Enhanced Model Order Estimation in Colored Noise Scenarios via Noise Prewhitening

João Paulo Carvalho Lustosa da Costa, Florian Roemer and Rafael Timóteo de Sousa Jr.

***Resumo*— Cenários com ruído colorido são freqüentes em aplicações de processamento de sinais. Na presença do ruído colorido, a maior parte das técnicas de seleção da ordem do modelo são severamente degradadas, pois elas assumem o ruído como sendo branco.**

Neste artigo, nós propomos uma técnica para seleção da ordem do modelo para cenários com ruído colorido. Na técnica proposta, nós combinamos o Modified Exponential Fitting Test (M-EFT) com a etapa de prewhitening. A partir da incorporação do prewhitening, a suposição necessária para o M-EFT que o perfil dos autovalores de ruído podem ser aproximados a uma curva exponencial torna-se novamente válida. Nós mostramos que o M-EFT combinado com o prewhitening alcança a mais elevada probabilidade de correta detecção da ordem do modelo para cenários com ruído colorido em comparação ao estado da arte das técnicas de seleção da ordem do modelo.

***Palavras-Chave*— Seleção da ordem do modelo, Prewhitening, Modified Exponential Fitting Test.**

***Abstract*— Colored noise scenarios are frequent in signal processing applications. In the presence of colored noise, most model order schemes are severely degraded, since they assume white noise.**

In this paper, we propose a model order selection scheme for colored noise. In our proposed scheme, we combine the Modified Exponential Fitting Test (M-EFT) with the prewhitening step. By incorporating the prewhitening step, the assumption necessary for the M-EFT that the noise eigenvalue profile can be approximated by an exponential curve becomes valid. We show that the M-EFT combined with the prewhitening step achieves a very high probability of correct model order detection for colored noise scenarios in comparison with the state-of-the-art model order selection schemes.

***Keywords*— Model Order Selection, Prewhitening, Modified Exponential Fitting Test.**

I. INTRODUCTION

The estimation of the number of main components, also known as model order selection, is required in several scientific applications, such as psychometrics [2], chemistry [1], radar [13], sonar [18], communications, medical imaging, and channel modeling [5], [17]. For most applications the assumption that the noise is uncorrelated may not be valid. For instance, the underwater noise components of a sonar system are in general spatially correlated [18], noise sources in audio applications are spatially non-uniformly separated [9], mutual coupling between sensors and oversampling also produce a spatial and temporal noise correlation, respectively.

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For colored noise scenarios, the parameter estimation [6] without prewhitening can be severely degraded. Similar degradation happens for model order selection schemes in such scenarios [13]. Moreover, usually when the model order is estimated, the statistics of the colored noise are not taken into account. However, in some applications, an estimate of the noise covariance matrix $\boldsymbol{R}_{ww}$ can be available by collecting measurement samples in the absence of signal components. For instance, in speech processing applications, the noise can be recorded in speechless frames [9].

As shown in [5], [4], [3], [7], for white noise scenarios, the modified exponential fitting test (M-EFT) outperforms the state-of-the-art model order selection techniques independently of the array size. However, the M-EFT is restricted to white noise applications. In addition, for multidimensional scenarios, it is frequent to stack all the dimensions of the tensor into only one dimension such that we have a highly structured matrix, where one dimension is much greater than the other dimension. For such highly structured matrices the M-EFT provides a huge gain in comparison to the state-of-the-art model order selection schemes [5], [4], [3]. Therefore, the multidimensional gain of the R-dimensional exponential fitting test (R-D EFT) is only possible since the R-D EFT is based on the M-EFT. It is also known in the literature that for colored noise scenarios with severe correlation, the RADOI outperforms the state-of-the-art model order selection techniques [13], [3].

In this paper, we propose to extend the M-EFT for colored noise scenarios by taking into account noise samples without signal components. With the statistics of the colored noise, the data can be prewhitened through stochastic prewhitening schemes [9], [8], [14]. Applying the M-EFT on the prewhitened data high probabilities of correct detection of the model order are achievable.

The remainder of this paper is organized as follows. After reviewing the notation in Section II, the data model is presented in Section III. Then the modified Exponential Fitting Test (M-EFT) in conjunction with the stochastic prewhitening is proposed in Section IV. The simulation results in Section V confirm the improved performance of M-EFT in conjunction with the stochastic prewhitening. Conclusions are drawn in Section VI.

