High-SNR Second-Order Statistics of Amplify-and-Forward Relaying with Variable Gains and Multiple Hops

Reginaldo N. de Souza, Edgar E. Benitez Olivo, Lucas C. F. Ferreira, Carlos R. N. da Silva, and José C. S. Santos Filho

***Abstract*— We analyze the high-SNR behavior of the level crossing rate and average fade duration for a variable-gain amplify-and-forward relaying system composed of an arbitrary number of hops under Rayleigh fading. Most studies on the second-order statistics of multi-hop amplify-and-forward relaying assume fixed gains at the relays and analyze the end-to-end channel gain (from source to destination). Such analysis leads to the so-called cascaded product channel, being mathematically attractive, yet ignoring the accumulation of noise along the hops. Here, in turn, we take the noise into account by analyzing the end-to-end SNR, which ultimately governs the system performance. Also, aiming at the benchmark for transparent relaying, we assume variable gains at the relays. We obtain simple closed-form asymptotic expressions for the investigated second-order statistics and, as a byproduct, for the associated outage probability.**

***Keywords*— Amplify-and-forward, average fade duration, level crossing rate, multi-hop systems, variable-gain relaying.**

I. Introduction

Second-order statistics are used to characterize the dynamic nature of the wireless channel in mobile communications systems, serving essential design issues such as packet length, symbol rate, and transmission time interval [1]. Those statistics complement the static characterization given by their first-order counterparts, e.g., outage probability and bit-error rate. Key second-order statistics are the level crossing rate (LCR) and the average fade duration (AFD). The former provides the temporal rate at which the fading channel crosses a given threshold, either upwards or downwards; the latter, the average amount of time the channel remains below that threshold.

Despite their practical importance, very little is known about second-order statistics in relaying networks. Due to tractability reasons, most related studies are limited to the dual-hop scenario [2]–[5]. Very few works have addressed the general scenario with multiple hops [6]–[10]. Although the results in [7], [9], [10] are noteworthy contributions on the second-order statistics for multi-hop fixed-gain amplify-and-forward (AF) relaying, those works analyzed the LCR and AFD for the end-to-end channel gain (a.k.a. the cascaded product channel), which is mathematically appealing while neglecting the accumulation of noise along the multiple hops. Even in [8], where the LCR and AFD are given for the end-to-end signal-to-noise ratio (SNR), the resulting noise at the destination was assumed to be Gaussian — an approximation for rendering the problem more tractable. Anyway, the second-order statistics for variable-gain (VG) AF relaying systems with multiple hops remain open for investigation. Our primary aim here is to help fill this gap.

Even for the simplest scenario with two hops, an exact analysis of the second-order statistics for AF relaying systems proves intricate. Existing solutions appear usually in multi-fold integral form, bringing little or no insight into the system performance. In this work, we aim to shed light on the second-order statistics of multi-hop VG-AF relaying by deriving simple closed-form asymptotic solutions at high SNR.

II. System Model

Consider a multi-hop AF relaying system composed of N hops, as depicted in Fig. 1, in which the communication process between the source S and the destination D occurs through $N-1$ VG-AF relays $\{R_n\}_{n=1}^{N-1}$. There is no direct link between S and D. Each relay R_n receives the information signal coming from the preceding relay R_{n-1} (or from S, in the case of R_1), amplifies the signal by a factor of G_n, and then forwards it to the next node. This process is carried on hop by hop, up to the destination D. All terminals, including the source, are assumed to transmit with the same average power P_{T}. The amplification factor at the nth relay is given by $G_n^2 = \left(\alpha_n^2 + \Gamma_0^{-1}\right)^{-1}$ [11, eq. (9)], where α_n is the channel amplitude of the nth hop, and $\Gamma_0 \triangleq P_{\mathrm{T}}/N_0$ is the average transmit SNR at the source and relays, with N_0 being the mean power of the additive white Gaussian noise at the relays and destination. In such a case, the end-to-end SNR is obtained as $\Gamma_e = [\prod_{n=1}^{N}\left(1+\Gamma_n^{-1}\right) - 1]^{-1}$, where $\Gamma_n = \Gamma_0\alpha_n^2$ is the received SNR at the nth hop [12]. A widely used upper bound on Γ_e is [12]

$$\Gamma \triangleq \left[\sum_{n=1}^{N}\frac{1}{\Gamma_n}\right]^{-1} \simeq \Gamma_e, \quad \text{for} \quad \Gamma_0 \to \infty, \tag{1}$$

where "$\simeq$" denotes asymptotic equivalence, indicating that Γ increasingly approaches Γ_e at high SNR. The upper bound in (1) serves as a benchmark for practical transparent relaying

