Applications of Enumerative Techniques in Communications

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***Abstract*—Many problems in communications and computer science require the characterization and enumeration of sequences. The purpose of this paper is to demonstrate the application of enumeration techniques to two such problems, enumeration of run length limited sequences, as used in computer memory systems, and the characterization error sequences as they occur in the analysis of bursty or fading channels.**

I. Introduction

Many problems in communications and computer science require the enumeration and characterization of sequences. Two such problems are considered here: i) the enumeration of run length limited (RLL) sequences in universal usage for data storage on hard disks and tape drives and ii) the enumeration and characterization of error sequences that occur with modulation systems on fading and bursty channels. The analysis of these systems critically depends on the solution to these enumeration and characterization problems.

In this paper we apply powerful enumeration techniques to enumerate and characterize sequences associated with the problems mentioned. While the techniques are standard in current combinatorial usage, their application to the problems mentioned is novel and provides solutions that are superior and more elegant than those previously applied, often obtaining simpler and closed form solutions to problems that previously had to rely on recursions and simulations.

The capacity of constrained (e.g. RLL) codes is an important parameter in their application to computer memory devices. The capacity of such a code $\mathcal{S}$, denoted as C, was defined by Shannon [1] as:

$$C = \lim_{n\to\infty} \frac{\log_2 c_n}{n}, \tag{1}$$

where c_n is the number of sequences in $\mathcal{S}$ of length n satisfying the constraint. Shannon proposed a technique for determining c_n based on a finite state diagram (FSD) representation of the code $\mathcal{S}$. The standard method of evaluating the capacity is to take $\log_2 \lambda$, where λ is the largest real eigenvalue of the adjacent matrix. In the first part of this paper a standard enumeration technique is given to find a simple formula for a formal power series, the generating series, for some classes of constrained codes, such that the number of binary sequences in $\mathcal{S}$, c_n, are the coefficients of appropriate power of indeterminates.

In the second part of this paper we develop mathematical tools to analyze a broad class of communication channel models known as finite state channels (FSC). These channel models are used effectively for a variety of bursty and fading channels [2]- [5].

We follow the theory of enumeration of constrained sequences described in [6] to enumerate a particular subset of error sequences generated by the FSC model. We will show that the probability of this subset can be obtained by acting on the generating series with a linear mapping extended as a homomorphism to the whole of the ring of all formal power series.

We adopt throughout this work the following notation. Given a matrix $\mathbf{A}$, the superscripts $\mathbf{A}^T$ and $\mathbf{A}^{-1}$ represent respectively the transpose, and the inverse of a matrix. The matrices $\boldsymbol{I}$ and $\mathbf{1}$ stand for the identity matrix and a column vector of ones, whose dimensionality is clear from the context. $\mathbf{E}\{E_k\}$ stands for the expected value of the random variable E_k.

A generic binary sequence will be denoted by $\boldsymbol{\sigma} = \sigma_1\sigma_2\cdots$. If s and z are commutative indeterminates, $[s^k\, z^n]\, T(s,z)$ denotes the coefficient of $s^k\, z^n$ in the formal power series $T(s,z)$. If $\mathcal{A}$ is a set of sequences, $\mathcal{A}^\star$ is the set of all sequences formed by concatenating any number of sequences in $\mathcal{A}$, that is, $\mathcal{A}^\star = \phi \cup \mathcal{A} \cup \mathcal{A}^2 \cup \mathcal{A}^3 \ldots$, where ϕ is the empty sequence. 1^r denotes a sequence of r consecutive ones, for example, 1^3=111.

II. Sequence Enumeration and Channel Capacity

We define $\boldsymbol{\sigma}$ to be an RLL(r,s) sequence if all runs of zeros and ones of $\boldsymbol{\sigma}$ have length at least r and at most s. An equivalent formulation is to consider the class binary sequences for which between any two ones there are at least d zeros and at most k zeros, the so-called (d,k) sequences. These types of constraints are a result of the timing mechanisms and readback heads of many systems. For example, the timing of the sampling on the readback is often derived from the sequence transitions of the read sequence. If there is too long a run of zeros being read the timing circuit will not be updated sufficiently often, leading to possible errors in the read process. Similarly the the minimum constraint is used to mitigate the effects of intersymbol interference. For example, $\boldsymbol{\sigma} = (0001100111100)$ is an RLL(2,4) sequence of length 13. The required generating series for a generic set $\mathcal{S}$ with respect to the weight function length of $\boldsymbol{\sigma}$ is:

$$F_{\mathcal{S}}(x) = \sum_{n=0}^{\infty} c_n x^n.$$

For the remainder of this section we consider examples of the calculation of $F_{\mathcal{S}}(x)$ for some classes of constrained codes. The key point to solve problems of this type is to find a bijection that expresses the set $\mathcal{S}$ in terms of concatenation products of binary strings.

