Sparse Solutions for Antenna Arrays

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***Abstract*— In the field of communication warfare, the use of transceivers with frequency hopping to avoid intensional interference is usual. As for the immunity to interference, such electronic countermeasure has excellent efficiency, but greatly increases the electromagnetic emission and, therefore, the station indiscretion. To overcome this problem, it is possible to make use of antenna arrays and interference cancellation algorithms that introduce nulls in the directions of interferers while keeping a constant gain in the direction of interest. Under some conditions, in this example or in any other antenna array application, the solution can be sparse. In this paper, we discuss the use of L_0-nom or L_1-norm constraints to obtain sparse solutions in antenna arrays.**

***Keywords*— Antenna array, Sparse Solutions, L_0-norm minimization, L_1-norm minimization, LASSO, Convex Optimization, Second-Order Cone Programming (SOCP), Beampattern.**

I. INTRODUCTION

The ability of antenna arrays to provide efficient and flexible ways to synthesize several beampatterns makes their applications suitable for canceling interference signals, while the gain toward the signal of interest remains unchanged. The classical and usual way to avoid intentional interferences is employing transceivers with frequency hopping [1]. Although such electronic counter-measure method ensures the communication between the stations, its use increases electromagnetic indiscretion, allowing easy detection.

In order to allow the interference cancellation algorithms to run on battery consumption critical devices, shrinks coefficients to zero is particularly important for its reduced computational requirements and consequent battery consumption [2], therefore, it is desirable to have a reduced number of coefficients for the synthesis of the beampattern. We may make use of antenna arrays and interference cancellation algorithms [3] with L_0-norm or L_1-norm constraints, whose solutions, under some conditions, can be sparse. Usually, a large number of antennas is necessary, therefore, the algorithms shall introduce null gains in the directions of interferences and maximize the number of coefficients equal to zero.

Without considering sparsity solutions, the synthesis of the beampattern could be expressed by $\mathbf{A}\mathbf{w} = \mathbf{b}$ subject to $\mathbf{C}^H\mathbf{w} = \mathbf{f}$, where $\mathbf{w}$, $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{C}$, are the coefficient vector (solution), the array manifold matrix, the beampattern vector and the constraint matrix, respectively. Matrix $\mathbf{C}$ contains the linear constraints to assure nulls in directions of the jammers and a constant gain in the direction of interest.

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II. BACKGROUND OVERVIEW

The idea to overcome the problem of the large number of antennas, necessary to perform interference cancellation and still maintain a suitable beampattern to communication system requirements, is to use algorithms for solving systems of type $\mathbf{A}\mathbf{w} = \mathbf{b}$ subject to the restriction of L_1 (or L_0) norm and additional linear constraints of type $\mathbf{C}^H\mathbf{w} = \mathbf{f}$. The resulting solution has sparsity inversely proportional to the imposed L_1 norm. Greater sparsity implies a smaller number of sensors (antennas). Mathematically, the problem can be expressed as:

$$\min_{(\mathbf{w}\in\mathbb{C}^N)} \;\|\mathbf{A}\mathbf{w}-\mathbf{b}\|_2^2 \;\text{ st: } \begin{cases}\|\mathbf{w}\|_1 \leq t\\ \mathbf{C}^H\mathbf{w} = \mathbf{f}\end{cases} \tag{1}$$

or

$$\min_{(\mathbf{w}\in\mathbb{C}^N)} \;\|\mathbf{w}\|_1 \;\text{ st: } \begin{cases}\|\mathbf{b}-\mathbf{A}\mathbf{w}\|_2^2 \leq \epsilon\\ \mathbf{C}^H\mathbf{w} = \mathbf{f}\end{cases} \tag{2}$$

The first approach, expressed by the Eq. (1), leads to the LASSO solution [4]. The second approach is a regularization problem [5] considering $J(\mathbf{w}) = \|\mathbf{w}\|_1$ and linear additional constraints. The classical solution method considers $J(\mathbf{w}) = \|\mathbf{b} - \mathbf{A}\mathbf{w}\|_2^2$, which corresponds to the Wiener solution [5], except for the constraints $\mathbf{C^H}\mathbf{w} = \mathbf{f}$.

