Parafac-based approach for direction-finding using a high-order virtual array

Carlos E. R. Fernandes[(1,2)], Gérard Favier[(1)], João C. M. Mota[(2)]
cfernand@i3s.unice.fr, favier@i3s.unice.fr, mota@gtel.ufc.br

***Resumo*— Neste artigo, o problema de localização de múltiplos usuários é tratado de forma cega, considerando-se a hipótese de campo distante. Explorando-se o conceito de Arranjos Virtuais, propõe-se um algoritmo de localização de fontes de alta resolução, usando para tanto uma técnica iterativa de otimização de mínimos quadrados com uma única etapa. O método proposto baseia-se na decomposição Parafac de um tensor de cumulantes de 4^a ordem combinada com um algoritmo de tipo MUSIC. A unicidade da solução proposta é discutida, permitindo uma análise da quantidade máxima de fontes que o algoritmo pode tratar.**

***Palavras-Chave*— Identificação cega de canal, algoritmo MUSIC, decomposição Parafac, localização de fontes, tensores, arranjo virtual de antenas.**

***Abstract*— In this paper, we treat the problem of blind multiuser localization, under the far-field assumption. Exploiting the Virtual Array (VA) concept, we use an iterative single-step least-squares (SS-LS) technique to propose a high-resolution direction finding (DF) algorithm that utilizes the Parafac decomposition of a 4th-order cumulant tensor combined with a MUSIC-like algorithm. The uniqueness issue is addressed to assess the maximum number of sources handled by the proposed method.**

***Keywords*— Blind channel identification, MUSIC algorithm, Parafac decomposition, source localization, tensors, virtual array of antennas.**

I. INTRODUCTION

High-resolution subspace-based direction finding (DF) methods, such as the well-known MUSIC [1], [2] and ESPRIT [3] algorithms, have become very popular in narrowband (NB) array processing. Exploiting the orthogonality between the signal and noise subspaces, these methods based on the second-order statistics (SOS) provide asymptotically infinite resolution and are very interesting solutions for localizing multiple sources when the spatial correlation of the additive noise is known [4]. However, the performance of SOS-based methods can be seriously deteriorated when dealing with several sources with low signal-to-noise ratio (SNR) and small angular separation using finite data sample sequences [5] or in presence of spatial noise with unknown correlation function. In addition, they can only treat overdetermined mixtures (more sensors than sources).

Source localization is a crucial aspect in sensor array processing. Determining the location of signal emitters allows for the implementation of source separation techniques as well as beamforming for interference suppression. During the last two decades, the use of high-order statistics (HOS) has been widely considered for the estimation of direction-of-arrival (DOA) in the context of multiuser NB array processing. Several solutions to the source localization and DF problems have been proposed for non-Gaussian signals based on the 4th-order cumulants of the array output data (c.f. [6] and references therein). Extensions of the MUSIC algorithm to the 4th- and higher (even) orders gave rise to the 4-MUSIC [7] and, more recently, the 2κ-MUSIC ($\kappa \geq 2$) methods [8]. In addition to noise robustness, these methods offer better resolution and allow for an increased number of sources to be localized, including certain underdetermined cases. Although characterized by a higher variance [9], the HOS-based MUSIC-like algorithms increase the number of virtual sensors and the effective aperture of the receive antenna array at the cost of an increased complexity due to the estimation of high-order statistical information [10].

In this paper, we are interested in the problem of blind multiuser localization in the context of multiple antenna array processing. Assuming that the sources are located at the far-field of the antenna array, our goal is to estimate signal DOAs using only the array output signals. We propose a new high-resolution DF algorithm that artificially adds sensors to a virtual antenna array without resorting to statistics of order higher than fourth. In fact, using the 4th-order cumulants only, the proposed method estimates the array matrix and, exploiting the structure of the cumulant tensor, creates an enhanced Virtual Array (VA) that yields an augmented observation space, thus providing additional degrees of freedom to the antenna array and allowing for improved resolution. Based on an iterative single-step least-squares (SS-LS) Parallel Factor (Parafac) decomposition technique introduced in [11], [12], the new source localization algorithm exploits an array having a double Kronecker structure, which commonly only arises when using 6th-order statistics. However, since we do not need to estimate cumulants of order higher than fourth, our approach keeps the variance of the cumulant estimators at a moderate level, even for quite short output data sequences. Uniqueness and identifiability conditions are discussed allowing us to study the capacity of the proposed technique in terms of the maximum number of resolvable sources. Computer simulations are provided to illustrate the performance of the proposed method compared with the classical MUSIC approaches.

The main contributions of this paper are: *i.*) derivation of

[(1)] Université de Nice Sophia Antipolis, CNRS, Laboratoire I3S, 2000 route des Lucioles, 06903, Sophia Antipolis, France.

[(2)] Universidade Federal do Ceará, Grupo de Telecomunicações Sem Fio (GTEL/UFC), Campus do Pici, 60455-760, Fortaleza, Brazil.