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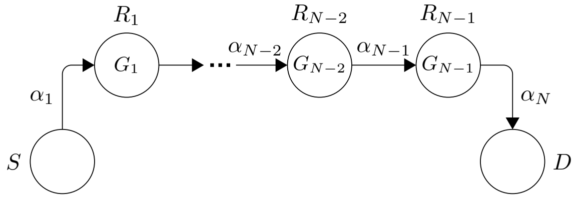

Fig. 1. Variable-gain amplify-and-forward relaying system with multiple hops.

systems and will provide a basis for the high-SNR analysis attained in the next section.

We assume that the channel amplitudes $\{\alpha_n\}_{n=1}^N$ of the hops are independent, non-necessarily identically distributed Rayleigh random variables (RV). Hence, each received SNR Γ_n follows an exponential distribution with mean value $\bar{\Gamma}_n = \Gamma_0\Omega_n$, where $\Omega_n = \mathrm{E}\left[\alpha_n^2\right]$ is the corresponding average channel power gain, with $\mathrm{E}\left[\cdot\right]$ denoting statistical expectation. Finally, considering an isotropic propagation environment, it is known that the time derivative $\dot{\alpha}_n$ of α_n is statistically independent of it and follows a zero-mean Gaussian distribution with variance $\sigma_{\dot{\alpha}_n}^2 = \pi^2 f_{m,n}^2 \Omega_n$, where $f_{m,n}$ is the maximum Doppler shift at the nth hop [1], [7].

III. High-SNR Second-Order Statistics

A. Preliminaries

A system outage occurs whenever the end-to-end SNR Γ_e drops below a certain threshold γ_{th}. The LCR $N_{\Gamma_e}\left(\gamma_{\text{th}}\right)$ gives the average temporal rate at which outage events take place, which can be obtained from Rice's formula [1]

$$N_{\Gamma_e}\left(\gamma_{\text{th}}\right) = \int_0^\infty \dot{\gamma}\, f_{\Gamma_e,\dot{\Gamma}_e}\left(\gamma_{\text{th}}, \dot{\gamma}\right)\, d\dot{\gamma} \tag{2}$$

in terms of the joint probability density function (PDF) $f_{\Gamma_e,\dot{\Gamma}_e}\left(\cdot,\cdot\right)$ of Γ_e and its time derivative $\dot{\Gamma}_e$. The AFD $T_{\Gamma_e}\left(\gamma_{\text{th}}\right)$, in turn, gives the average duration of an outage event, being obtained as [1]

$$T_{\Gamma_e}\left(\gamma_{\text{th}}\right) = \frac{P_{\Gamma_e}\left(\gamma_{\text{th}}\right)}{N_{\Gamma_e}\left(\gamma_{\text{th}}\right)}, \tag{3}$$

where

$$P_{\Gamma_e}\left(\gamma_{\text{th}}\right) = \int_0^{\gamma_{\text{th}}} f_{\Gamma_e}\left(\gamma\right) d\gamma \tag{4}$$

is the outage probability, with $f_{\Gamma_e}(\cdot)$ denoting the PDF of Γ_e.

B. Main Contributions

It is worth noting that there is no closed-form exact solution for either the first- or second-order statistics of VG-AF relaying systems. Regarding the outage probability, a two-fold integral-form solution was provided in [13] for an arbitrary number of hops, when the channels undergo Rayleigh, Rice, or Nakagami-m fading. Regarding the LCR and AFD, a two-fold integral-form solution was provided in [4] considering the existence of a direct link between source and destination, but for two hops only. In light of the mathematical complexity associated with VG-AF relaying, and aiming at better insights into the impact of each channel parameter on the system performance, we provide here a high-SNR analysis that yields simple closed-form expressions for the LCR, AFD, and, as a byproduct, outage probability. Our solutions apply to an arbitrary number of hops.

