A Code for Correcting Quantum Erasures

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***Abstract*— Quantum states are very delicate, so quantum error-correction procedures are believed to be essential for the successful implementation of quantum communications or computations. To erroneous qubits where the position is known, Grassl et al. [Phys. Rev. A 56, 33 (1997)] proposed an error model which they called quantum erasure channel. In this paper we present a quantum code capable of protecting n-qubit of information against the occurrence of t erasures. This code is based on GHZ states.**

Keywords*— quantum error-correction code; quantum erasure; GHZ states.

I. Introduction

Quantum computers are expected to harness the strange properties of quantum mechanics such as superposition and entanglement for enhanced ways of information processing. Arguably, the most formidable hurdle is the unavoidable decoherence caused by the coupling of the quantum computers to the environment, which destroys the fragile quantum information rapidly. It is thus of crucial importance to find ways to reduce the decoherence and carry out coherent quantum operations in the presence of noise.

There is a significant source of error - the loss of qubits in quantum computers. The qubit, which is the basic element of standard quantum computation (QC), is supposed to be an isolated two-level system consisting of a pair of orthonormal quantum states. However, most proposed quantum hardware are in fact multilevel systems, and the states of qubits are defined in a two-level subspace, which may leak out of the desired qubit space and into a larger Hilbert space [1]. This problem is common in practical QC with various qubits candidates, such as Josephson junctions [2], neutral atoms in optical lattices [3], and, most notoriously, single photons that can be lost during processing or owing to inefficient photon sources and detectors [4]– [5]. A special class of quantum erasure-correction code was proposed by Grassl et al. [1], which considered a situation in which the position of the erroneous qubits is known. According to classical coding theory, they called this model the quantum erasure channel (QEC). Some physical scenarios to determine the position of an error have been given [1].

In general, alteration of information is not *a priori* obvious for the observer, which should encode the information in a special way to detect such change. One way that can be explored to perform this encoding is the use of Greenberger-Horne-Zeilinger (GHZ) state. The GHZ state (also called *cat* state) was introduced by Daniel M. Greenberger, Michael A. Horne and Anton Zeilinger [6] as a new way of proving Bell's Theorem [7]. One example of n-party version of the GHZ state is given by

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\Big(|\overbrace{00\ldots0}^{n}\rangle + |\overbrace{11\ldots1}^{n}\rangle\Big) = \frac{1}{\sqrt{2}}\Big(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}\Big).$$

As the most frequently used multi-party entangled state, the GHZ state has appeared in applications such as non-locality [8], comunications complexity [9] and multi-party cryptography [10].

Yang et al. [11] presented a quantum error correction code which protects three qubits of quantum information against one erasure using GHZ states. Santos et al. [12] presented an extension of this code to protect five-qubits of information against one erasure. Also Santos et al. [13] have shown a generalization of this scheme to protect n-qubits of information against one erasure. However, a code that protects n-qubit information against one erasure is not robust enough for many applications, such as quantum secret sharing. Thus, based on the works cited above, we present here a code capable of protecting n-qubit of information against t erasures.

II. A Quantum Erasure-Correcting Code via GHZ states

An arbitrary state of n qubits can be written as follows:

$$|\psi\rangle = \sum_{i=0}^{2^n-1} \lambda_i |i\rangle, \tag{1}$$

where $\sum_{i=0}^{2^n-1} |\lambda_i|^2 = 1$; and $|i\rangle$ represents a general basis state of n qubits with the integer i corresponding to its binary decomposition. To protect n-qubit quantum information, we can use $t = \lfloor n/3 \rfloor$ blocks of n ancillary qubits to encode the state (1) into

$$|\psi\rangle_L = \sum_{i=0}^{2^n-1} \lambda_i \bigotimes_{d=0}^{t} |\psi^{(i)}\rangle_{1(d)2(d)\ldots n(d)}, \tag{2}$$

where $|\psi^{(i)}\rangle_{1(d)2(d)\ldots n(d)}$ are the $(t+1)$ n-qubit GHZ states which are given by

$$|\psi^{(i)}\rangle_{1(d)2(d)\dots n(d)} = \frac{1}{\sqrt{2}}\Big[|u^{(i)}_{1(d)}u^{(i)}_{2(d)}\dots u^{(i)}_{n(d)}\rangle \pm|\hat{u}^{(i)}_{1(d)}\hat{u}^{(i)}_{2(d)}\dots\hat{u}^{(i)}_{n(d)}\rangle\Big] \quad (3)$$

where $d=0$ corresponding to the block of n "message" qubits and the $d=1,\dots,t$ corresponding to t blocks of n ancillary qubits, respectively (here, $|u^{(i)}_m\rangle$ and $|\hat{u}^{(i)}_m\rangle$ represent two orthogonal states of the qubit $m(d)$, $\hat{u}^{(i)}_m = 1-u^{(i)}_m$ and $u^{(i)}_m \in \{0,1\}$).

