are generally continuous and without eccentricities, the edge detection process is equivalent to the detection of inflection points or singularities. To illustrate the relation between CWT and edge characterization, a smoothing function $\theta(t)$ is defined as a function whose integral is equal to 1 and converges to 0 at infinity. Let $\theta(t)$ be equal to a Gaussian function and twice differentiable. Thus, it can be defined the following functions:

$$\psi^a(t) = \frac{d\theta(t)}{dt} \quad and \quad \psi^b(t) = \frac{d^2\theta(t)}{dt^2}. \tag{7}$$

By definition, $\psi^a(t)$ e $\psi^b(t)$ can be considered to be wavelets because [6]

$$\int_{-\infty}^{\infty} \psi^a(t)\,dt = 0 \quad and \quad \int_{-\infty}^{\infty} \psi^b(t)\,dt = 0. \tag{8}$$

The CWT is computed by convolution of the signal with a dilated wavelet. Thus, the CWT of the signal $r(t)$ computed with respect to the wavelet $\psi^a(t)$ is given by

$$W^a r(t,s) = r(t) * \psi_s^a(t) \tag{9}$$

where

$$\psi_s(t) = \frac{1}{\sqrt{s}}\bar{\psi}\left(\frac{-t}{s}\right), \tag{10}$$

and $\bar{\psi}$ means the complex conjugate of ψ, and the scale factor s can be set to the dyadic scales $s = 2^j, j = 1, 2, ..., J$.

The same can be defined with respect to $\psi^b(t)$:

$$W^b f(t,s) = f(t) * \psi_s^b(t). \tag{11}$$

It can be derived that

$$W^a f(t,s) = f(t) * \left[s\frac{d\theta_s(t)}{dt}\right] = s\frac{d}{dt}\left[f(t) * \theta_s(t)\right] \tag{12}$$

and

$$W^b f(t,s) = f(t) * \left[s^2\frac{d^2\theta_s(t)}{dt^2}\right] = s^2\frac{d^2}{dt^2}\left[f(t) * \theta_s(t)\right]. \tag{13}$$

It is shown in [6] that the CWTs defined by Equations (12) and (13) are, respectively, the first and the second derivative of the signal smoothed at the scale s. The local extrema of $W^a f(t,s)$ correspond to the zero crossings of $W^b f(t,s)$ and to the inflection points of $f(t) * \theta_s(t)$. When the scale s is large, the convolution with $\theta_s(t)$ removes small signal fluctuations; it, therefore, only detects the sharp variations of large structures, i.e., only those modulus maxima or zero crossings that propagate to coarser scales are retained.

B. Singularity Analysis and Echo Identification

For echo identification purposes, the local extrema approach, which corresponds to Canny [10] edge detection in particular case where $\theta_s(t)$ is a Gaussian function, has some advantages in relation to zero-crossing detection. The local extrema can be both positive and negative peaks over the inflection points. Positive peaks are related to variations from lower amplitudes to higher ones and, conversely, negative peaks are related to variations from higher to lower amplitudes. The zero-crossing approach does not immediately return this information.

It is important to note that, differently from Canny algorithm, in CWT approach is not necessary set the scale that better approximate the solution. The CWT allows evaluate all variations along the scales. The location and features of a detected singularity is determined by analysis of trends in each scale.

Thus, it can be defined the following propositions:

1) Singularities or inflection points, in a reflectogram $r(t)$, correspond to edges of the echoes. Consequently, the echo edges can be acquired from $r(t)$ by picking the local extrema of its wavelet modulus in (12) for each scale;
2) The kind of edges – rise or fall – can be characterized by sign of the local extrema in CWT modulus;
3) All echoes in the reflectogram have the same time length of the input signal.

From the first proposition, basically, the method for detection and estimation of singularities in a TDR trace relies on fact that a singularity in a signal cause a disturbance on the amplitudes of the energy density function of the CWT, called *scalogram*, by creating large amplitudes on wavelet coefficients generating a kind of *cone of influence* of this point. The singularities are detected by finding the abscissa where the wavelet modulus extrema converge at finest scales [5].

Once the singularities were detected, it is now necessary to relate the features of these singularities to edges of the echoes and determine which set of singularities correspond to each echo. In according with proposition 2, rise and fall edge can be identified by wavelet analysis. Positive peaks on CWT indicate rise edges, whereas negative peaks indicate fall edges, as illustrated by Figure 1 for a part $f(t)$ of a reflectogram. On the other hand, just identifying the type of edges in a reflectogram is not sufficient to identify the echoes. In TDR traces, it is possible to have reflectograms where one or more echoes are mixed, as illustrated in Figure 2 for a function $f(t)$ obtained from a reflectogram of a $50m$-length loop. In cases like that, it is necessary to add another parameter to decide which edges define each echo in a reflectogram. In this respect, the proposition 3 gives us an important common feature among echoes that can be used as a parameter to generate a decision rule in the echo identification. Thus, it is sufficient to select as "echo interval" the pair of singularities that generates the minimum square error between the time interval of the pair and the width of the input pulse [8].