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a 3rd-order VA model based on an unfolded representation of a 4th-order cumulant tensor; *ii.*) proposition of an algorithm for estimating the VA using an iterative SS-LS approach; *iii.*) uniqueness and capacity analysis of the proposed method in terms of the maximum number of separable sources for a given number of receive antennas. The remaining of the paper is organized as follows: in section II, we formulate the array output signal model along with the basic definitions of signal and noise subspaces; we also discuss the VA concept and present a survey of classic MUSIC-like algorithms, including the general formulation for the case of statistics of any even order; in section III, we derive a new high-resolution DF algorithm exploiting the double Kronecker structure of an unfolded representation of a 4th-order cumulant tensor; in this section, we revisit the iterative SS-LS approach for the Parafac decomposition of the cumulant tensor and the uniqueness issue is addressed; finally, in section IV, we provide simulation results illustrating the proposed method and assessing its performance under different channel configurations. Conclusions are drawn in section V, along with some future work perspectives.

II. Source localization in NB array processing

Let us consider a linear antenna array with M identical NB sensors receiving the contribution of Q zero-mean stationary sources, assumed to be placed at the far-field of the array. Denoting by $\mathbf{y}(n) \in \mathbb{C}^{M\times 1}$ the vector of complex signals measured at the output of the antenna array, we have:

$$\begin{aligned}\mathbf{y}(n) &= \sum_{q=1}^{Q} s_q(n)\mathbf{a}(\theta_q) + \boldsymbol{\upsilon}(n)\\ &= \mathbf{A}\mathbf{s}(n) + \boldsymbol{\upsilon}(n)\end{aligned} \tag{1}$$

where the vector $\mathbf{s}(n) \in \mathbb{C}^{Q\times 1}$ is formed of the complex amplitudes of the source signals $s_q(n)$, which are stationary, ergodic and mutually independent with symmetric distribution, zero-mean and non-zero kurtosis γ_{4,s_q}, $q \in [1, Q]$, with azimuth angles given by θ_q and no elevation angle. Moreover, the signals $s_q(n)$ are assumed to be independent of the additive Gaussian noise $\boldsymbol{\upsilon}(n) \in \mathbb{C}^{M\times 1}$, which is stationary with zero-mean and unknown spatial correlation.

Matrix $\mathbf{A} \in \mathbb{C}^{M\times Q}$ concatenates the steering vectors $\mathbf{a}(\theta_q) \in \mathbb{C}^{M\times 1}$, containing the DOA information θ_q associated with each source $q \in [1, Q]$. The array matrix $\mathbf{A}$ can therefore be written as

$$\mathbf{A} = \Big[\mathbf{a}(\theta_1)\ldots\mathbf{a}(\theta_Q)\Big] \in \mathbb{C}^{M\times Q}, \tag{2}$$

where the mth element of vector $\mathbf{a}(\theta_q)$ corresponds to the spatial response of the mth array element with respect to the source q. We further consider that the sources are spaced far enough apart from each other so that the steering vectors are mutually independent. Assuming a planewave propagation with no coupling between sensors [13], we can write:

$$a_m(\theta_q) = \exp\left\{\frac{\jmath\, 2\,\pi\, x_m \cos\theta_q}{\lambda}\right\}, \tag{3}$$

where $\jmath = \sqrt{-1}$ and x_m is the distance of each array element $m \in [1, M]$ with respect to a given reference sensor, assumed by convention to be the first antenna, i.e. $x_1 = 0$. The signal wavelength λ is given by $\lambda = c/f_c$, where f_c is the carrier frequency and the constant c is the propagation speed of the light. Due to (3), matrix $\mathbf{A}$ has a particular unit-modulus property and, since $x_1 = 0$, it also has a all-one first row, i.e. $\mathbf{A}_{1\cdot} = [1, 1, \ldots, 1]$. In the case of Uniform Linear Antenna (ULA) arrays, the sensors are equally spaced from each other along the array axis and distanced of Δx with respect to adjacent sensors, so that (3) becomes:

$$a_m(\theta_q) = \exp\left\{\frac{\jmath\, 2\,\pi\,(m-1)\Delta x \cos\theta_q}{\lambda}\right\}, \tag{4}$$

and the spatial response array matrix $\mathbf{A}$ has the following Vandermonde structure:

$$\mathbf{A} = \begin{pmatrix} 1 & \ldots & 1\\ a_2(\theta_1) & \ldots & a_2(\theta_K)\\ a_2^2(\theta_1) & \ldots & a_2^2(\theta_K)\\ \vdots & \ddots & \vdots\\ a_2^{M-1}(\theta_1) & \ldots & a_2^{M-1}(\theta_K)\end{pmatrix}, \tag{5}$$

where the second row is the generating vector, from which the whole matrix can be deduced.