II. NOTATION

In order to facilitate the distinction between scalars and matrices, the following notation is used: scalars are denoted as italic letters $(a, b, \ldots, A, B, \ldots, \alpha, \beta, \ldots)$, column vectors as lower-case bold-face letters $(\boldsymbol{a}, \boldsymbol{b}, \ldots)$ and matrices as bold-face capitals $(\boldsymbol{A}, \boldsymbol{B}, \ldots)$. Lower-order parts are consistently named: the (i,j)-element of the matrix $\boldsymbol{A}$, is denoted as $a_{i,j}$.

We use the superscripts ${}^{\mathrm{T}}, {}^{\mathrm{H}}, {}^{-1}, {}^{+}$ and * for transposition, Hermitian transposition, matrix inversion, the Moore-Penrose pseudo inverse of matrices, and complex conjugation, respectively.

III. Data model

We consider a linear mixture of d time series $s_i(n)$ where $i = 1, \ldots, d$ and $n = 1, \ldots, N$ with mixing coefficients $a_i(m)$ where $m = 1, \ldots, M$. Additionally, our measurements are contaminated by colored noise samples $w_m^{(\mathrm{c})}(n)$. Therefore, the measured samples $x_m(n)$ are modeled by

$$x_m(n) = \sum_{i=1}^{d} a_i(m) s_i(n) + w_m^{(\mathrm{c})}(n). \tag{1}$$

The scalars $s_i(n)$, whose variance is σ_s^2, and $w_m^{(\mathrm{c})}(n)$, whose variance is σ_w^2, model the source symbols and the additive correlated noise component inherent in the measurement process, respectively. In the context of array signal processing, each of scalar $a_i(m)$ represents the elements of the mixing matrix $\boldsymbol{A}$, and the dimensions M and N can be the number of sensors and the number of subsequent time instants, respectively.

In matrix form, we can represent (1) in the following way

$$\boldsymbol{X} = \boldsymbol{A} \cdot \boldsymbol{S} + \boldsymbol{W}^{(\mathrm{c})}, \tag{2}$$

where $\boldsymbol{A} \in \mathbb{C}^{M \times d}$ contains the mixing vectors $\boldsymbol{a}_i \in \mathbb{C}^{M \times 1}$ for each of the d sources, $\boldsymbol{S} \in \mathbb{C}^{d \times N}$ contains the symbols $s_i(n)$, and $\boldsymbol{X}$ is corrupted by some correlated noise matrix $\boldsymbol{W}^{(\mathrm{c})} \in \mathbb{C}^{M \times N}$. We can model the noise $\boldsymbol{W}^{(\mathrm{c})}$ as $\boldsymbol{W}^{(\mathrm{c})} = \boldsymbol{L} \cdot \boldsymbol{W}$, where $\boldsymbol{L}$ correlates the white noise matrix $\boldsymbol{W}$. The noise elements $w_m(n)$ of $\boldsymbol{W}$ are modeled as ZMCSCG (zero-mean circularly-symmetric complex Gaussian) random variables. Note that the ranks of $\boldsymbol{A}$ and $\boldsymbol{S}$ are equal to the model order d. Therefore, rank of $\boldsymbol{A} \cdot \boldsymbol{S}$ is equal to d. The rank of $\boldsymbol{X}$ is $\alpha = \min\{M, N\} \geq d$. Therefore, our goal is given $\boldsymbol{X}$ to determine d.

The covariance matrix of the data model (2) is given by

$$\begin{aligned} \boldsymbol{R}_{xx} &= \mathrm{E}\{\boldsymbol{X} \cdot \boldsymbol{X}^{\mathrm{H}}\} \\ &= \boldsymbol{A} \cdot \boldsymbol{R}_{\mathrm{ss}} \cdot \boldsymbol{A}^{\mathrm{H}} + \sigma_w^2 \cdot \boldsymbol{R}_{ww}, \end{aligned} \tag{3}$$

where $\boldsymbol{R}_{\mathrm{ss}}$ is the signal covariance matrix, $\boldsymbol{R}_{ww}$ is the noise covariance matrix, such that $\mathrm{tr}(\boldsymbol{R}_{ww}) = M$, and $\mathrm{E}\{\cdot\}$ is the expected value operator. In practice, $\boldsymbol{R}_{xx}$ in (3) can be estimated from a finite set of realizations via

$$\hat{\boldsymbol{R}}_{xx} = \frac{1}{N} \cdot \boldsymbol{X} \cdot \boldsymbol{X}^{\mathrm{H}}. \tag{4}$$

In the absence of signals, i.e., $\boldsymbol{X} = \boldsymbol{W}^{(\mathrm{c})} \in \mathbb{C}^{M \times N_l}$, the noise covariance matrix $\boldsymbol{R}_{ww}$ can be estimated by using (4) and by replacing N by N_l. Note that we assume that N_l samples without signal components are available in order to estimate the noise statistics.