In the high-SNR regime ($\Gamma_0 \to \infty$), each investigated metric can be expressed asymptotically as $k_{\Gamma_e}(\gamma_{\text{th}}) \simeq (c_k\Gamma_0)^{-d_k}$, $k \in \{P, N, T\}$, as required, where d_k is called the diversity gain, and c_k is called the coding gain [14]. After a lengthy derivation process outlined in Section III-D, we obtain remarkably simple high-SNR expressions for the outage probability, LCR, and AFD of multi-hop VG-AF relaying systems operating over Rayleigh fading:

$$P_{\Gamma_e}(\gamma_{\text{th}}) \simeq \left(\frac{1}{\gamma_{\text{th}}\sum_{n=1}^N \frac{1}{\Omega_n}}\Gamma_0\right)^{-1} \tag{5}$$

$$N_{\Gamma_e}(\gamma_{\text{th}}) \simeq \left[\frac{1}{2\pi\gamma_{\text{th}}\left(\sum_{n=1}^N \frac{f_{m,n}}{\sqrt{\Omega_n}}\right)^2}\Gamma_0\right]^{-1/2} \tag{6}$$

$$T_{\Gamma_e}(\gamma_{\text{th}}) \simeq \left[\frac{2\pi}{\gamma_{\text{th}}\left(\frac{\sum_{n=1}^N \frac{1}{\Omega_n}}{\sum_{n=1}^N \frac{f_{m,n}}{\sqrt{\Omega_n}}}\right)^2}\Gamma_0\right]^{-1/2}. \tag{7}$$

C. Remarks

The asymptotic expressions in (5)–(7) reveal how the diversity and coding gains of each investigated metric are influenced by the average channel powers $\{\Omega_n\}_{n=1}^N$ and maximum Doppler shifts $\{f_{m,n}\}_{n=1}^N$ at the various hops. To our best knowledge, these expressions are new.

The parameters $\{\Omega_n\}_{n=1}^N$ and $\{f_{m,n}\}_{n=1}^N$ represent the channels' strength and the nodes' mobility, respectively. Note that these parameters play no role in the diversity gains of the outage probability, LCR, and AFD, namely, $d_P = 1$, $d_N = 1/2$, and $d_T = 1/2$. In particular, as known, the diversity gain of the outage probability mirrors the number of independent copies of information that achieve the destination [14], [15]. On the other hand, the impact on the coding gains vary for each metric: the higher the values of $\{\Omega_n\}_{n=1}^N$ (the stronger the channels), the higher the values of c_P (less likely fades), c_N (less frequent fades), and c_T (shorter fades); the higher the values of $\{f_{m,n}\}_{n=1}^N$ (the higher the mobility of the nodes), the smaller the values of c_N (more frequent fades) and the higher the values of c_T (shorter fades). As expected, the Doppler shifts (a dynamic feature) have no impact on the outage probability (a static metric).

D. Derivation Outline

In this section, we outline the derivation process of (5)–(7). First, we derive integral-form expressions for the LCR and AFD of Γ, the upper bound on Γ_e defined in (1). Then we simplify those expressions under the assumption of a high-SNR regime. For the lack of space, some details are omitted.

1) Approximate Analysis: To calculate the LCR of Γ, we start by rewriting (2) as

$$N_{\Gamma}\left(\gamma_{\mathrm{th}}\right)=\int_{0}^{\infty} \dot{\gamma}\, f_{\dot{\Gamma}|\Gamma}\left(\dot{\gamma}|\gamma_{\mathrm{th}}\right) f_{\Gamma}\left(\gamma_{\mathrm{th}}\right)\, d\dot{\gamma}, \tag{8}$$

where $f_{\dot{\Gamma}|\Gamma}(\cdot|\cdot)$ is the PDF of $\dot{\Gamma}$ conditioned on Γ, and $f_{\Gamma}(\cdot)$ is the PDF of Γ. The latter can be obtained as follows. In [16], Brennan proposed a method to calculate the PDF of a generic sum of N non-negative, arbitrarily correlated, arbitrarily distributed RVs. In our case, let us consider the sum

$$Z=\sum_{n=1}^{N} Z_{n}, \tag{9}$$

where $\{Z_n\}_{n=1}^{N}$ are non-negative, independent RVs. From Brennan's results, the PDF of Z is given by [16]

$$\begin{aligned} f_{Z}(z)=&\int_{0}^{z}\int_{0}^{z-z_N}\cdots\int_{0}^{z-\sum_{n=3}^{N} z_n} \\ &\times f_{Z_1}\left(z-\sum_{n=2}^{N} z_n\right)\prod_{n=2}^{N} f_{Z_n}\left(z_n\right) dz_2\ldots dz_{N-1}dz_N. \end{aligned} \tag{10}$$