Since any basis state in (1) is encoded into a product of $(t+1)$ n-qubit GHZ states, it is straightforward to show that for the encoded state (2), the density operator of each qubit is given by $\frac{1}{2}(|0\rangle\langle 0|+|1\rangle\langle 1|)$. This result means that the n-qubit quantum information, originally carried by the n "message" qubits, is distributed over each qubit after encoding the state (1) into (2).

The encoding can be easily done by using Hadamard gates and Controled-NOT (CNOT) gates. Considering that each basis state in (1) is encoded into a product of $(t+1)$ n-qubit GHZ states all taking the same form, the encoding operation is given by

$$U_{enc} = \prod_{d=0}^{t}\left(\prod_{i=1}^{n-1} C_{n(d),i(d)}\right)\otimes\prod_{d=0}^{t} H_{n(d)} \otimes\prod_{d=1}^{t}\left(\prod_{i=1}^{n} C_{i(0),i(d)}\right), \quad (4)$$

where the $t=\lfloor n/3\rfloor$ blocks of n ancillary qubits are initially in the state $|00\dots 0\rangle$; H_n is the Hadamard transformation operation acting on the qubit n; and $C_{n,i}$ is a CNOT operation acting on qubit n (control bit) and on the qubit i (target bit).

It should be mentioned that a general theory about how quantum information can be distributed has been proposed [14]. Although we deal with a special case that a single party cannot gain any information about the state, our main purpose is to wish to present a concrete encoding scheme for protect n-qubit information.

To illustrate it consider the following situations:

(i) An arbitrary state of 3 qubits is written in binary decomposition as follows

$$|\psi\rangle_{1(0)2(0)3(0)} = \lambda_0|000\rangle+\lambda_1|001\rangle+\lambda_2|010\rangle +\lambda_3|011\rangle+\lambda_4|100\rangle+\lambda_5|101\rangle +\lambda_6|110\rangle+\lambda_7|111\rangle. \quad (5)$$

As $n=3$ then we'll have to $t=\lfloor 3/3\rfloor=1$, so we have one block of three ancillary qubits. Thus, the encoding operation is given by

$$U_{enc} = \prod_{d=0}^{1}\left(\prod_{i=1}^{2} C_{3(d),i(d)}\right)\otimes\prod_{d=0}^{1} H_{3(d)} \otimes\left(\prod_{i=1}^{3} C_{i(0),i(1)}\right) \quad (6)$$

$$= C_{3(0),1(0)}C_{3(0),2(0)}C_{3(1),1(1)}C_{3(1),2(1)}H_{3(0)}H_{3(1)} C_{1(0),1(1)}C_{2(0),2(1)}C_{3(0),3(1)}, \quad (7)$$

thus we have

$$U_{enc}\Big(|\psi\rangle_{1(0)2(0)3(0)}|000\rangle_{1(1)2(1)3(1)}\Big) = |\psi\rangle_L. \quad (8)$$

Throughout this work, every joint operation, as outlined above, will follow the sequence from right to left. Using three ancillary qubits 1(1), 2(1) and 3(1), we encode the original state into

$$|\psi\rangle_L = \lambda_0|0\rangle_L+\lambda_1|1\rangle_L+\lambda_2|2\rangle_L+\lambda_3|3\rangle_L +\lambda_4|4\rangle_L+\lambda_5|5\rangle_L+\lambda_6|6\rangle_L+\lambda_7|7\rangle_L, \quad (9)$$

where the eight logical state are

$$\begin{aligned}
|0\rangle_L &= (|000\rangle+|111\rangle)_{1(0)2(0)3(0)}\otimes(|000\rangle+|111\rangle)_{1(1)2(1)3(1)}\\
|1\rangle_L &= (|000\rangle-|111\rangle)_{1(0)2(0)3(0)}\otimes(|000\rangle-|111\rangle)_{1(1)2(1)3(1)}\\
|2\rangle_L &= (|010\rangle+|101\rangle)_{1(0)2(0)3(0)}\otimes(|010\rangle+|101\rangle)_{1(1)2(1)3(1)}\\
|3\rangle_L &= (|010\rangle-|101\rangle)_{1(0)2(0)3(0)}\otimes(|010\rangle-|101\rangle)_{1(1)2(1)3(1)}\\
|4\rangle_L &= (|100\rangle+|011\rangle)_{1(0)2(0)3(0)}\otimes(|100\rangle+|011\rangle)_{1(1)2(1)3(1)}\\
|5\rangle_L &= (|100\rangle-|011\rangle)_{1(0)2(0)3(0)}\otimes(|100\rangle-|011\rangle)_{1(1)2(1)3(1)}\\
|6\rangle_L &= (|110\rangle+|001\rangle)_{1(0)2(0)3(0)}\otimes(|110\rangle+|001\rangle)_{1(1)2(1)3(1)}\\
|7\rangle_L &= (|110\rangle-|001\rangle)_{1(0)2(0)3(0)}\otimes(|110\rangle-|001\rangle)_{1(1)2(1)3(1)}.
\end{aligned}$$