IV. Modified Exponential Fitting Test with Prewhitening (M-EFT PWT)

Some model order selection techniques such as AIC and MDL assume that all noise eigenvalues are identical. This assumption is only fulfilled asymptotically for $N \rightarrow \infty$ and therefore implicitly requires $N >> M$. On the other hand, the Exponential Fitting Test (EFT) [12] has been proposed for scenarios where the number of independent temporal snapshots N is small. Basically the EFT [12] is based on the observation that, in a white Gaussian noise-only case, the profile of the ordered eigenvalues can be well approximated by a decaying exponential. For $N > M$, the EFT outperforms the state-of-the-art model order selection schemes in the literature [7].

For $N < M$, the EFT is outperformed by several model order selection techniques in the literature [7]. The Modified Exponential Fitting Test (M-EFT) [5], [4], [7] has been proposed for any values of M and N. By applying the M-EFT, a very high probability of correct detection is returned for any white noise scenario. Due to the impressive results of the M-EFT for white noise scenarios, an extension of M-EFT for colored noise scenarios is very appealing.

For severely colored noise scenarios, the assumption that the noise eigenvalues fit an exponential profile is not valid. Hence, we propose here the M-EFT in conjunction with a stochastic noise prewhitening (M-EFT PWT) as an attractive model order selection scheme, since the prewhitened noise eigenvalues fit again an exponential curve.

Similarly to the M-EFT, the M-EFT PWT is composed of two steps: the preprocessing step and the model order estimation step. In the preprocessing step in Subsection IV-A, we compute the thresholds η_P and also the prewhitening matrix $\boldsymbol{L}$, while in the model order estimation step in Subsection IV-B, the thresholds η_P and the prewhitening matrix $\boldsymbol{L}$ are used to estimate the model order.

A. M-EFT PWT: Preprocessing Step

This preprocessing step assumes that no signal components are present in our measurements. Therefore, similarly to (4), an estimate of the covariance noise matrix $\hat{\boldsymbol{R}}_{ww}$ is given by

$$\hat{\boldsymbol{R}}_{ww} = \frac{1}{N_l} \cdot \boldsymbol{W}^{(\mathrm{c})\prime} \cdot \left(\boldsymbol{W}^{(\mathrm{c})\prime}\right)^{\mathrm{H}}, \tag{5}$$

where $\boldsymbol{W}^{(\mathrm{c})\prime}$ are the noise samples without signal components.

By applying the Cholesky decomposition of $\hat{\boldsymbol{R}}_{ww}$, we obtain the prewhitening matrix $\boldsymbol{L}$ as shown

$$\hat{\boldsymbol{R}}_{ww} = \boldsymbol{L} \cdot \boldsymbol{L}^{\mathrm{H}}. \tag{6}$$

Once $\boldsymbol{L}$ is obtained the colored noise samples can be prewhitened as follows.

$$\boldsymbol{W}' = \boldsymbol{L}^{-1} \boldsymbol{W}^{(\mathrm{c})\prime}, \tag{7}$$

where $\boldsymbol{W}' \in \mathbb{C}^{M \times N}$ are the prewhitened noise samples with the same dimensions of $\boldsymbol{X}$. Since the noise samples $\boldsymbol{W}'$ are not colored, we can use them to compute the threshold coefficients η_P necessary for the M-EFT. Note that $\boldsymbol{L} \in \mathbb{C}^{M \times M}$ is full rank. Therefore, we assume that N_l is greater than M.