IV. Example of Echo Identification

In this Section, some echo identification results using the wavelet-based echo identification algorithm are presented. First, the TDR traces are measured by means of a differential TDR setup. Figure 3 illustrates a block diagram of the TDR setup. The objective is to increase the common mode rejection capability of TDR and to improve the connection between the source and the loop using differential probing. Basically, a pulse generator sends, on each output channel, two synchronized and unbalanced rectangular pulses with the same width and magnitude, but opposite signals. How these pulse generator channels use BNC connectors, converters, represented by the squares A and B, are used to transpose the BNC output to the "tip-ring"connection. The two wires of the pair under test are connected to signal conductors in A and B, whereas the ground wires on A and B are shorted. A digital oscilloscope is used to measure the echo response and the resulting TDR trace is obtained by difference between traces detected by channels Ch1 and Ch2.

It is important to point out that in the neighborhood of an edge on the raw TDR measurements there may be many points where the signal is changing rapidly due to measurement error

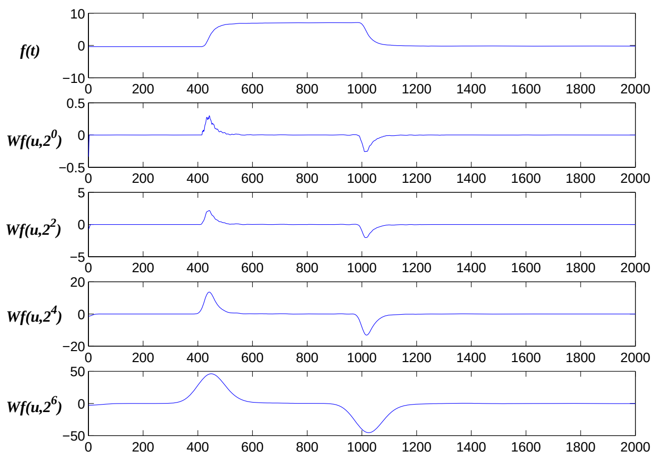

Fig. 1. CWT of signal $f(t)$ at scales 2^j, $j = 0, 2, 4, 6$.

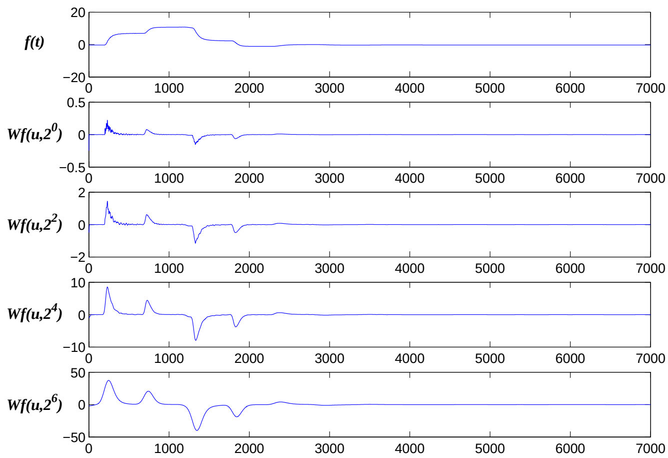

Fig. 2. CWT of a reflectogram $f(t)$ obtained from a $500m$-length loop at scales 2^j, $j = 0, 2, 4, 6$.

and environmental noise, represented in Equation 1 by $n(t)$. However, we only want to identify one of them as an echo edge. In this way, it is adopted a noise filtering method based on Discrete Wavelet Transform (DWT) proposed by Donoho and Jonhstone in [11] and [12]. The principle of the wavelet de-noising is to identify and remove the elements in each coefficient of the DWT of the signal that probably contain the most part of noise and, in the follow, calculate the inverse DWT of the "cleaned"wavelet coefficients to obtain a denoised signal.

The use of this method is justified by the fact that the energy of the TDR signal is more concentrated in some scales (frequency ranges) than in others which is different from noise that is spread along scales. It is important to note that the desired signal contains important features well localized in time and distributed on the scales. In addition, the wavelet coefficient related to the low frequency information contains the most part of information about the shape of the TDR trace and is not significantly immersed in noise. Therefore, this coefficient was not "denoised".