To simplify the notation, normalization factors are omitted here and in the remainder of this work.

(ii) An arbitrary state of 7 qubits is written in binary decomposition as follows

$$|\psi\rangle = \lambda_0|0000000\rangle+\lambda_1|0000001\rangle+\lambda_2|0000010\rangle+\lambda_3|0000011\rangle +\dots+\lambda_{124}|1111100\rangle+\lambda_{125}|1111101\rangle+\lambda_{126}|1111110\rangle +\lambda_{127}|1111111\rangle. \quad (10)$$

As $n=7$ then we'll have to $t=\lfloor 7/3\rfloor=2$, so we have two blocks of seven ancillary qubits. Thus, the encoding operation is as follows

$$\begin{aligned}
U_{enc} &= \prod_{d=0}^{2}\left(\prod_{i=1}^{6} C_{7(d),i(d)}\right)\otimes\prod_{d=0}^{2} H_{7(d)}\otimes\prod_{d=1}^{2}\left(\prod_{i=1}^{7} C_{i(0),i(d)}\right)\\
&= (C_{7(0),1(0)}C_{7(0),2(0)}C_{7(0),3(0)}C_{7(0),4(0)}C_{7(0),5(0)} C_{7(0),6(0)})\otimes(C_{7(1),1(1)}C_{7(1),2(1)}C_{7(1),3(1)}C_{7(1),4(1)}\\
&\quad C_{7(1),5(1)}C_{7(1),6(1)})\otimes(C_{7(2),1(2)}C_{7(2),2(2)}C_{7(2),3(2)} C_{7(2),4(2)}C_{7(2),5(2)}C_{7(2),6(2)})\otimes H_{7(0)}H_{7(1)}H_{7(2)}\\
&\quad \otimes(C_{1(0),1(1)}C_{2(0),2(1)}C_{3(0),3(1)}C_{4(0),4(1)}C_{5(0),5(1)} C_{6(0),6(1)}C_{7(0),7(1)})\otimes(C_{1(0),1(2)}C_{2(0),2(2)}C_{3(0),3(2)}\\
&\quad C_{4(0),4(2)}C_{5(0),5(2)}C_{6(0),6(2)}C_{7(0),7(2)}),
\end{aligned} \quad (11)$$

thus we have

$$U_{enc}\left(|\psi\rangle_{(0)}\otimes|0000000\rangle_{(1)}\otimes|0000000\rangle_{(2)}\right)=|\psi\rangle_L. \qquad (12)$$

Using two blocks of seven ancillary qubits (in a total of fourteen ancillary qubits), we encode the original state into

$$\begin{aligned}|\psi\rangle_L &= \lambda_0|0\rangle_L+\lambda_1|1\rangle_L+\lambda_2|2\rangle_L+\lambda_3|3\rangle_L+\ldots+\lambda_{124}|124\rangle_L\\ &\quad+\lambda_{125}|125\rangle_L+\lambda_{126}|126\rangle_L+\lambda_{127}|127\rangle_L,\end{aligned} \qquad (13)$$