Let λ_i be the i-th eigenvalue of the sample covariance matrix given by $\hat{\boldsymbol{R}}'_{ww} = \frac{1}{N}\boldsymbol{W}' \cdot \left(\boldsymbol{W}'\right)^{\mathrm{H}}$.[1] By assuming the exponential model, we can express each eigenvalue in the following fashion [5]

$$\mathrm{E}\{\lambda_i\} = \mathrm{E}\{\lambda_1\} \cdot q(\alpha,\beta)^{i-1}, \quad (8)$$

where we assume that the eigenvalues are sorted so that λ_1 is the largest. The term $q(\alpha,\beta)$ is defined as

$$q(\alpha,\beta) = \exp\left\{-\sqrt{\frac{30}{\alpha^2+2} - \sqrt{\frac{900}{(\alpha^2+2)^2} - \frac{720\alpha}{\beta(\alpha^4+\alpha^2-2)}}}\right\}, \quad (9)$$

where $\alpha = \min(M,N)$ and $\beta = \max(M,N)$ [5], [4], [7]. The predicted noise eigenvalue can be estimated by taking the previous eigenvalues into account as follows [5]

$$\hat{\lambda}_{\alpha-P} = (P+1)\,\frac{1-q(P+1,\beta)}{1-q(P+1,\beta)^{P+1}}\,\hat{\sigma}^2 \quad (10)$$

$$\hat{\sigma}^2 = \frac{1}{P}\sum_{i=0}^{P-1}\lambda_{\alpha-i}, \quad (11)$$

where $\hat{\sigma}^2$ is the estimated noise power using the previous eigenvalues.

To decide whether the $(\alpha-P)$-th noise eigenvalue $\lambda_{\alpha-P}$ fits to the exponential profile we measure its relative distance to the predicted eigenvalue $\hat{\lambda}_{\alpha-P}$. By setting a threshold η_P we can formulate the following hypotheses:

$$\begin{aligned} H_{P+1}:\ \lambda_{\alpha-P} \text{ is a noise EV }, \ \frac{\lambda_{\alpha-P}-\hat{\lambda}_{\alpha-P}}{\hat{\lambda}_{\alpha-P}} \le \eta_P \\ \bar{H}_{P+1}:\ \lambda_{\alpha-P} \text{ is a signal EV }, \ \frac{\lambda_{\alpha-P}-\hat{\lambda}_{\alpha-P}}{\hat{\lambda}_{\alpha-P}} > \eta_P. \end{aligned} \quad (12)$$

By repeating the hypotheses for several realizations, we obtain the mapping between a set of η_P and the probability of false alarm. Some practical values for the probability of false alarm are between 10^{-3} and 10^{-6}. Since M and N are usually constant, the preprocessing step is performed only once.

B. M-EFT PWT: Model Order Estimation

In this step, both signals and noise are present and we desire to estimate the model order d. Note that once the prewhitening matrix $\boldsymbol{L}$ and the set of η_P are computed, we can apply them several times to estimate the model order d. Hence, the preprocessing step is performed only once.

First we can compute the prewhitened covariance matrix of $\boldsymbol{X}$ as follows

$$\hat{\boldsymbol{R}}'_{xx} = \boldsymbol{L}^{-1} \cdot \hat{\boldsymbol{R}}_{xx} \cdot \boldsymbol{L}^{-\mathrm{H}}. \quad (13)$$

Note that $\boldsymbol{R}_{xx}$ can be replaced by its covariance model given in (3).

$$\boldsymbol{R}'_{xx} = \boldsymbol{L}^{-1} \cdot \boldsymbol{A} \cdot \boldsymbol{R}_{\mathrm{ss}} \cdot \boldsymbol{A}^{\mathrm{H}} \boldsymbol{L}^{-\mathrm{H}} + \boldsymbol{R}'_{ww}. \quad (14)$$

[1]For $M >> N$, the computational complexity of the eigenvalue decomposition (EVD) of $\boldsymbol{R_{xx}} \in \mathbb{C}^{M\times M}$ is much greater than the computatonal complexity of the the singular value decomposition (SVD) of $\boldsymbol{X} \in \mathbb{C}^{M\times N}$.

Note that the model order d, which is given by the rank $\boldsymbol{L}^{-1} \cdot \boldsymbol{A} \cdot \boldsymbol{R}_{\mathrm{ss}} \cdot \boldsymbol{A}^{\mathrm{H}} \boldsymbol{L}^{-\mathrm{H}}$ remains equal to d.

By using the eigenvalues of $\boldsymbol{R}'_{xx}$ and the threshold coefficients η_P in the hypotheses in (12), we can estimate the model order d.