Now, let it considers the topology illustrated in Figure 4, where a single serial loop of length L has just two discontinuities: the point number 1, which represents the impedance mismatch between the TDR equipment and the loop; and the point 2, which is the open termination of the loop. Consequently, the correspondent TDR trace has just two real echoes. The first is the equivalent to the term $e^{(0)}(t)$ in (1) and the second is due to the end of the loop. The spurious echoes are consecutive to these echoes.

From Figure 5 to Figure 8, four denoised TDR measurements are shown for a probing signal of 10V of amplitude and $1.17\mu s$ of pulse duration. These figures show results for the topology represented in Figure 4 with $L = 50m$, $100m$, $500m$, and $1000m$, respectively. In all cases, the wires had $0.5mm$ of diameter. The reflectograms in Figure 5 and Figure 6 are examples of traces with mixed echoes, whereas Figures

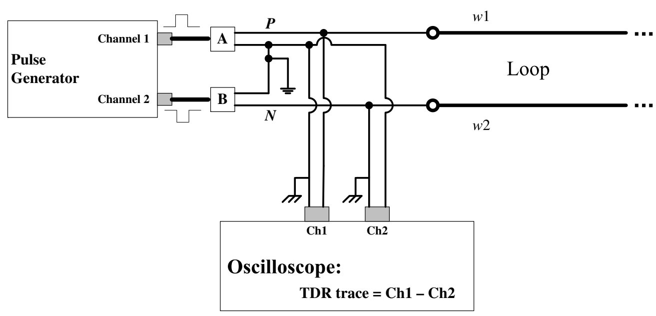

Fig. 3. Schematic diagram of the TDR differential configuration used to measure the test loops.

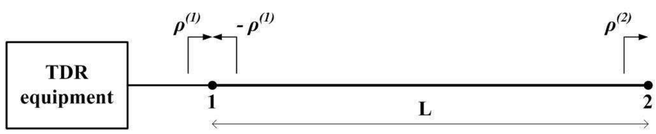

Fig. 4. Loop topology consisting of a single serial section connected in the TDR equipment through the point 1 and terminated in the point 2.

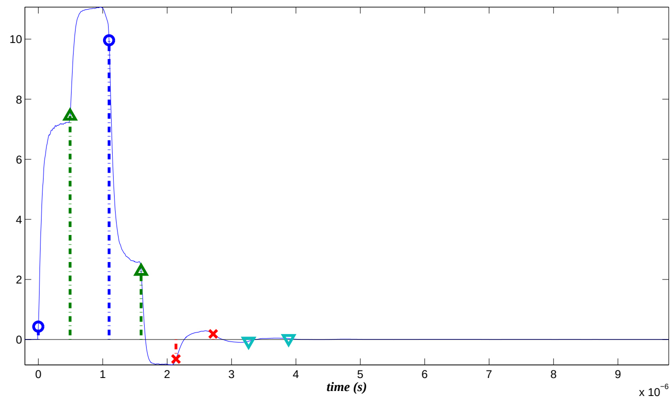

Fig. 5. Echoes identified in a TDR trace of a $50m$-length loop.

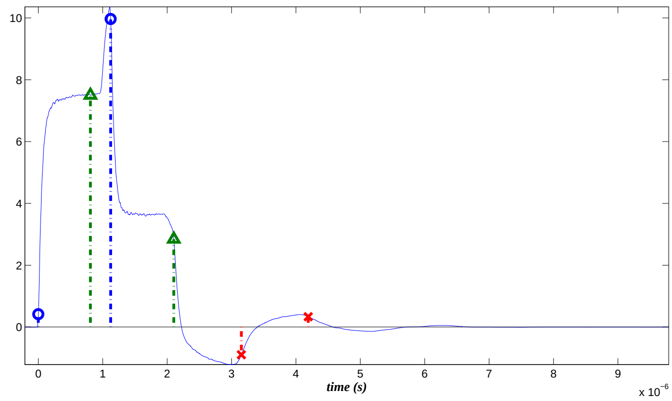

Fig. 6. Echoes identified in a TDR trace of a $100m$-length loop.

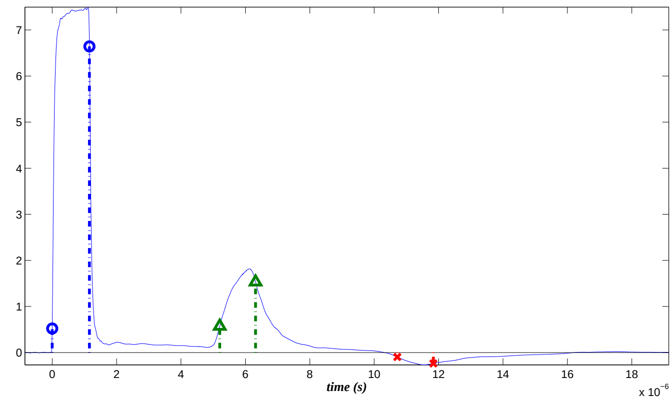

Fig. 7. Echoes identified in a TDR trace of a $500m$-length loop.