where

$$\begin{aligned}
|0\rangle_L &= (|0000000\rangle+|1111111\rangle)_{(0)}\otimes(|0000000\rangle+|1111111\rangle)_{(1)}\\ &\quad\otimes(|0000000\rangle+|1111111\rangle)_{(2)}\\
|1\rangle_L &= (|0000000\rangle-|1111111\rangle)_{(0)}\otimes(|0000000\rangle-|1111111\rangle)_{(1)}\\ &\quad\otimes(|0000000\rangle-|1111111\rangle)_{(2)}\\
|2\rangle_L &= (|0000010\rangle+|1111101\rangle)_{(0)}\otimes(|0000010\rangle+|1111101\rangle)_{(1)}\\ &\quad\otimes(|0000010\rangle+|1111101\rangle)_{(2)}\\
|3\rangle_L &= (|0000010\rangle-|1111101\rangle)_{(0)}\otimes(|0000010\rangle-|1111101\rangle)_{(1)}\\ &\quad\otimes(|0000010\rangle-|1111101\rangle)_{(2)}\\
\vdots\ &\ \vdots\ \ \vdots\\
|124\rangle_L &= (|1111100\rangle+|0000011\rangle)_{(0)}\otimes(|1111100\rangle+|0000011\rangle)_{(1)}\\ &\quad\otimes(|1111100\rangle+|0000011\rangle)_{(2)}\\
|125\rangle_L &= (|1111100\rangle-|0000011\rangle)_{(0)}\otimes(|1111100\rangle-|0000011\rangle)_{(1)}\\ &\quad\otimes(|1111100\rangle-|0000011\rangle)_{(2)}\\
|126\rangle_L &= (|1111110\rangle+|0000001\rangle)_{(0)}\otimes(|1111110\rangle+|0000001\rangle)_{(1)}\\ &\quad\otimes(|1111110\rangle+|0000001\rangle)_{(2)}\\
|127\rangle_L &= (|1111110\rangle-|0000001\rangle)_{(0)}\otimes(|1111110\rangle-|0000001\rangle)_{(1)}\\ &\quad\otimes(|1111110\rangle-|0000001\rangle)_{(2)}.
\end{aligned} \qquad (14)$$

In most cases, physical systems (particles or solid-state devices) may have many levels, such as atoms, ions, and SQUIDs [11]. If a qubit is represented by a two-dimensional (2D) subspace of the Hilbert space of a multilevel physical system, the interaction with the environment may lead to the leakage out of the 2D qubit space (i.e., the space spanned by the two states $|0\rangle$ and $|1\rangle$ of a qubit). The decoherence process, therefore, is given by

$$\begin{aligned}|e_0\rangle|0\rangle &\longrightarrow |\varepsilon_0\rangle|0\rangle+|\varepsilon_1\rangle|1\rangle+\sum_{i\neq 0,1}|\varepsilon_i\rangle|i\rangle,\\ |e_0\rangle|1\rangle &\longrightarrow |\varepsilon_0^{'}\rangle|0\rangle+|\varepsilon_1^{'}\rangle|1\rangle+\sum_{i\neq 0,1}|\varepsilon_i^{'}\rangle|i\rangle,\end{aligned} \qquad (15)$$

where $\{|i\rangle\}$, together with $|0\rangle$ and $|1\rangle$, forms a complete orthogonal basis of a multilevel system, and $|\varepsilon_i\rangle$, $|\varepsilon_i^{'}\rangle$ are environment states. As will be shown below, during the restoration operation, there is no need to perform any operations on the "bad" qubit. For simplicity, we can rewrite (15) as

$$\begin{aligned}|e_0\rangle|0\rangle &\longrightarrow |\tilde{0}\rangle\\ |e_0\rangle|1\rangle &\longrightarrow |\tilde{1}\rangle,\end{aligned} \qquad (16)$$

where the above environment states $|\varepsilon_i\rangle$ and $|\varepsilon_i^{'}\rangle$ have been included in $|\tilde{0}\rangle$ end $|\tilde{1}\rangle$.

To extract the original state, we first perform a unitary transformation in block of qubits that have not suffered decoherence, which we regard as the partial decoding operation (since that blocks which suffer decoherence are not involved in the decoding operation). After that, we need to perform an error recovery operation in order to extract the original state.

It is considered here that can occur at most $t=\lfloor n/3\rfloor$ erasures in a block of n qubits.

Assuming that the erasure t_j $(j=1,\ldots,t)$ occurred in the block $k\in K$, where K is the set of blocks that were detected erasures, the decoding operator is given by

$$U_{dec}=\prod_{d=0(d\neq k)}^{t}\left(H_{n(d)}\otimes\prod_{i=1}^{n-1}C_{n(d),i(d)}\right). \qquad (17)$$

To illustrate, let us now consider the case where the qubit 1 block (0) and qubit 2 block (1) undergoes decoherence (erasure) and we see what will happen to the encoded state $|\psi_L\rangle$ of (13). After decoherence, it goes to

$$\begin{aligned}|\tilde{\psi}\rangle_L &= |\psi\rangle_L\otimes|e_0\rangle=\lambda_0|\tilde{0}\rangle_L+\lambda_1|\tilde{1}\rangle_L+\lambda_2|\tilde{2}\rangle_L+\lambda_3|\tilde{3}\rangle_L+\\ &\quad\ldots+\lambda_{124}|1\tilde{2}4\rangle_L+\lambda_{125}|1\tilde{2}5\rangle_L+\lambda_{126}|1\tilde{2}6\rangle_L\\ &\quad+\lambda_{127}|1\tilde{2}7\rangle_L,\end{aligned} \qquad (18)$$