The L_0 norm constrained problem, defined as

$$(P_0) : \min_{(\mathbf{w}\in\mathbb{C}^N)} \|\mathbf{w}\|_0 \;\text{ st: } \|\mathbf{b}-\mathbf{A}\mathbf{w}\|_2^2 \leq \epsilon, \tag{3}$$

could be another important method to be investigated. The computational complexity to solve (P_0) by exhaustive search is huge, exponential in N, where N is the number of coefficients. In [5] , the (P_0) problem is solved by smoothing the L_0-norm in various forms and, then, handling the revised problem as a smooth optimization. The main idea under (P_0) problem is to minimize the cardinality of feasible solution set, whose results are sparse. Unfortunately, the algorithms in [5] do not cover the beampattern problem with additional linear constraints of type $\mathbf{C}^H\mathbf{w} = \mathbf{f}$ and, therefore, it is necessary to investigate in deep these algorithms so one can carry out necessary changes to meet the additional linear constraints.

Alternatively, a small number of sensors (antennas) can be obtained from the LASSO solution, through a projection on the L_1 Ball [6]-[7], whose result is projected onto the intersection of hyperplanes of the constraints $\mathbf{C^H}\mathbf{w} - \mathbf{f} = \mathbf{0}$. This intersection is feasible only for certain values of L_1 norm and, therefore, the algorithm shall find points with lower MSE on the polyhedral surface $g(\mathbf{w}) = \|\mathbf{w}\|_1 - t = 0$ as function of L_1 norm. This intuitive method could be called as "Successive L_1 Ball and Hyperplane Projection Method". Fig. 1 illustrates a simple example in $\mathbb{R}^3$, where the sparse solution $\mathbf{X}_{\text{SP}} = [0.000\ 0.000\ 1.250]^T$ is obtained from the Least Squares (LS) solution $\mathbf{X}_{\text{LS}} = [-0.512\ -0.400\ 0.808]^T$ using the *Successive L_1 Ball and Hyperplane Projection Method*. The L_1 norm and linear constraints are maintained. The main weakness of this algorithm is the need to conduct a line search over a non-continuous feasible set (L_1-norm values), which results in a huge computational effort.

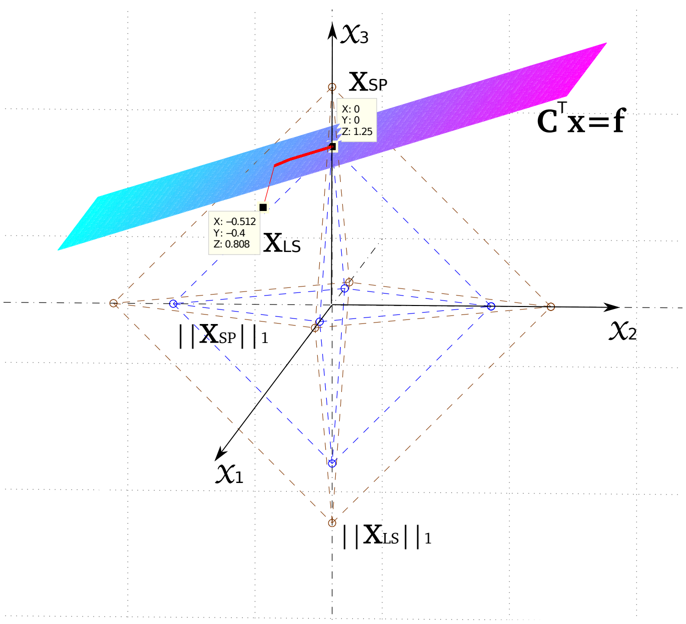

Fig. 1. Solution subject to L_1-norm and linear constraints calculated from LS solution.