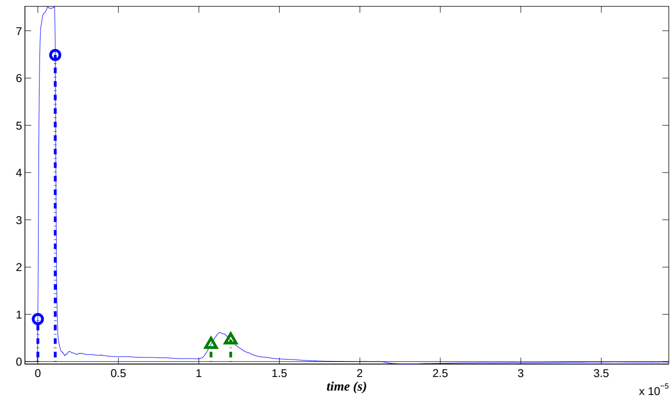

Fig. 8. Echoes identified in a TDR trace of a $1000m$-length loop.

7 and 8 others are reflectograms with non-mixed echoes.

In all tests, it was used the first derivative of Gaussian wavelet along with dyadic scales $s = 2^j, j = 0, 1, 2, 3, 4, 5, 6$, in the echo identification algorithm. The identified echoes are delimited by different pairs of markers, representing the detected singularities, in each figure.

In fact, through the identification of the echoes, the algorithm shall obtain an approximation of the reflectogram model presented in (1). As a consequence, after identification, (1) can be rewritten as

$$\begin{aligned} r\left(t\right) = e^{(0)}\left(t - \xi^{(0)}\right) + e^{(1)}\left(t - \xi^{(1)}\right) \\ + e^{(2)}\left(t - \xi^{(2)}\right) + ... + e^{(M)}\left(t - \xi^{(M)}\right), \end{aligned} \quad (14)$$

where M is the number of main echoes (echoes with significant level of energy) and $\xi^{(i)}$, with $i = 0, 1, 2, ..., M$, is the time of arrival of the echo $e^{(i)}$. It is also possible to rewrite (1) as the sum of convolutions between the ith echo path impulse response $h_r^{(i)}$ and input signal $s(t)$ shifted by $\xi^{(i)}$, as given by

$$\begin{aligned} r\left(t\right) = h^{(0)}\left(t\right) * s\left(t - \xi^{(0)}\right) + h^{(1)}\left(t\right) * s\left(t - \xi^{(1)}\right) \\ + h^{(2)}\left(t\right) * s\left(t - \xi^{(2)}\right) + ... \\ + h^{(M)}\left(t\right) * s\left(t - \xi^{(M)}\right). \end{aligned} \quad (15)$$

In conclusion, the expression in (15) results in an approximate model structure for the TDR reflectogram using the terms $h^{(i)}(t)$ as system parameters. The discrimination between real and spurious echoes can be done by analysis explained in Section II. However, finding the values of the system parameters and the relation of them with the topology is a function of the next step of the algorithm: *classification*.

V. Final Considerations

In this paper, the feature extraction problem in pattern recognition for loop makeup identification by TDR measurements is addressed. Specifically, this work formulates the echo identification task in a TDR reflectogram as a feature extraction problem involving singularity detection and characterization. It introduces a multi-scale edge detection wavelet approach to identify all singularities that occur in reflectograms and relates them to their respective echoes. Hence, once the algorithm identifies the echoes, it also provides an approximation of the auto-regressive TDR trace mathematic model using the impulse responses of the echo paths like model parameters, supplying features for classification.

Other contribution of this work is the ability of the proposed algorithm to identify echoes both in reflectograms with mixed echoes and with non-mixed echoes due to decision rules based on assumptions about the linearity of the subscriber line channel.

In addition, although the proposed algorithm has been applied specifically to TDR measurements, this methodology could be adapted to analysis of any signals with transient features. Other research lines in reflectometry can be favored with the utilization of this tool, like applications that analyze impulse response, scattering parameters and transfer function curves to extract features for pattern identification in a loop.

The next step of this research is the development of a complete pattern identification technique through the development of a classification tool. This tool shall take advantage of the features wavelet-based echo identification algorithm introduced in this paper to infer about the topology of subscriber loops.

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