where

$$\begin{aligned}
|\tilde{0}\rangle_L &= (|\tilde{0}000000\rangle+|\tilde{1}111111\rangle)_{(0)}\otimes(|0\tilde{0}00000\rangle+|1\tilde{1}11111\rangle)_{(1)}\\ &\quad\otimes(|0000000\rangle+|1111111\rangle)_{(2)}\\
|\tilde{1}\rangle_L &= (|\tilde{0}000000\rangle-|\tilde{1}111111\rangle)_{(0)}\otimes(|0\tilde{0}00000\rangle-|1\tilde{1}11111\rangle)_{(1)}\\ &\quad\otimes(|0000000\rangle-|1111111\rangle)_{(2)}\\
|\tilde{2}\rangle_L &= (|\tilde{0}000010\rangle+|\tilde{1}111101\rangle)_{(0)}\otimes(|0\tilde{0}00010\rangle+|1\tilde{1}11101\rangle)_{(1)}\\ &\quad\otimes(|0000010\rangle+|1111101\rangle)_{(2)}\\
|\tilde{3}\rangle_L &= (|\tilde{0}000010\rangle-|\tilde{1}111101\rangle)_{(0)}\otimes(|0\tilde{0}00010\rangle-|1\tilde{1}11101\rangle)_{(1)}\\ &\quad\otimes(|0000010\rangle-|1111101\rangle)_{(2)}\\
\vdots\ &\ \vdots\ \ \vdots\\
|1\tilde{2}4\rangle_L &= (|\tilde{1}111100\rangle+|\tilde{0}000011\rangle)_{(0)}\otimes(|1\tilde{1}11100\rangle+|0\tilde{0}00011\rangle)_{(1)}\\ &\quad\otimes(|1111100\rangle+|0000011\rangle)_{(2)}\\
|1\tilde{2}5\rangle_L &= (|\tilde{1}111100\rangle-|\tilde{0}000011\rangle)_{(0)}\otimes(|1\tilde{1}11100\rangle-|0\tilde{0}00011\rangle)_{(1)}\\ &\quad\otimes(|1111100\rangle-|0000011\rangle)_{(2)}\\
|1\tilde{2}6\rangle_L &= (|\tilde{1}111110\rangle+|\tilde{0}000001\rangle)_{(0)}\otimes(|1\tilde{1}11110\rangle+|0\tilde{0}00001\rangle)_{(1)}\\ &\quad\otimes(|1111110\rangle+|0000001\rangle)_{(2)}\\
|1\tilde{2}7\rangle_L &= (|\tilde{1}111110\rangle-|\tilde{0}000001\rangle)_{(0)}\otimes(|1\tilde{1}11110\rangle-|0\tilde{0}00001\rangle)_{(1)}\\ &\quad\otimes(|1111110\rangle-|0000001\rangle)_{(2)}.
\end{aligned} \qquad (19)$$

Comparing (19) with (14), we can see that for each "bad" logical state in (19), the portion of the product, corresponding to the block (2), is intact. We can first perform a unitary transformation on the qubits in the block (2) which we call partial decoding operation (since the qubits of blocks

(0) and (1) are not involved in the decoding operation). The decoding operation is shown as follows

$$\begin{aligned} U_{dec} &= \prod_{d=0(d\neq 0,1)}^{2}\left(H_{7(d)}\otimes\prod_{i=1}^{6}C_{7(d),i(d)}\right) \\ &= H_{7(2)}C_{7(2),1(2)}C_{7(2),2(2)}C_{7(2),3(2)}C_{7(2),4(2)}C_{7(2),5(2)} \\ &\quad C_{7(2),6(2)}. \end{aligned} \quad (20)$$