One is usually interested in the construction, and number, of sequences of a finite length n, that meet a certain constraint, and which can be concatenated together so that arbitrary catenations also satisfy the constraint. Since capacity is an asymptotic quantity, this version is not considered here.

The generating series for RLL(r,s) codes is obtained directly from the following decomposition of the set $\{0,1\}^\star$:

$$\{0,1\}^\star = 1^\star\{00^\star 11^\star\}^\star 0^\star,$$

since any binary sequence begins with a sequence (possibly empty) of 1's, then alternating block's of 0's and 1's, and ends with a sequence (possibly empty) of 0's. Here, the superscript $\star$ represents an arbitrary and repetitive choice from the set or symbol containing it. It is now clear that the set of all RLL(r,s) sequences, denoted as $\mathcal{S}_{r,s}$ can be expressed as:

$$\begin{aligned}\mathcal{S}_{r,s} = \{\phi, 1^r, 1^{r+1}, \cdots, 1^s\}\{0^r, 0^{r+1}\cdots, 0^s, 1^r,\\ 1^{r+1}, \cdots, 1^s\}^\star\{\phi, 0^r, 0^{r+1}, \cdots, 0^s\}.\end{aligned} \tag{2}$$

If the indeterminate x marks the length of the sequence, note that the generating series for the sets $\{\phi, 1^r, 1^{r+1}, \cdots, 1^s\}$ or $\{\phi, 0^r, 0^{r+1}, \cdots, 0^s\}$ is $1 + x^r + x^{r+1} + \cdots + x^s$. Therefore, $F_{\mathcal{S}_{r,s}}(x)$ is given by:

$$\begin{aligned} F_{\mathcal{S}_{r,s}}(x) &= (1 + x^r + x^{r+1} + \cdots + x^s)^2 \\ &\quad \times \left(1 - (x^r + x^{r+1} + \cdots + x^s)^2\right)^{-1} \\ &= \frac{(1+\frac{x^r - x^{s+1}}{1-x})^2}{1-(\frac{x^r - x^{s+1}}{1-x})^2} \\ &= \frac{1-x+x^r-x^{s+1}}{1-x-x^r+x^{s+1}}, \end{aligned} \tag{3}$$

where we have used the fact that the generating series for the set $\mathcal{S}^\star$ is $(1 - F_{\mathcal{S}}(x))^{-1}$. The series expansion of Equation (3) yields the coefficients c_n, as we can see in the example below, for $r = 3$ and $s = 7$:

$$F_{\mathcal{S}_{3,7}}(x) = 1 + 2x^3 + 2x^4 + 2x^5 + 4x^6 + 6x^7 + \cdots$$

We can also readily seen from Equation (3) that the coefficients c_n for the RLL(r,s) code satisfy the following recurrence equation:

$$c_n - c_{n-1} - c_{n-r} + c_{n-s-1} = 0 \tag{4}$$

for $n > s + 1$, with initial conditions:

$$\begin{aligned} &c_0 = 1 \\ &c_i = 0, && \text{if } 1 \le i \le r-1 \\ &c_r = 2 + c_{r-1} \\ &c_i = c_{i-1} + c_{i-r}, && \text{if } r+1 \le i \le s \\ &c_{s+1} = -2 + c_s + c_{s+1-r}. \end{aligned}$$

We now consider the generating series for the set $\mathcal{S}_{r,\infty}$, that is, sequences containing runs of zeros and ones of length at least r. When s goes to infinity the term $(1+x^r+x^{r+1}+\cdots)$ has a closed form given by $1 + x^r/(1 - x)$. Therefore

$$F_{\mathcal{S}_{r,\infty}}(x) = \frac{(1 + \frac{x^r}{1-x})^2}{1 - (\frac{x^r}{1-x})^2} = \frac{1 - x + x^r}{1 - x - x^r}. \tag{5}$$

The asymptotic behavior of c_n is the main subject of the next subsection. More example of the calculation of generating series will also be given.