A. Array output statistics

Let us define the spatial covariance matrix $\mathbf{C}^{(2,y)} \in \mathbb{C}^{M\times M}$, so that $[\mathbf{C}^{(2,y)}]_{i,j} = C_{2,y}(i,j) \triangleq \mathit{cum}\left[y_i(n), y_j^*(n)\right]$, $i,j \in [1, M]$. From (1), we have $\mathbf{C}^{(2,y)} = \mathbb{E}\left\{\mathbf{y}(n)\mathbf{y}^{\mathsf{H}}(n)\right\}$ and hence:

$$\mathbf{C}^{(2,y)} = \mathbf{A}\boldsymbol{\Gamma}_{2,s}\mathbf{A}^{\mathsf{H}} + \mathbf{C}^{(2,\upsilon)} \tag{6}$$

where $\boldsymbol{\Gamma}_{2,s} = \mathbb{E}\left\{\mathbf{s}(n)\mathbf{s}^{\mathsf{H}}(n)\right\}$ and $\mathbf{C}^{(2,\upsilon)} = \mathbb{E}\left\{\boldsymbol{\upsilon}(n)\boldsymbol{\upsilon}^{\mathsf{H}}(n)\right\}$. Due to the assumption of mutual independence of the sources, it follows that $\boldsymbol{\Gamma}_{2,s}$ is a diagonal matrix with diagonal entries given by $\gamma_{2,s_q} = \mathbb{E}\left\{|s_q(n)|^2\right\}$, $q \in [1, Q]$.

Moreover, by defining the 4th-order tensor $\mathcal{C}^{(4,y)} \in \mathbb{C}^{M\times M\times M\times M}$ with scalar component given by $C_{4,y}(i,j,k,l) \triangleq \mathit{cum}\left[y_i^*(n), y_j(n), y_k^*(n), y_l(n)\right]$, we can build the Quadricovariance matrix $\mathbf{C}^{(4,y)} \in \mathbb{C}^{M^2\times M^2}$, as $[\mathbf{C}^{(4,y)}]_{(j-1)M+i,\,(k-1)M+l} = C_{4,y}(i,j,k,l)$, yielding the structure given below [14]:

$$\mathbf{C}^{(4,y)} = \Big(\mathbf{A}\diamond\mathbf{A}^*\Big)\boldsymbol{\Gamma}_{4,s}\Big(\mathbf{A}\diamond\mathbf{A}^*\Big)^{\mathsf{H}}, \tag{7}$$

where $\boldsymbol{\Gamma}_{4,s} = \mathit{Diag}\left(\gamma_{4,s_1}, \ldots, \gamma_{4,s_Q}\right)$ and $\diamond$ denotes the Khatri-Rao product, i.e. the column-wise Kronecker product, defined for $\mathbf{X} \in \mathbb{C}^{m\times n}$ and $\mathbf{Y} \in \mathbb{C}^{p\times n}$ as $\mathbf{X}\diamond\mathbf{Y} \triangleq [\mathbf{X}_{\cdot 1}\otimes\mathbf{Y}_{\cdot 1}\ldots\mathbf{X}_{\cdot n}\otimes\mathbf{Y}_{\cdot n}] \in \mathbb{C}^{mp\times n}$ [15]. The rank of $\mathbf{C}^{(4,y)}$ equals Q when $Q \le M^2$.

Comparing (7) with (6), we note strong similarities in the structures of $\mathbf{C}^{(4,y)}$ and (the noiseless part of) $\mathbf{C}^{(2,y)}$. Both are diagonal quadratic forms, the latter one being built from the source steering vectors, while $\mathbf{C}^{(4,y)}$ involves a column-wise Kronecker product of those vectors. This structural analogy is the basic idea allowing for extending some array processing methods based on SOS to the 4th-order [14]. In addition, since the above analysis only evokes the linearity and the additivity properties of cumulants, it can be extended to statistics of any (even) order. In fact, complex-valued

2κth-order output cumulants, defined as $C_{2\kappa,y}(i_1,\ldots,i_{2\kappa}) \triangleq \textit{cum}\left[y_{i_1}(n),\ldots,y_{i_\kappa}(n),y^*_{i_{\kappa+1}}(n),\ldots,y^*_{i_{2\kappa}}(n)\right]$, $\kappa \geq 2$, can be represented by a Hermitian matrix $\mathbf{C}_\ell^{(2\kappa,y)} \in \mathbb{C}^{M^\kappa \times M^\kappa}$, which admits the following decomposition:

$$\mathbf{C}_\ell^{(2\kappa,y)} = \left(\mathbf{A}^{\diamond\ell} \diamond \mathbf{A}^{*^{\diamond\kappa-\ell}}\right)\mathbf{\Gamma}_{2\kappa,s}\left(\mathbf{A}^{\diamond\ell} \diamond \mathbf{A}^{*^{\diamond\kappa-\ell}}\right)^{\mathsf{H}}, \quad (8)$$

$\ell \in [1,\kappa]$, where $\mathbf{\Gamma}_{2\kappa,s} = \textit{Diag}\left(\gamma_{2\kappa,s_1},\ldots,\gamma_{2\kappa,s_Q}\right)$ and $\gamma_{2\kappa,s_q}$ is the 2κ th-order cumulant of the input signal $s_q(n)$. The notation $\mathbf{X}^{\diamond n}$ stands for a multiple Khatri-Rao product involving a matrix $\mathbf{X}$ so that $\mathbf{X}^{\diamond n} = \mathbf{X} \diamond \mathbf{X} \diamond \ldots \diamond \mathbf{X}$, where the Khatri-Rao product symbol $\diamond$ appears $n-1$ times. Throughout the rest of this paper, we omit the index ℓ, choosing by convention $\ell = \kappa/2$ when κ is even and $\ell = (\kappa+1)/2$ for κ odd.