V. Simulation results

In this section we present simulation results demonstrating the performance of the M-EFT in conjunction with stochastic prewhitening (M-EFT PWT). Following the CFAR approach, the probability of false alarm is set to a constant for all signal to noise ratios. For simplicity, we set $P_{\mathrm{fa}}(P) = 10^{-6}$ for all values of P.

We also assume that the noise samples are zero mean circularly symmetric complex Gaussian distributed with variance equal to σ_n^2. The noise covariance matrix for $M = 3$ is given by

$$\boldsymbol{R}_{ww} = \begin{bmatrix} 1 & \rho^* & (\rho^*)^2 \\ \rho & 1 & \rho^* \\ \rho^2 & \rho & 1 \end{bmatrix}, \quad (15)$$

where ρ is the noise correlation coefficient.

The source symbols as well as the mixing matrix elements are zero mean i.i.d. circularly symmetric complex Gaussian distributed. The power of the source symbols is equal to σ_s^2 for all the sources. The SNR at the receiver can then be defined as

$$\mathrm{SNR} = 10 \cdot \log_{10}\left(\frac{\sigma_{\mathrm{s}}^2}{\sigma_{\mathrm{n}}^2}\right). \quad (16)$$

The variance of the mixing matrix elements $a_i(m)$ is one and $a_i(m)$ varies for each realization.

In the simulations, we consider the following state-of-the-art model order selection techniques: Akaike's information theoretic criterion (AIC) [16], the Minimum Description Length (MDL) criterion [16], Efficient Detection Criterion (EDC) [19], RADOI [13], KN [10][2], M-EFT [5], [4], [7], EFT [12], Stein's Unbiased Risk Estimate (SURE) [15], and Nadakuditi Edelman Model Order (NEMO) selection scheme [11].

Figure 1 depicts a scenario where $d = 4$ sources are mixed by a matrix $\boldsymbol{A} \in \mathbb{C}^{M\times d}$, where $M = 70$, $\rho = 0.9$ and we collect $N = 10$ snapshots. We can clearly see that the M-EFT PWT outperforms all state-of-the-art model order selection techniques. Note that since $M > N$, the gap between M-EFT and all the other schemes is really significant. In addition, the M-EFT without prewhitening fails completely, while the RADOI without prewhitening outperforms RADOI with prewhitening (RADOI PWT). Such behavior of RADOI scheme is due to the fact that RADOI expression already takes into account the colored noise. For all the other model order schemes, we also incorporate the prewhitening step in order to have a fair comparison. Otherwise they would also fail.

In Figure 2, we verify the performance of the state-of-the-art techniques for low correlation levels, where $\rho = 0.2$. For

[2]The KN model order selection program can be downloaded at http://www.wisdom.weizmann.ac.il/~nadler/.

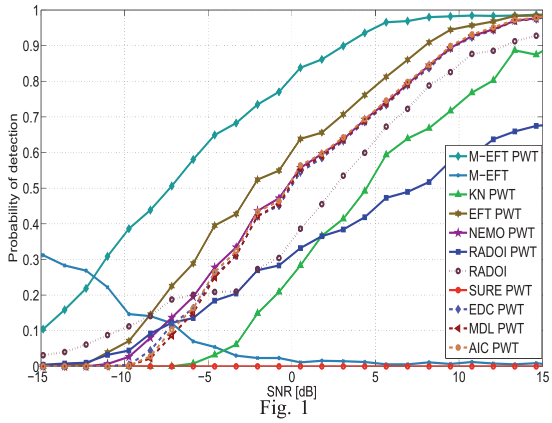

Fig. 1

PROBABILITY OF DETECTION VS. SNR FOR $M = 70$ COMPARING SOME STATE-OF-THE-ART MODEL ORDER SELECTION TECHNIQUES. THE NUMBER OF SNAPSHOTS N IS SET TO 8 AND THE NUMBER OF SOURCES $d = 4$. THE CORRELATION FACTOR ρ IS SET TO 0.9. THE NUMBER OF SAMPLES WITHOUT SIGNAL COMPONENTS N_l IS SET TO 1000.