To express (1) as in (9), we introduce the relationships

$$Z \triangleq \Gamma^{-1} \tag{11}$$

$$Z_n \triangleq \Gamma_n^{-1}, \tag{12}$$

with the respective PDFs being related as $f_{\Gamma}(\gamma)=\gamma^{-2} f_Z(\gamma^{-1})$ and $f_{Z_n}(z_n)=z_n^{-2} f_{\Gamma_n}(z_n^{-1})$. Using this into (10), the PDF of Γ is then obtained as

$$\begin{aligned} f_{\Gamma}(\gamma)=&\frac{1}{\gamma^2}\int_{0}^{\frac{1}{\gamma}}\int_{0}^{\frac{1}{\gamma}-z_N}\cdots\int_{0}^{\frac{1}{\gamma}-\sum_{n=3}^{N} z_n} \\ &\times\frac{1}{\left(\frac{1}{\gamma}-\sum_{n=2}^{N} z_n\right)^2} f_{\Gamma_1}\left(\frac{1}{\frac{1}{\gamma}-\sum_{n=2}^{N} z_n}\right) \\ &\times\prod_{n=2}^{N}\frac{1}{z_n^2} f_{\Gamma_n}\left(\frac{1}{z_n}\right) dz_2\ldots dz_{N-1}dz_N. \end{aligned} \tag{13}$$

Now, to obtain $f_{\dot{\Gamma}|\Gamma}(\cdot|\cdot)$, we first determine from (1) the time derivative of Γ:

$$\dot{\Gamma}=\frac{\partial\Gamma}{\partial t}=\frac{\sum_{n=1}^{N}\frac{1}{\Gamma_n^2}\dot{\Gamma}_n}{\left(\sum_{n=1}^{N}\frac{1}{\Gamma_n}\right)^2}=\frac{2\sqrt{\Gamma_0}\sum_{n=1}^{N} Z_n^{3/2}\dot{\alpha}_n}{\left(\sum_{n=1}^{N} Z_n\right)^2}, \tag{14}$$

since $\dot{\Gamma}_n=\partial\Gamma_n/\partial t=2\sqrt{\Gamma_0}\sqrt{\Gamma_n}\dot{\alpha}_n$ and $\Gamma_n=Z_n^{-1}$. Hence, it becomes apparent that $\dot{\Gamma}$ is a weighted sum of $\{\dot{\alpha}_n\}_{n=1}^{N}$, zero-mean Gaussian RVs with variances given by $\sigma_{\dot{\alpha}_n}^2=\pi^2 f_{m,n}^2\Omega_n$. The weights depend on $\{Z_n\}_{n=1}^{N}$. As a result, conditioned on Γ and $Z_2,\ldots,Z_N$, $\dot{\Gamma}$ is also a Gaussian RV, with zero mean and variance computed from (14) as

$$\begin{aligned} \sigma_{\dot{\Gamma}|\Gamma,Z_2,\ldots,Z_N}^2=&4\Gamma_0\gamma_{\mathrm{th}}\left[\left(1-\gamma_{\mathrm{th}}\sum_{n=2}^{N} z_n\right)^3\sigma_{\dot{\alpha}_1}\right. \\ &\left.+\gamma_{\mathrm{th}}^3\sum_{n=2}^{N} z_n^3\sigma_{\dot{\alpha}_n}\right]. \end{aligned} \tag{15}$$

By integrating (8) with respect to $\dot{\gamma}$ while taking into account the Gaussianity of $f_{\dot{\Gamma}|\Gamma,Z_2,\ldots,Z_N}(\cdot|\cdot,\ldots,\cdot)$, we have [17]

$$\int_{0}^{\infty}\dot{\gamma}\, f_{\dot{\Gamma}|\Gamma,Z_2,\ldots,Z_N}(\cdot|\cdot,\ldots,\cdot)\, d\dot{\gamma}=\sqrt{\frac{\sigma_{\dot{\Gamma}|\Gamma,Z_2,\ldots,Z_N}^2}{2\pi}}. \tag{16}$$