After decoding, we have

$$\begin{aligned} |\tilde{0}\rangle_L &= (|\tilde{0}000000\rangle + |\tilde{1}111111\rangle)_{(0)} \otimes (|0\tilde{0}00000\rangle + |1\tilde{1}11111\rangle)_{(1)} \\ &\quad \otimes |0000000\rangle_{(2)} \\ |\tilde{1}\rangle_L &= (|\tilde{0}000000\rangle - |\tilde{1}111111\rangle)_{(0)} \otimes (|0\tilde{0}00000\rangle - |1\tilde{1}11111\rangle)_{(1)} \\ &\quad \otimes |0000001\rangle_{(2)} \\ |\tilde{2}\rangle_L &= (|\tilde{0}000010\rangle + |\tilde{1}111101\rangle)_{(0)} \otimes (|0\tilde{0}00010\rangle + |1\tilde{1}11101\rangle)_{(1)} \\ &\quad \otimes |0000010\rangle_{(2)} \\ |\tilde{3}\rangle_L &= (|\tilde{0}000010\rangle - |\tilde{1}111101\rangle)_{(0)} \otimes (|0\tilde{0}00010\rangle - |1\tilde{1}11101\rangle)_{(1)} \\ &\quad \otimes |0000011\rangle_{(2)} \\ &\vdots \\ |1\tilde{2}4\rangle_L &= (|\tilde{1}111100\rangle + |\tilde{0}000011\rangle)_{(0)} \otimes (|1\tilde{1}11100\rangle + |0\tilde{0}00011\rangle)_{(1)} \\ &\quad \otimes |1111100\rangle_{(2)} \\ |1\tilde{2}5\rangle_L &= (|\tilde{1}111100\rangle - |\tilde{0}000011\rangle)_{(0)} \otimes (|1\tilde{1}11100\rangle - |0\tilde{0}00011\rangle)_{(1)} \\ &\quad \otimes |1111101\rangle_{(2)} \\ |1\tilde{2}6\rangle_L &= (|\tilde{1}111110\rangle + |\tilde{0}000001\rangle)_{(0)} \otimes (|1\tilde{1}11110\rangle + |0\tilde{0}00001\rangle)_{(1)} \\ &\quad \otimes |1111110\rangle_{(2)} \\ |1\tilde{2}7\rangle_L &= (|\tilde{1}111110\rangle - |\tilde{0}000001\rangle)_{(0)} \otimes (|1\tilde{1}11110\rangle - |0\tilde{0}00001\rangle)_{(1)} \\ &\quad \otimes |1111111\rangle_{(2)}. \end{aligned} \quad (21)$$

For the recovery operation $\mathscr{R}$, assume that the t_j erasure ($j = 1,\ldots,t$) occurred in the qubit b of the block k, denoted by $t_j = b(k)$, where $b \in \{1,\ldots,n\}$ and $k \in K$. Thus, the recovery operator, when $b \neq n$, is then given by

$$\begin{aligned} U_{rec}^{b(k)} &= \prod_{d=0(d\neq k)}^{t}\left(T_{b(d),n(d),n(k)-b(k)}\otimes Z_{n(d),n(k)-b(k)}\right. \\ &\quad \otimes T_{b(d),n(d),n(k)-b(k)}\otimes\prod_{i=1(i\neq b)}^{n-1}C_{i(d),i(k)} \\ &\quad \left.\otimes\prod_{i=1(i\neq b)}^{n}C_{b(d),i(k)}\right), \end{aligned} \quad (22)$$

where $T_{a,b,c}$ is a Toffoli gate operation [15], and $Z_{b,c}$ is a controlled Pauli σ_Z operation. A Toffoli gate $T_{a,b,c}$ has two control bits corresponding to the first two subscripts (a,b), and the target bit c. When the two control bits are in the state $|11\rangle$, the state of the target bit will change, following $|0\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |0\rangle$, while when the two control bits are in the state $|00\rangle$, $|01\rangle$ or $|10\rangle$, the state of the target bit will be invariant. The controlled Pauli σ_Z operation $Z_{b,c}$ has the control bit b and target bit c, which sends the state of the target bit $|0\rangle \rightarrow |0\rangle$ and $|1\rangle \rightarrow -|1\rangle$ when the control bit is in the state $|1\rangle$; otherwise, when the control bit is in the $|0\rangle$, the state of the target bit will not change.

The recovery operator, when $b = n$, is given by

$$U_{rec}^{b(k)} = \prod_{d=0(d\neq k)}^{t}\left(Z_{n(d),n(k)-1}\otimes\prod_{i=1(i\neq b)}^{n-1}C_{i(d)i(k)}\right). \quad (23)$$

Assuming that the erasures $t_1,\ldots,t_f$ ($f = \lfloor n/3 \rfloor$) occurred, the recovery operation is given by

$$\mathscr{R} = U_{rec}^{t_f}\Big(\ldots\Big(U_{rec}^{t_1}\big(U_{dec}(|\tilde{\Psi}\rangle_L)\big)\Big)\ldots\Big). \quad (24)$$