III. The Antenna Array Pattern

A. Signal Model

Consider a uniform linear array (ULA) composed by N receiving antennas (sensors) and q receiving narrowband signals coming from different directions $\phi_1, \cdots, \phi_q$. The output signal observed from N sensors during M snapshots can be denoted as $\mathbf{x}(t_1), \mathbf{x}(t_2), \cdots, \mathbf{x}(t_M)$. The $N \times 1$ signal vector is then written as:

$$\mathbf{x}(t) = \sum_{k=1}^{q} \mathbf{a}(\phi_k)\mathbf{s}_k(t) + \mathbf{n}(t)\ ,\ t = t_1, t_2, \cdots, t_M \tag{4}$$

or, using matrix notation,

$$\mathbf{X} = [\mathbf{x}(t_1)\mathbf{x}(t_2)\ \cdots\ \mathbf{x}(t_M)] = \mathbf{\Psi}\mathbf{S} + \mathbf{N}, \tag{5}$$

where matrix $\mathbf{X}$ is the input signal matrix of dimension $N \times M$. Matrix $\mathbf{\Psi} = [\mathbf{a}(\phi_1), \cdots, \mathbf{a}(\phi_q)]$ is the steering matrix of dimension $N \times q$, whose columns are denoted by

$$\boldsymbol{a}(\phi) = [1, e^{-j(2\pi/\lambda)d\cos(\phi)}, \cdots, e^{-j(2\pi/\lambda)d(N-1)\cos(\phi)}]^T. \tag{6}$$

$\mathbf{S}$, in (5), is a $q \times M$ signal matrix, whose rows refer to snapshots. $\mathbf{N}$ is the noise matrix of dimension $N \times M$; λ and d are the wavelength of the signal and the distance between antenna elements (sensors), respectively.

Based on definition above, the covariance matrix $\mathbf{R}$, defined as $\mathrm{E}[\mathbf{x}(t)\mathbf{x}^H(t)]$, can be estimated as

$$\hat{\mathbf{R}} = \mathbf{X}\mathbf{X}^H = \mathbf{\Psi}\mathbf{S}\mathbf{S}^H\mathbf{\Psi}^H + \mathbf{N}\mathbf{N}^H, \tag{7}$$

which, except for a multiplicative constant, corresponds to the time average

$$\hat{\mathbf{R}} = \frac{1}{M}\sum_{k=1}^{M} \mathbf{x}(t_k)\mathbf{x}^H(t_k). \tag{8}$$

From the formulation above and imposing linear constraints, it is possible to obtain a closed-form expression to $\mathbf{w}_{opt}$ [3], the LCMV solution

$$\mathbf{w}_{opt} = \mathbf{R}^{-1}\mathbf{C}\Big(\mathbf{C}^H\mathbf{R}^{-1}\mathbf{C}\Big)^{-1}\mathbf{f}, \tag{9}$$

such that $\mathbf{C}^H\mathbf{w}_{opt} = \mathbf{f}$, $\mathbf{C}$ and $\mathbf{f}$ given by [8]

$$\mathbf{C} = [1, e^{-j(2\pi/\lambda)d\cos(\phi_I)}, \cdots, e^{-j(2\pi/\lambda)d(N-1)\cos(\phi_I)}]^T \tag{10}$$

and

$$\mathbf{f} = 1. \tag{11}$$

In Eq. (10), the angle ϕ_I indicates the interest direction. In order to consider interfering signal directions ϕ_J, new constraints shall be considered as additional rows in $\mathbf{C}$ and $\mathbf{f}$.

Fig. (2) shows the geometry of a Uniform Linear Array (ULA) of sensors.