Then, by applying (13) and (16) into (8), we obtain an approximate analytical expression for the LCR of a multi-hop VG-AF relaying system over Rayleigh fading as

$$\begin{aligned} N_{\Gamma}\left(\gamma_{\mathrm{th}}\right)=&\frac{\sqrt{2\pi\gamma_{\mathrm{th}}}}{\Gamma_0^{N-1/2}}\int_{0}^{\frac{1}{\gamma_{\mathrm{th}}}}\int_{0}^{\frac{1}{\gamma_{\mathrm{th}}}-z_N}\cdots\int_{0}^{\frac{1}{\gamma_{\mathrm{th}}}-\sum_{n=3}^{N} z_n} \\ &\times\sqrt{f_{m,1}^2\Omega_1\left(1-\gamma_{\mathrm{th}}\sum_{n=2}^{N} z_n\right)^3+\gamma_{\mathrm{th}}^3\sum_{n=2}^{N} f_{m,n}^2\Omega_n z_n^3} \\ &\times\frac{\exp\left[-\frac{\gamma_{\mathrm{th}}}{\Omega_1\Gamma_0\left(1-\gamma_{\mathrm{th}}\sum_{n=2}^{N} z_n\right)}-\sum_{n=2}^{N}\frac{1}{\Gamma_0\Omega_n z_n}\right]}{\left(1-\gamma_{\mathrm{th}}\sum_{n=2}^{N} z_n\right)^2\Omega_1\prod_{n=2}^{N} z_n^2\Omega_n} \\ &\times dz_2\ldots dz_{N-1}dz_N. \end{aligned} \tag{17}$$

To calculate the AFD of Γ as in (3), we need to find the associated outage probability. By substituting (13) into (4), we obtain an approximate analytical expression for the outage probability of a multi-hop VG-AF relaying system over Rayleigh fading as

$$\begin{aligned} P_{\Gamma}\left(\gamma_{\mathrm{th}}\right)=&\frac{1}{\Gamma_0^N}\int_{0}^{\frac{1}{\gamma_{\mathrm{th}}}}\int_{0}^{\frac{1}{\gamma}}\int_{0}^{\frac{1}{\gamma}-z_N}\cdots\int_{0}^{\frac{1}{\gamma}-\sum_{n=3}^{N} z_n} \\ &\times\frac{\exp\left[-\frac{\gamma}{\Omega_1\Gamma_0\left(1-\gamma\sum_{n=2}^{N} z_n\right)}-\sum_{n=2}^{N}\frac{1}{\Gamma_0\Omega_n z_n}\right]}{\left(1-\gamma\sum_{n=2}^{N} z_n\right)^2\Omega_1\prod_{n=2}^{N} z_n^2\Omega_n} \\ &\times dz_2\ldots dz_{N-1}dz_N d\gamma. \end{aligned} \tag{18}$$

Then, by replacing (17) and (18) into (3), we obtain a corresponding expression for the AFD.

2) Asymptotic Analysis: The final step is to derive asymptotic representations for (17), (18), and the corresponding AFD as $\Gamma_0\to\infty$. In such cases, a popular approach is to replace the exponential function in the integrands by its Maclaurin series expansion, to drop the terms beyond the second one (i.e., $\exp(-x_i)\simeq 1-x_i$), and to solve and simplify the integral. But this approach alone does not work in our case; the resulting integrals turn out to diverge, since their integrands contain poles on each integration limit. To overcome this, we split the integration interval into two parts containing a single pole each, and then we perform certain mathematical manipulations to ensure the integral will converge when the exponential function is approximated.

But applying the above method to (17) and (18) proves quite involved if done directly for an arbitrary number of hops N. Rather, we apply the method for two and three hops first, and then, building on these results, we generalize them by induction for any number of hops. Next, due to the lack of space, we present the derivation steps only for the LCR under two hops. The same rationale can be also applied for the outage probability, as well as for a larger number of hops. Finally, using (3), those results can be combined into corresponding AFD expressions.