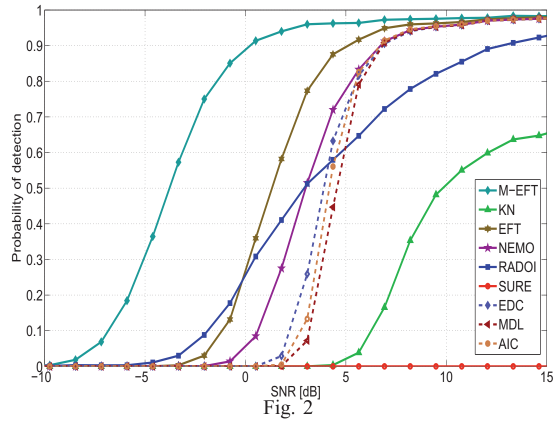

Fig. 2

PROBABILITY OF DETECTION VS. SNR FOR $M = 70$ COMPARING THE STATE-OF-THE-ART MODEL ORDER SELECTION SCHEMES WITHOUT USAGE OF THE NOISE STATISTICS. THE NUMBER OF SNAPSHOTS N IS SET TO 8 AND THE NUMBER OF SOURCES $d = 4$. THE CORRELATION FACTOR ρ IS SET TO 0.2.

such colored noise scenarios, the M-EFT is the best approach and no information about the noise is necessary.

In Figure 3, we observe the effect of varying the noise correlation, while in Figure 4, we observe the effect of varying the number of snapshots N_l without signal components. The M-EFT outperforms all the other approaches in both cases. Note that the error of the estimation of the noise statistics depends on the size of N_l. Therefore, since in Figure 4, for $N_l = 10^2$ to 10^5, the M-EFT PWT still outperforms significantly all the other MOS schemes, it implies that our scheme is also robust against errors in the estimation of the noise statistics.

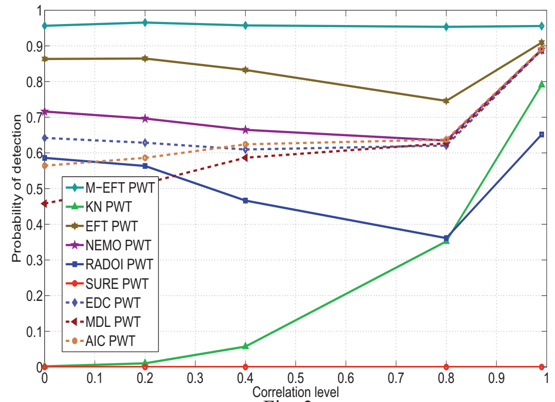

Fig. 3

PROBABILITY OF DETECTION VS. SNR FOR $M = 70$ COMPARING ALL STATE-OF-THE-ART TECHNIQUES FOR DIFFERENT VALUES OF ρ. THE NUMBER OF SNAPSHOTS N IS SET TO 8 AND THE NUMBER OF SOURCES $d = 4$. THE CORRELATION FACTOR SNR IS SET TO 4.5 DB.

VI. CONCLUSIONS

In this paper, we propose the M-EFT combined with the stochastic noise prewhitening (M-EFT PWT) for the estimation of the model order in colored noise scenarios. We modify the preprocessing step and the model order estimation step of the M-EFT by taking into account the colored noise statistics.

As shown in this paper, for applications where the colored noise is present and where samples without signal components are available, our proposed M-EFT PWT outperforms the state-of-the-art model order selection techniques.

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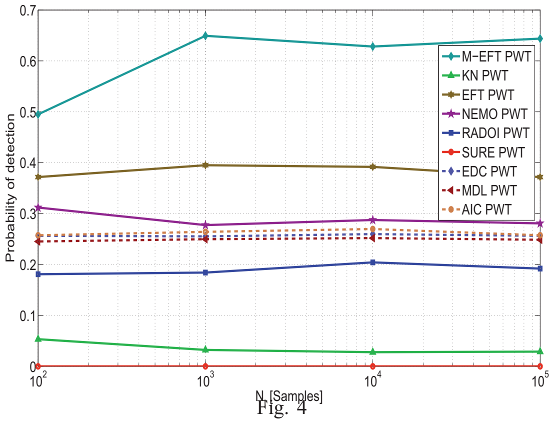

Fig. 4

PROBABILITY OF DETECTION VS. NUMBER OF SAMPLES WITHOUT SIGNAL COMPONENTS N_l FOR M = 70 COMPARING SEVERAL STATE OF THE ART MODEL ORDER SELECTION SCHEMES. THE NUMBER OF SNAPSHOTS N IS SET TO 8 AND THE NUMBER OF SOURCES d = 4. THE CORRELATION FACTOR ρ IS SET TO 0.9. THE SNR IS SET TO -4.6 DB.

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