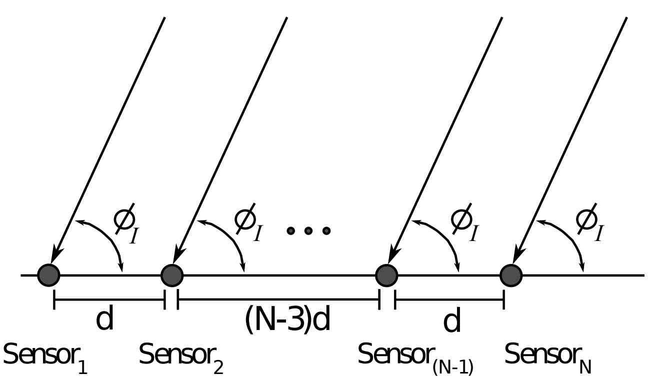

Fig. 2. The Geometry of a Uniform Linear Array (ULA) of sensors.

B. Antenna Beampattern

Considering a harmonic plane wave with wavelength λ incident from direction ϕ, that propagates across a linear array of N isotropic antennas at locations $p_1, p_2, \cdots, p_N \in \mathbb{R}^2$, the beampattern is given by

$$B(\phi) = \sum_{k=1}^{N} w_k e^{-j\frac{2\pi}{\lambda}p_k\cos(\phi)}\ , \tag{12}$$

where w_k is the k-th component os vector $\mathbf{w}$.

For a planar uniform rectangular array (URA) containing $N \times M$ antennas placed on a geometric grid with N columns and M rows, the beampattern could be expressed as

$$B(\phi) = \sum_{k=1}^{N} \sum_{l=1}^{M} w_{k,l} e^{-j\frac{2\pi}{\lambda}\left[p_k^x \cos(\phi) + p_l^y \sin(\phi)\right]}. \quad (13)$$

In order to make a link between beampattern definition given by Eq. (13) and the formulation stated by Eqs. (1) and (2), an equivalent matrix formulation can be stated.

$$\mathbf{b}(\phi) = \mathbf{A}(\phi, p)\mathbf{w}, \quad (14)$$

where elements of $\mathbf{w}$ and $\mathbf{A}(\phi, p)$ are complex quantities with $\mathbf{w}$ given by $\mathbf{w} = [w_1, w_2, \cdots, w_N]^T$.

IV. Beampattern Synthesis as a Convex Optimization Problem

An important and powerful beampattern synthesis tool is Convex Optimization [10]-[11]. This tool could be used to solve the problem stated by Eqs. (1) and (2). In reference [9], a comprehensive summary about this application is presented. In this section, based on the theory presented in [9], a formulation used in our problem will be derived. A technique to solve optimization problems with complex variables is known as Second-Order Cone Programming (SOCP) [11], therefore, the problem defined in (2) should be reformulated in order to be solved using SOCP.

In (2), all matrices and vectors are complex. The first step to re-write this problem is to convert the formulation to real variables

$$\underset{\sim}{\mathbf{w}} = \begin{bmatrix} \Re(\mathbf{w}) \\ \Im(\mathbf{w}) \end{bmatrix} \in \mathbb{R}^{2N\times 1} \quad (15)$$

$$\underset{\sim}{\mathbf{b}} = \begin{bmatrix} \Re(\mathbf{b}) \\ \Im(\mathbf{b}) \end{bmatrix} \in \mathbb{R}^{2M\times 1} \quad (16)$$

$$\underset{\sim}{\mathbf{f}} = \begin{bmatrix} \Re(\mathbf{f}) \\ \Im(\mathbf{f}) \end{bmatrix} \in \mathbb{R}^{2N_C\times 1}, \quad (17)$$

where M and N_C are the size of $\mathbf{b}$ and the number of linear constraints, respectively.

$$\underset{\sim}{\mathbf{T}} = \begin{bmatrix} \Re(\mathbf{A}) & -\Im(\mathbf{A}) \\ \Im(\mathbf{A}) & \Re(\mathbf{A}) \end{bmatrix} \in \mathbb{R}^{2M\times 2N} \quad (18)$$

$$\underset{\sim}{\mathbf{P}} = \begin{bmatrix} \Re(\mathbf{C}^H) & -\Im(\mathbf{C}^H) \\ \Im(\mathbf{C}^H) & \Re(\mathbf{C}^H) \end{bmatrix} \in \mathbb{R}^{2N_C\times 2N} \quad (19)$$