In practical applications, the array output statistics are not known and must be estimated from the received data sequences using the *ergodicity* assumption. Exact expressions exist for computing the variance of cumulant estimators of order 2κ, involving very complicated calculations with cumulants of order up to 4κ [16]. When using HOS, it is particularly important to note that short sample data sequences may lead to significant errors with respect to true cumulants [9].

B. The Virtual Array concept

By replacing the array response vectors by their Kronecker product, we actually increase the dimension of the observation space, thus allowing for the separation of a greater number of sources [17]. Actually, the element in position $(m_1-1)M + m_2$ of the 2nd-order steering vector $\mathbf{a}(\theta_q) \otimes \mathbf{a}^*(\theta_q)$ can be viewed as a *virtual sensor* (VS) distanced of $(m_1-m_2)\Delta x$ with respect to the reference sensor, for all $m_1, m_2 \in [1,M]$. The number of different VS depends on the array geometry since some VS positions may coincide. In the case of a ULA array with space diversity only, a 2nd-order VA has $2M-1$ different VS, meaning that it can deal with up to $2M-2$ independent sources [18]. In a general case, using an optimal array geometry, we can have up to M^2-M+1 different VS.

The theory of Virtual Arrays has been introduced independently in [10] and [17] using 4th-order statistics. The concept has been further developed in [18] and [19], for the cases of 4th- and higher-order cumulants, respectively. One major interest in using high-order (HO) VAs consists in exploiting the Kronecker structure that naturally arises in the HOS representations to improve the array angular resolution. Despite the increased variance of the HOS estimators, the HO VAs provide resolution gains, which can be measured by means of the spatial correlation between two sources, given by the normalized inner product of the respective steering vectors.

C. MUSIC-like DF algorithms

In its basic form, the MUSIC technique has been introduced to provide asymptotically unbiased estimates of the parameters of multiple wavefronts arriving at an antenna array [1], [2]. Exploiting the orthogonality between the signal and noise subspaces, the MUSIC algorithm aims to determine the number of sources, their location (DOAs) and the cross-correlations among the directional waveforms. The SOS-based MUSIC (2-MUSIC) algorithm is of particular interest in the noiseless case and when the additive noise is spatially white.

Assuming $M > Q$, the 2-MUSIC algorithm consists in searching for Q local maxima of the following localization function:

$$P_2(\theta) = \frac{1}{\left\|\mathbf{w}(\theta)^{\mathsf{H}}\mathbf{U}_n\right\|^2}, \quad (9)$$

where the orthogonal projector $\mathbf{w}(\theta) \in \mathbb{C}^{M\times 1}$ has the form of the steering vector $\mathbf{a}(\theta)$ defined in (3) and $\mathbf{U}_n \in \mathbb{C}^{M\times M-Q}$ is formed from the eigenvectors of $\mathbf{C}^{(2,y)}$ associated with its $M-Q$ smallest eigenvalues. The function $P_2(\theta)$ clearly measures the orthogonality between the signal and noise subspaces for the source q. Using this method, we can only localize $M-1$ sources, thus only overdetermined mixtures ($M \geq Q$) can be treated.

On the other hand, the Khatri-Rao structure of $\mathbf{C}_\ell^{(2\kappa,y)}$ given in (8) yields an increased number of virtual antenna elements, thus allowing for the localization of more sources than sensors, the amount of which varies in function of the array geometry. This is the main idea behind the extension of the MUSIC algorithm to the 4th- (and higher-) orders [7], [14]. In fact, using $\mathbf{C}^{(2\kappa,y)}$, a general localization function can be defined as:

$$P_{2\kappa}(\theta) = \frac{1}{\left\|\mathbf{w}_\kappa(\theta)^{\mathsf{H}}\,\mathbf{U}_n\right\|^2}, \quad (10)$$

where the orthogonal projector $\mathbf{w}_\kappa(\theta) \in \mathbb{C}^{M^\kappa\times 1}$ takes the form of the κth-order steering vector and $\mathbf{U}_n$ is the $M^\kappa \times (M^\kappa - Q)$ matrix that concatenates the eigenvectors of $\mathbf{C}^{(2\kappa,y)}$ associated with its $M^\kappa - Q$ smallest eigenvalues. Source DOAs are found by searching for the local maxima of $P_{2\kappa}(\theta)$. See [8] for a survey on the 2κ-MUSIC algorithms.