To illustrate the application of the recovery operation, we consider the situation shown in (21). Since the erasures were $t_1 = 1(0)$ and $t_2 = 2(1)$, then $b = 1 \neq n$, and also $b = 2 \neq n$ ($n = 7$). Therefore we'll use the recovery operator of (22). The recovery operators are as follows

$$\begin{aligned} U_{rec}^{1(0)} &= \prod_{d=0(d\neq 0,1)}^{2}\left(T_{1(d),7(d),7(0)-1(0)}\otimes Z_{7(d),7(0)-1(0)}\right. \\ &\quad \otimes T_{1(d),7(d),7(0)-1(0)}\otimes\prod_{i=1(i\neq 1)}^{6}C_{i(d),i(0)} \\ &\quad \left.\otimes\prod_{i=1(i\neq 1)}^{n}C_{1(d),i(0)}\right) \\ &= T_{1(2),7(2),6(0)}\otimes Z_{7(2),6(0)}\otimes T_{1(2),7(2),6(0)}\otimes C_{2(2),2(0)} \\ &\quad C_{3(2),3(0)}C_{4(2),4(0)}C_{5(2),5(0)}C_{6(2),6(0)}\otimes C_{1(2),2(0)} \\ &\quad C_{1(2),3(0)}C_{1(2),4(0)}C_{1(2),5(0)}C_{1(2),6(0)}C_{1(2),7(0)}; \end{aligned} \quad (25)$$

and

$$\begin{aligned} U_{rec}^{2(1)} &= \prod_{d=0(d\neq 0,1)}^{2}\left(T_{2(d),7(d),7(1)-2(1)}\otimes Z_{7(d),7(1)-2(1)}\right. \\ &\quad \otimes T_{2(d),7(d),7(1)-2(1)}\otimes\prod_{i=1(i\neq 2)}^{6}C_{i(d),i(1)} \\ &\quad \left.\otimes\prod_{i=1(i\neq 2)}^{n}C_{2(d),i(1)}\right) \\ &= T_{2(2),7(2),5(1)}\otimes Z_{7(2),5(1)}\otimes T_{2(2),7(2),5(1)}\otimes C_{1(2),1(1)} \\ &\quad C_{3(2),3(1)}C_{4(2),4(1)}C_{5(2),5(1)}C_{6(2),6(1)}\otimes C_{2(2),1(1)} \\ &\quad C_{2(2),3(1)}C_{2(2),4(1)}C_{2(2),5(1)}C_{2(2),6(1)}C_{2(2),7(1)}. \end{aligned} \quad (26)$$

Therefore, the recovery operation $\mathscr{R}$ in this case is as follows

$$\mathscr{R} = U_{rec}^{2(1)}\Big(U_{rec}^{1(0)}\Big(U_{dec}\big(|\tilde{\Psi}\rangle_L\big)\Big)\Big). \quad (27)$$

When we applying the recovery operator $U_{rec}^{1(0)}$ (25) in (21), we have

$$\begin{aligned} |\tilde{0}\rangle_L &= (|\tilde{0}000000\rangle + |\tilde{1}111111\rangle)_{(0)} \otimes (|0\tilde{0}00000\rangle + |1\tilde{1}11111\rangle)_{(1)} \\ &\quad \otimes |0000000\rangle_{(2)} \\ |\tilde{1}\rangle_L &= (|\tilde{0}000000\rangle + |\tilde{1}111111\rangle)_{(0)} \otimes (|0\tilde{0}00000\rangle - |1\tilde{1}11111\rangle)_{(1)} \\ &\quad \otimes |0000001\rangle_{(2)} \\ |\tilde{2}\rangle_L &= (|\tilde{0}000000\rangle + |\tilde{1}111111\rangle)_{(0)} \otimes (|0\tilde{0}00010\rangle + |1\tilde{1}11101\rangle)_{(1)} \\ &\quad \otimes |0000010\rangle_{(2)} \\ |\tilde{3}\rangle_L &= (|\tilde{0}000000\rangle + |\tilde{1}111111\rangle)_{(0)} \otimes (|0\tilde{0}00010\rangle - |1\tilde{1}11101\rangle)_{(1)} \\ &\quad \otimes |0000011\rangle_{(2)} \end{aligned}$$

$$\begin{aligned}
&\vdots \quad \vdots \quad \vdots \\
|1\tilde{2}4\rangle_L &= (|\tilde{0}000000\rangle + |\tilde{1}111111\rangle)_{(0)} \otimes (|1\tilde{1}11100\rangle + |0\tilde{0}00011\rangle)_{(1)} \\
&\quad \otimes |1111100\rangle_{(2)} \\
|1\tilde{2}5\rangle_L &= (|\tilde{0}000000\rangle + |\tilde{1}111111\rangle)_{(0)} \otimes (|1\tilde{1}11100\rangle - |0\tilde{0}00011\rangle)_{(1)} \\
&\quad \otimes |1111101\rangle_{(2)} \\
|1\tilde{2}6\rangle_L &= (|\tilde{0}000000\rangle + |\tilde{1}111111\rangle)_{(0)} \otimes (|1\tilde{1}11110\rangle + |0\tilde{0}00001\rangle)_{(1)} \\
&\quad \otimes |1111110\rangle_{(2)} \\
|1\tilde{2}7\rangle_L &= (|\tilde{0}000000\rangle + |\tilde{1}111111\rangle)_{(0)} \otimes (|1\tilde{1}11110\rangle - |0\tilde{0}00001\rangle)_{(1)} \\
&\quad \otimes |1111111\rangle_{(2)}. \qquad (28)
\end{aligned}$$