Then, Eq. (2) can be re-expressed as

$$\min \|t\|_1 \ \text{s.t.} \ \begin{cases} \|\underset{\sim}{\mathbf{T}}\underset{\sim}{\mathbf{w}} - \underset{\sim}{\mathbf{b}}\|_2 \le \|\underset{\sim}{\mathbf{b}}\|_2\delta_1, \\ \|\underset{\sim}{\mathbf{P}}\underset{\sim}{\mathbf{w}} - \underset{\sim}{\mathbf{f}}\|_2 \le \|\underset{\sim}{\mathbf{f}}\|_2\delta_2, \end{cases} \quad (20)$$

where $\|t\|_1 = \sum_{i=1}^{N} \sqrt{\underset{\sim}{w}_i^2 + \underset{\sim}{w}_{i+N}^2}$.

The formulation in (20) faithfully represents the complex L_1 norm of (2). Unfortunately, minimizing the L_1 norm of a real vector $\underset{\sim}{\mathbf{w}}$ does not guarantee sparsity of real and imaginary components simultaneously, similarly to the method presented in [12]. In order to guarantee simultaneous real and imaginary sparsity, the constraints should be re-written as follows

$$\min \|t\|_1 \ \text{s.t.} \begin{cases} \sqrt{[\Re(w_i)]^2 + [\Im(w_i)]^2} \le t_i, \ i = 1, ..., N \\ \|\underset{\sim}{\mathbf{T}}\underset{\sim}{\mathbf{w}} - \underset{\sim}{\mathbf{b}}\|_2 \le \|\underset{\sim}{\mathbf{b}}\|_2\delta_1 \\ \|\underset{\sim}{\mathbf{P}}\underset{\sim}{\mathbf{w}} - \underset{\sim}{\mathbf{f}}\|_2 \le \|\underset{\sim}{\mathbf{f}}|_2\delta_2 \end{cases} \quad (21)$$

Now considering the following auxiliary variables

$$\mathbf{y} = \begin{bmatrix} \underset{\sim}{\mathbf{w}} \\ t \end{bmatrix} \in \mathbb{R}^{3N\times 1} \quad (22)$$

$$\mathbf{S}_i = \begin{bmatrix} \mathbf{e}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{e}_i \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{3N\times 2} \quad (23)$$

$$\mathbf{q}_i = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{e}_i \end{bmatrix} \in \mathbb{R}^{3N\times 1} \quad (24)$$

$$\mathbf{S}_{LS} = \begin{bmatrix} \underset{\sim}{\mathbf{T}}^T \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{3N\times 2M} \quad (25)$$

$$\mathbf{S}_C = \begin{bmatrix} \underset{\sim}{\mathbf{P}}^T \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{3N\times 2N_C}, \quad (26)$$

where $\mathbf{e}_i \in \mathbb{R}^{N\times 1}$ is a vector composed by zeros, except the i-th component that is one, and defining $\mathbf{g} = [0, 0, \cdots, 0, 1, 1, \cdots, 1]^T_{3N\times 1}$, with the $2N$ first components equal to zero and the other ones equal to one, the problem could be reformulated as

$$\min_{\underset{\sim}{\mathbf{w}}} \ \mathbf{g}^T\mathbf{y} \ \text{st:} \quad \begin{cases} \|\mathbf{S}_i^T\mathbf{y}\|_2 \le \mathbf{q}_i^T\mathbf{y} \quad i = 1, \cdots, N \\ \|\mathbf{S}_{LS}^T\mathbf{y} - \underset{\sim}{\mathbf{b}}\|_2 \le \|\underset{\sim}{\mathbf{b}}\|_2\delta_1 \\ \|\mathbf{S}_C^T\mathbf{y} - \underset{\sim}{\mathbf{f}}\|_2 \le \|\underset{\sim}{\mathbf{f}}\|_2\delta_2 \end{cases} \quad (27)$$

The problem stated in (27) is known as Second Order Cone Programming (SOCP).