III. PARAFAC-BASED APPROACH FOR DIRECTION-FINDING

In this section, we propose a high-resolution DF algorithm that creates a 3rd-order virtual array, only exploiting the Khatri-Rao structure of a 4th-order cumulant tensor. Our solution is based on an iterative single-step least-squares (SS-LS) Parafac decomposition technique [11], [12], which exploits the symmetry properties of 4th-order output cumulants.

Let us rewrite the scalar representation of the 4th-order tensor $\mathcal{C}^{(4,y)}$, defined in section II-A, as follows:

$$C_{4,y}(i,j,k,l) = \sum_{q=1}^{Q} \gamma_{4,s_q}\, a_i^*(\theta_q)\, a_j(\theta_q)\, a_k^*(\theta_q)\, a_l(\theta_q) \quad (11)$$

for $1 \leq i,j,k,l \leq M$ and $q \in [1,Q]$, where the nonzero source Kurtoses γ_{4,s_q} are assumed unknown. It follows from (11) that $\mathcal{C}^{(4,y)}$ is a 4th-order tensor with rank Q that admits a Parafac decomposition of which the canonical components can be straightforwardly deduced and are all written in terms of the array matrix $\mathbf{A}$ and the diagonal Kurtosis matrix $\mathbf{\Gamma}_{4,s}$ [12]. Let us now define the unfolded tensor representation $\mathbf{C}_{[1]} \in \mathbb{C}^{M^3\times M}$, as follows:

$$\left[\mathbf{C}_{[1]}\right]_{(j-1)M^2+(k-1)M+l,\, i} = C_{4,y}(i,j,k,l), \quad (12)$$

which can be easily shown to be written as follows:

$$\begin{aligned}\mathbf{C}_{[1]} &= (\mathbf{A} \diamond \mathbf{A}^* \diamond \mathbf{A})\, \mathbf{\Gamma}_{4,s} \mathbf{A}^{\mathsf{H}} & (13)\\ &= \mathbf{A}^{(3)}\, \mathbf{\Gamma}_{4,s}\, \mathbf{A}^{\mathsf{H}} & (14)\end{aligned}$$

where $\mathbf{A}^{(3)}$ is the $M^3 \times Q$ 3rd-order VA matrix, defined as $\mathbf{A}^{(3)} = \mathbf{A} \diamond \mathbf{A}^* \diamond \mathbf{A}$, with $\mathbf{A}$ defined in (5).

A. The iterative SS-LS algorithm

Using the unfolded tensor representation $\mathbf{C}_{[1]}$, the SS-LS algorithm iteratively minimizes the following LS cost function:

$$\psi\big(\hat{\mathbf{A}}_{r-1}, \mathbf{A}\big) \triangleq \left\| \mathbf{C}_{[1]} - \hat{\mathbf{A}}^{(3)}_{r-1}\, \mathbf{\Gamma}_{4,s}\, \mathbf{A}^{\mathsf{H}} \right\|_F^2, \qquad (15)$$

with

$$\hat{\mathbf{A}}^{(3)}_{r-1} = \hat{\mathbf{A}}_{r-1} \diamond \hat{\mathbf{A}}^*_{r-1} \diamond \hat{\mathbf{A}}_{r-1}, \qquad (16)$$

where r is the iteration number and $\|\cdot\|_F$ denotes the Frobenius norm. The iterative minimization of $\psi(\hat{\mathbf{A}}_{r-1}, \mathbf{A})$ yields the following LS solution:

$$\begin{aligned}\hat{\mathbf{A}}^{\mathsf{H}}_r &\triangleq \arg\min_{\mathbf{A}} \psi\big(\hat{\mathbf{A}}_{r-1}, \mathbf{A}\big)\\ &= \mathbf{\Gamma}^{-1}_{4,s}\, \hat{\mathbf{A}}^{(3)\#}_{r-1}\, \mathbf{C}_{[1]}. & (17)\end{aligned}$$

Note that we only have to initialize $\hat{\mathbf{A}}_0$. In fact, at each iteration $r \geq 1$, we deduce $\hat{\mathbf{A}}^{(3)}_{r-1}$ from (16) and then, we compute $\hat{\mathbf{A}}_r$ from (17).

Iterative LS algorithms are known to be very sensitive to the initialization. Exploiting the unit-modulus property of the array steering matrix, the following modification of the SS-LS algorithm is expected to improve convergence. After initializing $\hat{\mathbf{A}}_0$ with an $M \times Q$ matrix drawn from a (complex) Gaussian distribution, we perform the following steps:

1) *At each $r \geq 1$, before computing $\hat{\mathbf{A}}_r$, divide each entry of the preceding estimate by its own magnitude, i.e.*

$$[\hat{\mathbf{A}}_{r-1}]_{mq} \leftarrow \frac{[\hat{\mathbf{A}}_{r-1}]_{mq}}{\left|[\hat{\mathbf{A}}_{r-1}]_{mq}\right|}, \qquad \textit{for } q = 1,\ldots,Q \ \textit{and } m = 1,\ldots,M;$$