After this, we apply the recovery operator $U_{rec}^{2(1)}$ (29) in (28) and obtain

$$\begin{aligned}
|\tilde{0}\rangle_L &= (|\tilde{0}000000\rangle + |\tilde{1}111111\rangle)_{(0)} \otimes (|0\tilde{0}00000\rangle + |1\tilde{1}11111\rangle)_{(1)} \\
&\quad \otimes |0000000\rangle_{(2)} \\
|\tilde{1}\rangle_L &= (|\tilde{0}000000\rangle + |\tilde{1}111111\rangle)_{(0)} \otimes (|0\tilde{0}00000\rangle + |1\tilde{1}11111\rangle)_{(1)} \\
&\quad \otimes |0000001\rangle_{(2)} \\
|\tilde{2}\rangle_L &= (|\tilde{0}000000\rangle + |\tilde{1}111111\rangle)_{(0)} \otimes (|0\tilde{0}00000\rangle + |1\tilde{1}11111\rangle)_{(1)} \\
&\quad \otimes |0000010\rangle_{(2)} \\
|\tilde{3}\rangle_L &= (|\tilde{0}000000\rangle + |\tilde{1}111111\rangle)_{(0)} \otimes (|0\tilde{0}00000\rangle + |1\tilde{1}11111\rangle)_{(1)} \\
&\quad \otimes |0000011\rangle_{(2)} \\
&\vdots \quad \vdots \quad \vdots \\
|1\tilde{2}4\rangle_L &= (|\tilde{0}000000\rangle + |\tilde{1}111111\rangle)_{(0)} \otimes (|0\tilde{0}00000\rangle + |1\tilde{1}11111\rangle)_{(1)} \\
&\quad \otimes |1111100\rangle_{(2)} \\
|1\tilde{2}5\rangle_L &= (|\tilde{0}000000\rangle + |\tilde{1}111111\rangle)_{(0)} \otimes (|0\tilde{0}00000\rangle + |1\tilde{1}11111\rangle)_{(1)} \\
&\quad \otimes |1111101\rangle_{(2)} \\
|1\tilde{2}6\rangle_L &= (|\tilde{0}000000\rangle + |\tilde{1}111111\rangle)_{(0)} \otimes (|0\tilde{0}00000\rangle + |1\tilde{1}11111\rangle)_{(1)} \\
&\quad \otimes |1111110\rangle_{(2)} \\
|1\tilde{2}7\rangle_L &= (|\tilde{0}000000\rangle + |\tilde{1}111111\rangle)_{(0)} \otimes (|0\tilde{0}00000\rangle + |1\tilde{1}11111\rangle)_{(1)} \\
&\quad \otimes |1111111\rangle_{(2)}. \qquad (29)
\end{aligned}$$

Therefore, after performing the decoding operator U_{dec} (20) and the recovery operators $U_{rec}^{1(0)}$ (25) and $U_{rec}^{2(1)}$ (29), the system composed of twenty eight qubits and the environment will be in state

$$\Big(|\tilde{0}000000\rangle + |\tilde{1}111111\rangle\Big)_{(0)} \otimes \Big(|0\tilde{0}00000\rangle + |1\tilde{1}11111\rangle\Big)_{(1)} \otimes |\psi\rangle_{(2)}. \quad (30)$$

As shown above, since the "damaged" particle is not involved in the recovery operations, the present code can still work in the case when the interaction with environment leads to the leakage of a qubit out of the qubit space.

III. Conclusion

We have presented a N-qubit code for protecting n-qubit quantum information against t erasures, where $N = n(t+1)$ and $t = \lfloor n/3 \rfloor$. There are already experimental studies that used schemes similar to that presented in this paper in order to protect a logical qubit from loss of a physical qubit, see Ref. [16]. The encoding, decoding and error recovery operations, as shown here, are relatively straightforward. A special feature of the erasure recovery scheme is that no measurement are required.

Finally we recall that erasure recovering algorithms like that introduced here have several interesting applications, e.g., concatenated quantum codes [13] and quantum secret sharing [17], [18].

Acknowledgments

We would like to thank Elloá B. Guedes and the anonymous reviewer for comments and suggestions. This work was partially supported by CNPq (309431/2006.9) and CAPES.

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