To illustrate how the formulation presented above allows to null both real and imaginary components of the coefficient vector $\mathbf{w}$, a simple development of Eq. (27), for $N = 2$, is presented bellow:

$$\min \ \{t_1 + t_2\} \ \text{s.t.} \quad \begin{cases} \sqrt{[\Re(w_1)]^2 + [\Im(w_1)]^2} \le t_1 \\ \sqrt{[\Re(w_2)]^2 + [\Im(w_2)]^2} \le t_2 \\ \|\mathbf{S}_{LS}^T\mathbf{y} - \underset{\sim}{\mathbf{b}}\|_2 \le \|\underset{\sim}{\mathbf{b}}\|_2\delta_1 \\ \|\mathbf{S}_C^T\mathbf{y} - \underset{\sim}{\mathbf{f}}\|_2 \le \|\underset{\sim}{\mathbf{f}}\|_2\delta_2 \end{cases} \quad (28)$$

V. Numerical Results

A. 2-dimensional Array Constrained Synthesis using "Successive L_1 Ball and Hyperplane Projection Method".

As the first numerical simulation example, we designed an L_1-norm constrained problem to a 2-dimensional constrained array with 36 elements spaced by 0.25λ. The direction of the signal of interest is $103°$ and the directions of the interferences are $15°$, $50°$, $140°$ and $160°$.

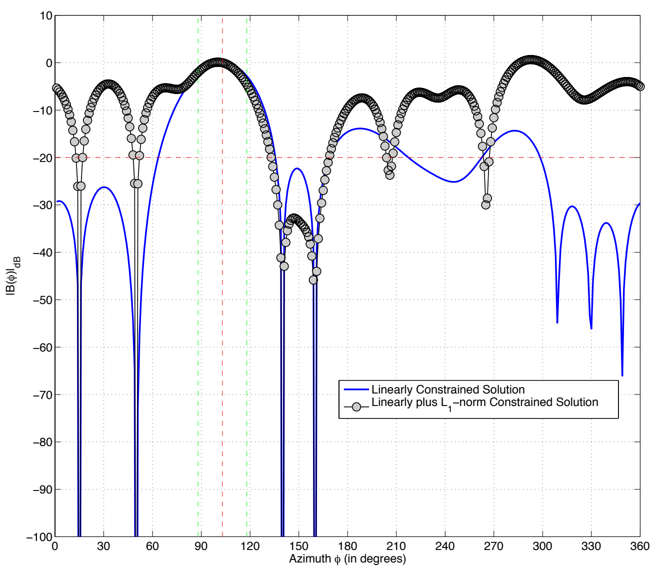

Fig. 3. 2D Constrained Array Synthesis via "Successive L_1 Ball and Hyperplane Projection Method".

The plots of Fig. 3 compare the design solutions with and without additional L_1-norm constraint. The L_1 norm constrained resulted in 17 null coefficients, according to Fig. 4. It is important to notice that even with 17 null coefficients, the constraints are satisfied. Unfortunately, depending on the array configuration and on the linear restrictions, a sparse solution may not be possible.