2) *Normalize each column by its first element:*

$$[\hat{\mathbf{A}}_{r-1}]_{\cdot q} \leftarrow \frac{[\hat{\mathbf{A}}_{r-1}]_{\cdot q}}{[\hat{\mathbf{A}}_{r-1}]_{1\,q}};$$

3) *Deduce $\hat{\mathbf{A}}^{(3)}_{r-1}$ from (16) and compute the array matrix estimate at iteration r as follows:*

$$\hat{\mathbf{A}}_r \leftarrow \left[\hat{\mathbf{A}}^{(3)\#}_{r-1}\, \mathbf{C}_{[1]}\right]^H. \qquad (18)$$

Notice that, due to the normalization step, the computation of $\hat{\mathbf{A}}_r$ becomes independent of the source Kurtosis matrix $\mathbf{\Gamma}_{4,s}$. The algorithm is stopped when $|e(r) - e(r-1)|^2 \leq \varepsilon$, where $e(r) = \|\hat{\mathbf{A}}_r - \hat{\mathbf{A}}_{r-1}\|_F / \|\hat{\mathbf{A}}_r\|_F$ and ε is an arbitrary small positive constant.

B. Uniqueness and identifiability

Due to the Vandermonde structure of the array matrix, given in (5), and assuming that the sources are not closely located, matrix $\mathbf{A}$ can be shown to be full k-rank [20], so that $k_A = r_A = \min(M, Q)$. In this case, the uniqueness of the Parafac decomposition of tensor $\mathcal{C}^{(4,y)}$ is ensured under the condition stated by the Kruskal Theorem [21], which yields $Q \leq (4M-3)/2$, for $M < Q$ [12]. This leads to the following sufficient uniqueness condition:

$$2 \leq Q \leq 2M - 2. \qquad (19)$$

Although (19) is not a necessary condition, it establishes an upper bound on the number of guaranteed resolvable sources. This bound limits the number of sources that we can treat using the 3rd-order VA matrix $\mathbf{A}^{(3)}$, regardless of the number of virtual sensors.

In the case of a ULA array with M sensors, the number of different virtual sensors associated with the κth-order VA is shown to be equal to $\kappa(M-1)+1$ [19]. In this context, the 3rd-order VA matrix $\mathbf{A}^{(3)}$ admits a maximum capacity of $3M-3$ sources. Since the SS-LS approach can only ensure a unique solution under the uniqueness condition (19), it should not be used to identify a VA with more than $2M-2$ sources. This ensures that the noise subspace has at least M free dimensions (i.e. linearly independent basis vectors). Moreover, when using an M-element ULA array, the capacity of the 4-MUSIC algorithm is associated with the number of VS sensors of a 2nd-order VA, which coincides with the upper bound of the SS-LS approach. However, if 4-MUSIC operates with maximal capacity, the noise subspace of the 2nd-order VA has only one free dimension.

C. DOA estimation

The source DOAs can be recovered from the VA matrix $\mathbf{A}^{(3)}$ by using a 6th-order MUSIC-like localization function $P_6(\theta)$, such as defined in (10) with $\kappa = 3$, i.e.

$$P_6(\theta) = \frac{1}{\left\| \mathbf{w}_3(\theta)^{\mathsf{H}}\, \mathbf{U}_n \right\|^2}, \qquad (20)$$

where $\mathbf{w}_3(\theta) = \mathbf{a}(\theta) \otimes \mathbf{a}^*(\theta) \otimes \mathbf{a}(\theta)$, with $\mathbf{a}(\theta)$ defined in (3), and $\mathbf{U}_n$ is a $M^3 \times (M^3 - Q)$ matrix representing the noise subspace and formed of the left singular vectors of $\mathbf{A}^{(3)}$ associated with its $M^3 - Q$ smallest singular values. The angles θ_q are obtained from the parameters of the orthogonal projectors $\mathbf{w}_3(\theta) \in \mathbb{C}^{M^3 \times 1}$ associated with the local maxima of the 6th-order localization function $P_6(\theta)$, defined in (20).

Although involving a channel estimation stage prior to source localization, the above described approach allows for improved resolution due to the use of a 3rd-order VA, obtained without resorting to 6th-order statistics. While keeping the cumulant estimation variance at a lower level compared with the 2κ-MUSIC algorithms, $\kappa > 2$, the proposed technique is robust to an additive Gaussian noise with unknown spatial correlation, contrary to the 2-MUSIC method. In addition, for ULA arrays, the SS-LS approach is shown to resolve as many sources as the 4-MUSIC algorithm.