The LCR for the dual-hop case is obtained by setting $N = 2$ in (17), yielding

$$N_\Gamma(\gamma_{\text{th}}) = \frac{\sqrt{2\pi}\sqrt{\gamma_{\text{th}}}}{\Omega_1\Omega_2\Gamma_0^{3/2}} \int_0^{\frac{1}{\gamma_{\text{th}}}} \times \frac{\sqrt{(1-z_2\gamma_{\text{th}})^3\Omega_1 f_{m,1}^2 + (z_2\gamma_{\text{th}})^3\Omega_2 f_{m,2}^2}}{(1-z_2\gamma_{\text{th}})^2 z_2^2} \times \exp\left[-\frac{\gamma_{\text{th}}}{(1-z_2\gamma_{\text{th}})\Gamma_0\Omega_1} - \frac{1}{z_2\Gamma_0\Omega_2}\right] dz_2. \tag{19}$$

A high-SNR asymptotic expression for (19) cannot be obtained by directly using the Maclaurin representation of the exponential function and dropping the terms beyond the first, since the resulting integral will diverge due to the poles on $z_2 = 0$ and $z_2 = 1/\gamma_{\text{th}}$. To overcome this, as mentioned before, we split the integration interval into two parts with only one pole each, and then a change of variable is made to ensure the integral will converge when the exponential function is approximated. Specifically, after the change of variables $u \triangleq z_2\gamma_{\text{th}}$, (19) can be split as

$$N_\Gamma(\gamma_{\text{th}}) = \frac{\sqrt{2\pi}\gamma_{\text{th}}^{3/2}}{\Omega_1\Omega_2\Gamma_0^{3/2}}(I_1 + I_2), \tag{20}$$

where

$$I_1 \triangleq \int_0^{1/2} \frac{\sqrt{(1-u)^3\Omega_1 f_{m,1}^2 + u^3\Omega_2 f_{m,2}^2}}{(1-u)^2 u^2} \times \exp\left[-\frac{\gamma_{\text{th}}}{\Gamma_0\Omega_1(1-u)} - \frac{\gamma_{\text{th}}}{\Gamma_0\Omega_2 u}\right] du \tag{21}$$

$$I_2 \triangleq \int_{1/2}^{1} \frac{\sqrt{(1-u)^3\Omega_1 f_{m,1}^2 + u^3\Omega_2 f_{m,2}^2}}{(1-u)^2 u^2} \times \exp\left[-\frac{\gamma_{\text{th}}}{\Gamma_0\Omega_1(1-u)} - \frac{\gamma_{\text{th}}}{\Gamma_0\Omega_2 u}\right] du. \tag{22}$$

By changing also $v \triangleq (\gamma_{\text{th}} - 2u\gamma_{\text{th}})/(\Gamma_0\Omega_2 u)$ in (21), and $w \triangleq (-\gamma_{\text{th}} + 2\gamma_{\text{th}}u)/(\Gamma_0\Omega_1 - \Gamma_0\Omega_1 u)$ in (22), we have

$$I_1 = \frac{\Gamma_0\Omega_2}{\gamma_{\text{th}}} \int_0^\infty \frac{(2\gamma_{\text{th}} + v\Gamma_0\Omega_2)^2}{(\gamma_{\text{th}} + v\Gamma_0\Omega_2)^2} \times \sqrt{\frac{(\gamma_{\text{th}} + v\Gamma_0\Omega_2)^3\Omega_1 f_{m,1}^2}{(2\gamma_{\text{th}} + v\Gamma_0\Omega_2)^3} + \frac{\gamma_{\text{th}}^3\Omega_2 f_{m,2}^2}{(2\gamma_{\text{th}} + v\Gamma_0\Omega_2)^3}} \times \exp\left[-v - \frac{2\gamma_{\text{th}}}{\Gamma_0\Omega_2} - \frac{2\gamma_{\text{th}}^2 + v\gamma_{\text{th}}\Gamma_0\Omega_2}{\Gamma_0\Omega_1(\gamma_{\text{th}} + v\Gamma_0\Omega_2)}\right] dv \tag{23}$$

$$I_2 = \frac{\Gamma_0\Omega_1}{\gamma_{\text{th}}} \int_0^\infty \frac{(2\gamma_{\text{th}} + w\Gamma_0\Omega_1)^2}{(\gamma_{\text{th}} + w\Gamma_0\Omega_1)^2} \times \sqrt{\frac{\gamma_{\text{th}}^3\Omega_1 f_{m,1}^2}{(2\gamma_{\text{th}} + w\Gamma_0\Omega_1)^3} + \frac{(\gamma_{\text{th}} + w\Gamma_0\Omega_1)^3\Omega_2 f_{m,2}^2}{(2\gamma_{\text{th}} + w\Gamma_0\Omega_1)^3}} \times \exp\left[-w - \frac{2\gamma_{\text{th}}}{\Gamma_0\Omega_1} - \frac{2\gamma_{\text{th}}^2 + w\gamma_{\text{th}}\Gamma_0\Omega_1}{\Gamma_0\Omega_2(\gamma_{\text{th}} + w\Gamma_0\Omega_1)}\right] dw. \tag{24}$$