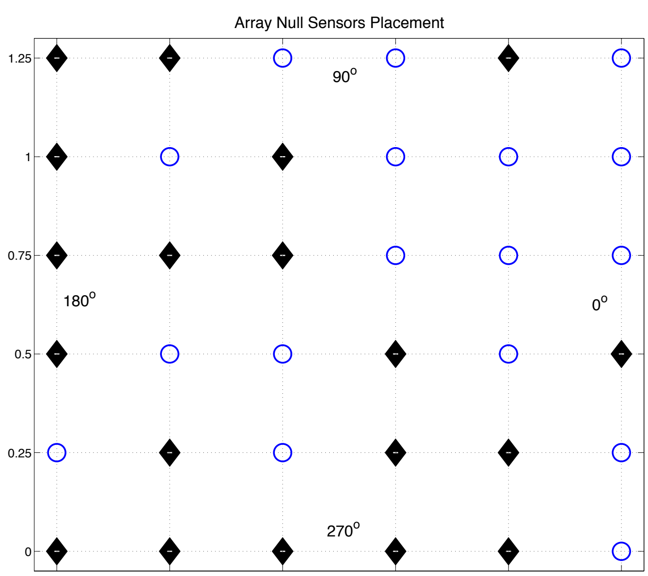

Fig. 4. 2D Constrained Array Null Sensors (O) Placement.

B. Constrained Uniform Linear Array Synthesis using SOCP.

The second simulation result is shown in Fig. 5. The antenna array considered was an ULA with $N = 200$ elements spaced by $\frac{\lambda}{2}$. The L_1-norm minimization constraint was considered. As linear constraints, we considered the signal of interest direction of $103°$ and the directions of interferences equal to: $15°$, $50°$, $140°$ and $160°$. A 3-dB bandwidth of $2°$ was considered. The L_1 norm constraint converged to unit. This simulation shows a typical application of Convex Optimization on synthesis of Antenna Arrays with sharp constraints as, for example, a 3-dB bandwidth of $2°$.

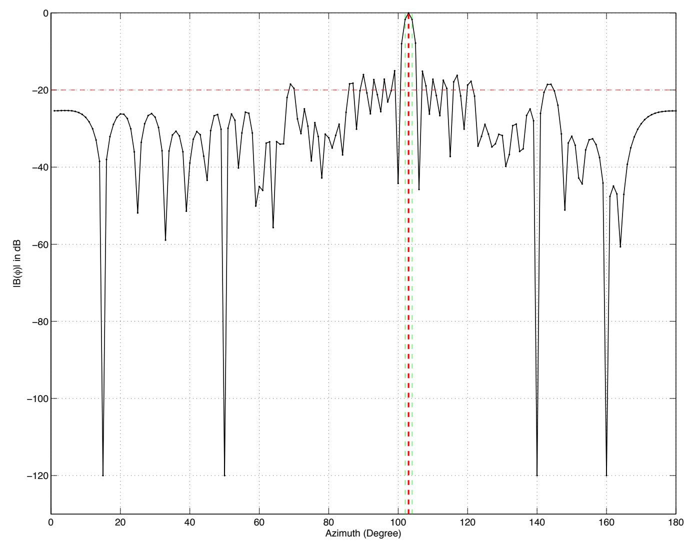

Fig. 5. Constrained Uniform Linear Array Synthesis using SOCP.

Analyzing the plot presented in Fig. 5, it is possible to observe that all linear constraints were satisfied, but, unfortunately, it was not possible to obtain a sparse solution. The L_1-norm constraint converged to unit and no null coefficient was obtained.

C. Constrained URA Synthesis using SOCP.

In a third simulation result, shown in Fig. 6, we imposed additional constraints to Eq. (2) in order to solve the problem described in subsection V-A. These constraints were imposed such that 16 coefficients, located according to Fig. 7, were forced to be null, which corresponding to 44% of total number of coefficients. As can be seen in the figure, the linear constraints, responsible for the interference cancellation, were satisfied, but the secondary lobe levels have increased destroying the beamforming original pattern. Depending on the application, it may be acceptable to lose the antenna radiation characteristic. For the focused application, the important requirement is to suppress all the jammers signals.