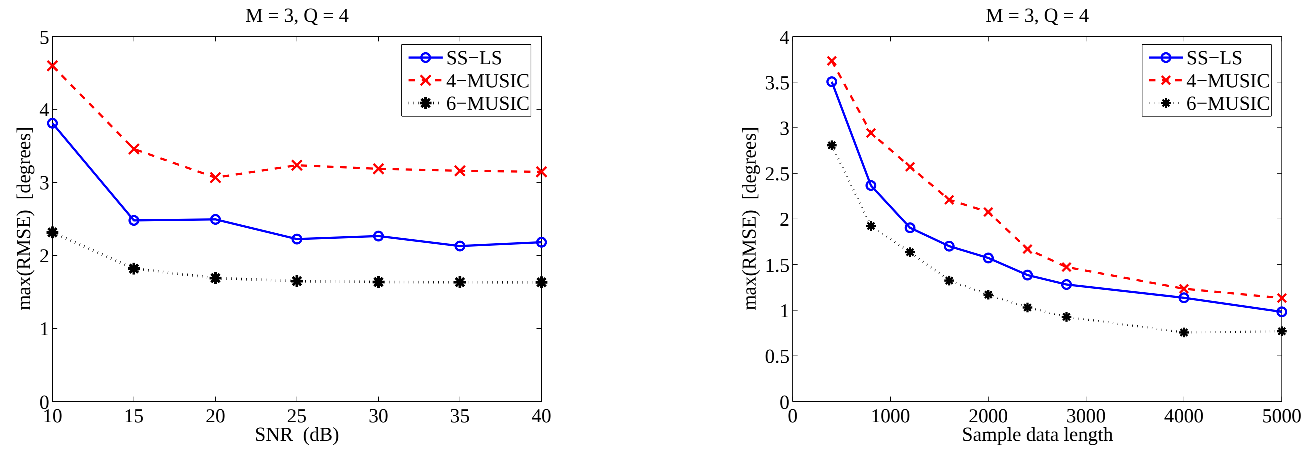

Fig. 1. Maximal RMSE as a function of the SNR with $N = 1000$ (left) and as a function of the sample data length with SNR=15dB (right).

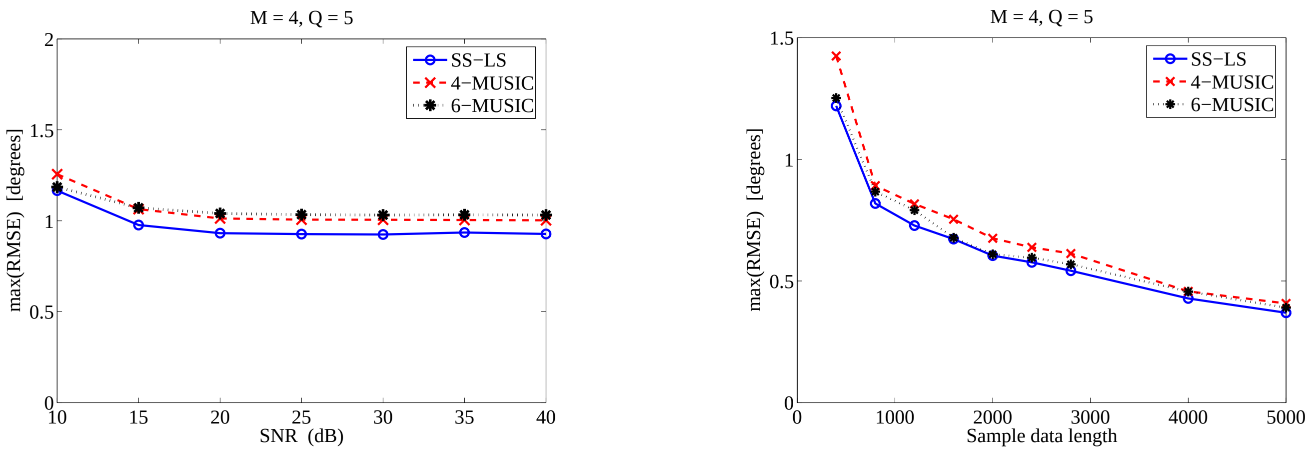

Fig. 2. Maximal RMSE as a function of the SNR for $N = 1000$ (left) and as a function of the sample data length with SNR=15dB (right).

IV. Simulation Results

In this section, we evaluate the performance of the proposed DOA estimation method, providing performance comparisons using the 2-, 4- and 6-MUSIC algorithms. We use the root mean-squared error (RMSE) performance criterion, defined for each source q as follows [8]:

$$\mathrm{RMSE}(q) \triangleq \sqrt{\frac{1}{R}\sum_{r=1}^{R}\left|\hat{\theta}_q^{\langle r\rangle} - \theta_q\right|^2}, \qquad q \in [1, Q], \qquad (21)$$

where R is the number of Monte Carlo simulations and $\hat{\theta}_q^{\langle r\rangle}$ is the estimation of θ_q for the simulation r. The DOA estimates $\hat{\theta}_q^{\langle r\rangle}$, $q \in [1, Q]$, are deduced from the angle arguments of the orthogonal projectors $\mathbf{w}_\kappa(\theta)$ associated with the local maxima of the corresponding localization function $P_{2\kappa}(\theta)$. Local maxima can be obtained by searching the critical points, i.e. where the first derivative is zero, with a negative second derivative.