Since $1/\Gamma_0 \to 0$ as $\Gamma_0 \to \infty$, we can apply the Maclaurin series to the whole integrand in (23) and (24), and take only the first term to track the asymptotic behavior:

$$I_1 \simeq \frac{\Omega_2\sqrt{\Omega_1 f_{m,1}^2}\Gamma_0}{\gamma_{\text{th}}} \int_0^\infty \exp[-v]\, dv = \frac{\Omega_2\sqrt{\Omega_1 f_{m,1}^2}\Gamma_0}{\gamma_{\text{th}}} \tag{25}$$

$$I_2 \simeq \frac{\Omega_1\sqrt{\Omega_2 f_{m,2}^2}\Gamma_0}{\gamma_{\text{th}}} \int_0^\infty \exp[-w]\, dw = \frac{\Omega_1\sqrt{\Omega_2 f_{m,2}^2}\Gamma_0}{\gamma_{\text{th}}}. \tag{26}$$

Then, by substituting (25) and (26) into (20), we obtain a high-SNR asymptotic expression for the LCR of a dual-hop VG-AF relaying system:

$$N_{\Gamma_e}(\gamma_{\text{th}}) \simeq \left[\frac{1}{2\pi\gamma_{\text{th}}\left(\frac{f_{m,1}}{\sqrt{\Omega_1}} + \frac{f_{m,2}}{\sqrt{\Omega_2}}\right)^2}\Gamma_0\right]^{-1/2}. \tag{27}$$

As already mentioned, the procedure described in (19)–(27) can be also applied for the outage probability and for any number of hops. For these cases, due to space constraints, we only reproduce the final expressions. The high-SNR asymptotic expression for the outage probability with $N = 2$ is obtained as

$$P_{\Gamma_e}(\gamma_{\text{th}}) \simeq \left[\frac{1}{\gamma_{\text{th}}\left(\frac{1}{\Omega_1} + \frac{1}{\Omega_2}\right)}\Gamma_0\right]^{-1}, \tag{28}$$

while the high-SNR asymptotic expressions for the LCR and outage probability with $N = 3$ are given respectively as

$$N_{\Gamma_e}(\gamma_{\text{th}}) \simeq \left[\frac{1}{2\pi\gamma_{\text{th}}\left(\frac{f_{m,1}}{\sqrt{\Omega_1}} + \frac{f_{m,2}}{\sqrt{\Omega_2}} + \frac{f_{m,3}}{\sqrt{\Omega_3}}\right)^2}\Gamma_0\right]^{-1/2} \tag{29}$$

$$P_{\Gamma_e}(\gamma_{\text{th}}) \simeq \left[\frac{1}{\gamma_{\text{th}}\left(\frac{1}{\Omega_1} + \frac{1}{\Omega_2} + \frac{1}{\Omega_3}\right)}\Gamma_0\right]^{-1}. \tag{30}$$

Expressions similar to (27)–(30) can be found for $N > 3$. All in all, building on these results, and using (3), we arrive by induction at the general solution given in (5)–(7).

IV. Numerical Results

The analytical expressions derived in the previous section are now evaluated for sample scenarios. To validate our analysis, simulations results are also provided for the exact and approximate end-to-end SNRs. For illustration purposes and without loss of generality, the SNR threshold and the average channel power at each hop are set to unity, i.e., $\gamma_{\text{th}} = 0$ dB and $\Omega_1 = \cdots = \Omega_N = 1$. Also, we consider that all nodes have the same mobility, so that $f_{m,1} = \cdots = f_{m,N} = f_m$.