VI. CONCLUSIONS

In this paper, we have presented some classical shrinkage numerical techniques and preliminary simulation results for obtaining solutions to the problem of sparse antenna array design. When using SOCP, depending on the configuration of the antenna array under investigation, one could achieve a sparse solution. One way to force a sparse solution is to

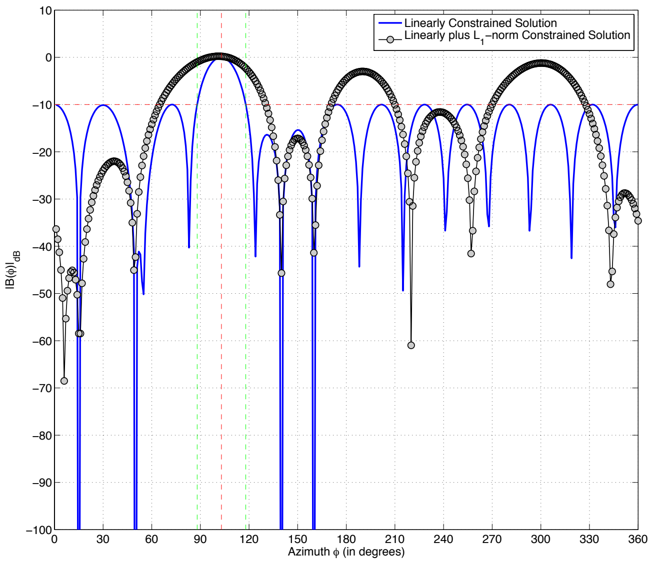

Fig. 6. 2D Constrained Array Synthesis via SOCP.

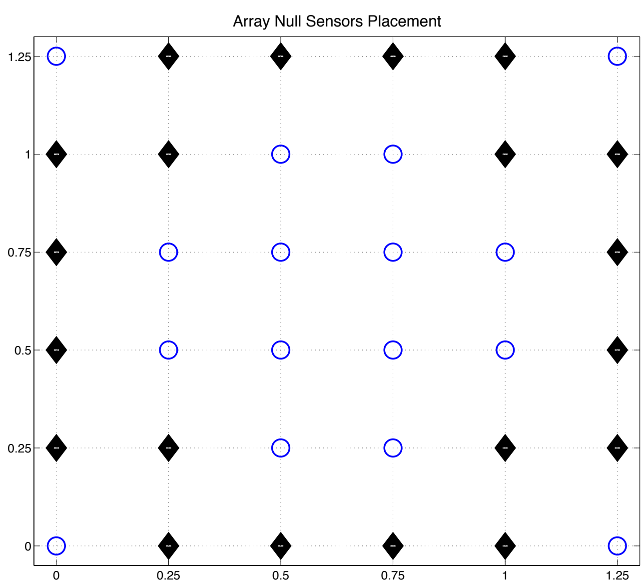

Fig. 7. 2D Constrained Array Null Sensors (O) Placement.

solve the problem applying a L_0-norm minimization. The solution for (P_0), unfortunately, has computational complexity that increases exponentially with the number of coefficients. To overcome the huge computational effort, techniques and Greedy algorithms defined in [5] are employed. Unfortunately, these algorithms do not cover the beampattern problem with additional linear constraints of type $\mathbf{C}^H\mathbf{w} = \mathbf{f}$ and, therefore, it is necessary to investigate in deep these algorithms so one can carry out necessary changes to meet the linear additional constraints.

Since the L_1-norm constrained problem associated to additional linear constraint is essentially a convex problem for some L_1-norm and linear constraint values, its application in the complex field should be better investigated.

As appropriate tool to solve problems of L_1-norm minimization, one can make use of convex optimization techniques. When dealing with problems in solving complex variables and additional linear constraints, the SOCP could be applied.

In practice, the use of L_1-norm constrained algorithms to cancel interfering signals allows to run beampattern synthesis softwares on systems with limited power capacity, since a reduced number of coefficients can be employed. As can be seen in Section (V-C), the price to be paid to a reduced number of coefficients is a temporary degradation of the radiation pattern of the array of antennas, that will occur during the presence of an interfering signal, when the anti-jamming algorithm shall be in use. Sometimes, in battlefield, it is better to receive a message with some noise, while canceling the jamming signals, than to be exposed by emissions of frequency hopping systems.

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