We first simulated the case of a ULA array with $M = 3$ narrowband sensors spaced of $\lambda/2$, receiving $Q = 4$ sources with azimuth angles given by $\theta_1 = -55°$, $\theta_2 = -25°$, $\theta_3 = 5°$, $\theta_4 = 50°$. The array output signals are corrupted by spatially white additive Gaussian noise. The curves in fig. 1 show the RMSE for the worst estimated sources, as a function of the SNR for $N = 1000$ (left), and also for several values of the sample data length (right), with a fixed SNR value of 15dB. In this case, the SS-LS and the 4-MUSIC algorithms operate with their maximal capacity in terms of the number of sources. By exploiting the larger noise subspace of the 3rd-order virtual array, the SS-LS approach provides better results than the 4-MUSIC algorithm, using the same output statistics. In this scenario, the 6-MUSIC algorithm is not at its identifiability bound and gives better results than the two other algorithms, at the cost of having to estimate 6th-order cumulants.

By adding a fourth sensor ($M = 4$) to the antenna array (with $\lambda/2$ spacing), we set up another simulation scenario with $Q = 5$ sources. In this case, the additional source arrives from the direction $\theta_5 = 20°$, with no elevation angle. In fig. 2, we show the maximal RMSE as a function of the SNR, for $N = 1000$ (left). These curves demonstrate that the three algorithms achieve better performance, with very similar

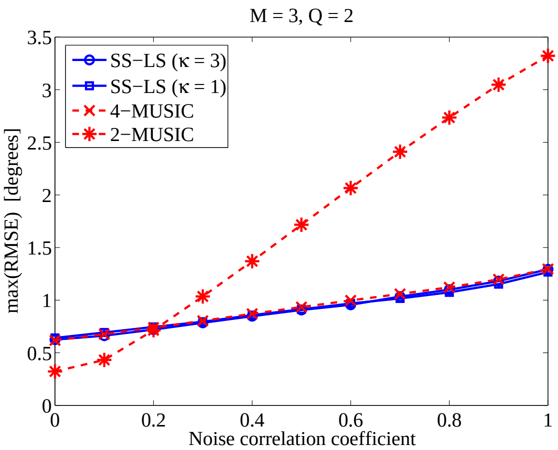

Fig. 3. Maximal RMSE vs. noise spatial correlation ($N = 1000$ and SNR=5dB).

results when the VAs do not operate with maximal capacity. The results for the worst estimated source are given at the right side of fig. 2 for several values of the sample data length, with a fixed SNR of 15dB. In this case, the 6-MUSIC algorithm does not yield any noticeable advantage.

We have also tested the algorithms in presence of Gaussian noise with unknown spatial correlation. In this case, we used a $\lambda/2$-spaced 3-element ULA array receiving $Q = 2$ sources with DOAs given by $\theta_1 = 5°$ and $\theta_2 = 50°$, respectively. Since this is an overdetermined case, we used the SS-LS approach to estimate the DOAs from both, the 3rd-order virtual array $\hat{\mathbf{A}}^{(3)}$ ($\kappa = 3$) and the estimated array matrix $\hat{\mathbf{A}}$ ($\kappa = 1$). The additive Gaussian noise has been simulated so that its spatial correlation matrix is given by $[\mathbf{R}_\upsilon]_{ij} = \sigma_\upsilon^2 \rho^{|i-j|}$, $i, j \in [1, M]$, where σ_υ^2 is the noise variance per antenna and ρ is the spatial correlation coefficient of the noise. In fig. 3, we compare our results with the 2- and 4-MUSIC algorithms using $N = 1000$ output symbols, with a SNR of 5dB, for different values of the noise spatial correlation. Note that, for $\kappa = 1$ as well as for $\kappa = 3$, the SS-LS approach performed very closely to the 4-MUSIC algorithm, showing good robustness with respect to spatially colored noise, as it should be expected. The 2-MUSIC algorithm, on the other hand, degrades as ρ increases, since the SOS are not able to handle an additive noise with unknown spatial correlation.

V. Conclusion

In this paper, we have considered the blind source localization problem in the context of multiuser narrowband array processing, under the assumption of sources located at the far-field of the antenna array. The DOA estimation problem has been treated using the 4th-order cumulants only. A high-resolution DF algorithm has been proposed, exploiting the structure of the cumulant tensor. This method is based on the estimation of an array matrix formed from a double (column-wise) Kronecker product, thus creating an enhanced virtual array that commonly only arises when using 6th-order statistics. This yields an augmented observation space, which provides a resolution improvement without resorting to statistics of order higher than fourth. Consequently, the proposed method works well even for relatively short output data sequences and it is robust with respect to an additive Gaussian noise with an unknown spatial correlation. Making use of the symmetry properties of 4th-order output cumulants, the estimation of the enhanced virtual array utilizes the iterative SS-LS technique to perform the Parafac decomposition of the cumulant tensor. In the case of ULA arrays, this yields as many resolvable sources as the 4-MUSIC algorithm but with better DOA estimation performance, as confirmed by our simulation results.

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