Figs. 2 and 3 show the normalized LCR (N_{Γ_e}/f_m) and normalized AFD ($T_{\Gamma_e} \times f_m$) versus the average transmit SNR (Γ_0), respectively, for a number of hops ranging from two to five. The approximate curves, labeled as “Approximate

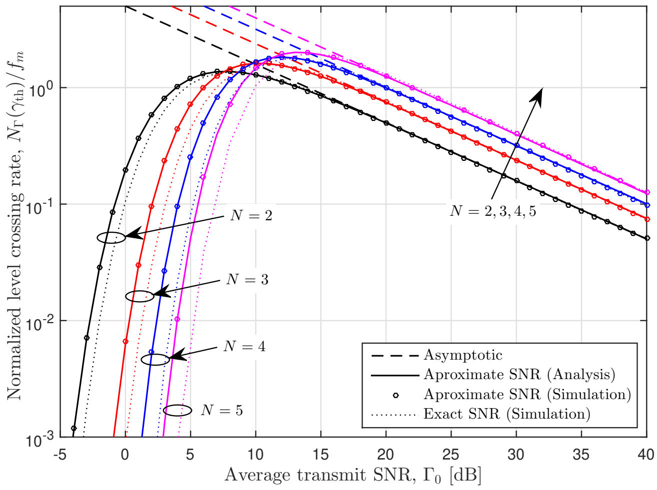

Fig. 2. Normalized level crossing rate for a multi-hop VG-AF relaying system: $\Omega_1 = \cdots = \Omega_5 = 1$ and $\gamma_{\text{th}} = 0$ dB.

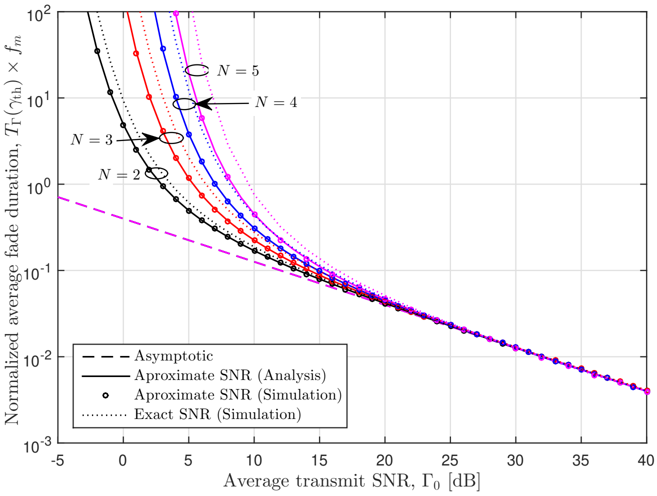

Fig. 3. Normalized average fade duration for a multi-hop VG-AF relaying system: $\Omega_1 = \cdots = \Omega_5 = 1$ and $\gamma_{\text{th}} = 0$ dB.

SNR (Analysis)", were obtained from (17) and (18), and the asymptotic curves, from (6) and (7). Note the tightness of the proposed approximations at medium to high SNR, regardless of the number of hops. As expected, the proposed approximations become less accurate at low SNR, and this mismatch increases with the number of hops. It can be also observed how the number of hops affects the performance of VG-AF relaying in itself. Although a broader coverage area is normally attained as the number of hops is increased, the outage events become more frequent (see Fig. 2). Yet, the penalty due to an extra hop diminishes with the number of hops. On the other hand, as shown in Fig. 3, the number of hops barely affects the duration of the outage events at medium to high SNR.

Note that the above observations can be straightforwardly assessed via our main analytical results in (5)–(7). In particular, for the homogeneous scenario addressed in the numerical examples, the coding gains therein specialize to $c_P = \Omega/(N\gamma_{\text{th}})$, $c_N = \Omega/(2\pi f_m^2 N^2 \gamma_{\text{th}})$, and $c_T = 2\pi f_m^2 \Omega/\gamma_{\text{th}}$.

V. Conclusions

We investigated the second-order statistics of a variable-gain amplify-and-forward relaying system containing an arbitrary number of hops subject to Rayleigh fading. Our main results are novel high-SNR expressions for the LCR, AFD, and, in passing, outage probability, of the end-to-end SNR. These asymptotic results circumvent the integral-form formulations that typically emerge in related exact analyses, while shedding light on the subject by means of simple closed-form expressions that unveil how each system parameter roughly affects the performance. In particular, our results show that the diversity gains of the LCR and AFD are both $1/2$, regardless of the number of hops. Also, when all the hops are identically distributed, the coding gain of the LCR is inversely proportional to the squared number of hops, whereas the coding gain of the AFD is not affected whatsoever.

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