A. Capacity Calculation

For the class of codes we are considering, the generating series $F_{\mathcal{S}}(x)$ can be expressed as the ratio of two polynomials $F_{\mathcal{S}}(x) = N(x)/D(x)$. On considering the partial fraction expansion of $F_{\mathcal{S}}(x)$, we would expect c_n to be expressed as the following polynomial equation:

$$c_n = A_1(n)\alpha_1^n + A_2(n)\alpha_2^n + \ldots + A_T(n)\alpha_T^n,$$

where T is the degree of $D(x)$, $A_1(n), \cdots, A_T(n)$ are polynomials in n, and the α_i's are the roots of the reciprocal polynomial $D^\star(x) = x^T D(1/x)$. The asymptotic behavior of the sequence $c_n^{\frac{1}{n}}$ is known in the literature [7, pp. 114]:

$$\lim_{n\to\infty} c_n^{\frac{1}{n}} = \theta, \tag{6}$$

where θ is the largest real root of the reciprocal polynomial $D^\star(x)$. It now follows from Equations (1) and (6) that the capacity can be written as:

$$C = \log_2 \theta.$$

Therefore, the capacity is given by the base two logarithm of the largest real root of the reciprocal polynomial $D^\star(x)$.

From Equation (3) we have that $D^\star(x)$ for the RLL(r,s) code is $x^{s+1} - x^s - x^{s-r+1} + 1$. The capacity of this code converges for large s to $\log_2(1.618) = 0.6942$, since 1.618 is the largest real root of the polynomial $D^\star(x) = x^2 - x - 1$ given by Equation (5).

In some optical recording channel applications the minimum and maximum run-length of 0's and 1's are required to be different [8]. This gives rise to a new constrained sequence called asymmetrical runlength-limited (ARLL) code [8] with parameters (r, s)-(e, l), where (r, s) represent the minimum and maximum runlength of 1's, and (e, l) are the analogous parameters of 0's. By modifying Equations (2) and (3) accordingly, we conclude that the generating series for ARLL(r, s)-(e, l) codes is:

$$F_{\mathcal{S}_{(r,s,e,l)}}(x) = \frac{(1 - x + x^r - x^{s+1})(1 - x + x^e - x^{l+1})}{1 - 2x + x^2 - x^{s+l+2} + x^{s+e+1} + x^{r+l+1} - x^{r+e}}.$$

The capacity follows readily from the denominator polynomial of the equation above. Moreover, the reciprocal denominator polynomial for the ARLL(r, ∞)-(e, ∞) code is $D^\star(x) = x^{e+r} - 2x^{e+r-1} + x^{e+r-2} - 1$.

We refer to $\boldsymbol{\sigma}$ to be an RLL$_o(d, k)$ code if between consecutive ones there are at least d and at most k zeros. The

IBM $\mathrm{RLL}_o(2,7)$ code [9] constitutes an example of practical use of these codes. The $\mathrm{RLL}_o(d,k)$ code is asymptotically equivalent to the set of sequences that ends with a 1, where each run of zeroes satisfies an $\mathrm{RLL}_o(d,k)$ constraint. This set is expressed as $\mathcal{S}_{d,k} = \{(0^d, \cdots, 0^k)1\}^\star$. So

$$F_{\mathcal{S}_{d,k}}(x) = \frac{1-x}{1-x-x^{d+1}+x^{k+2}}. \tag{7}$$

From Equations (3) and (7) we can check the known result that the capacity of $\mathrm{RLL}_o(d,k)$ and $\mathrm{RLL}(d+1,k+1)$ codes is the same [10].

Recently, practical applications of M-ary modulation schemes can be achieved on some optical and magnetic media [11]. This offers the possibility of achieving large information density on the channel. An M-ary $\mathrm{RLL}_o(d,k)$ code, denoted as $\mathrm{RLL}_o(M,d,k)$, is one where at least d and at most k zeroes occur between non-zero symbols. Similarly to the development above, the set of all sequences is $\mathcal{S}_{M,d,k} = \{(0^d, \cdots, 0^k)1, 2, \cdots (M-1)\}^\star$. Consequently, $F_{\mathcal{S}_{M,d,k}}(x)$ is written as:

$$\begin{aligned} F_{\mathcal{S}_{M,d,k}}(x) &= (1-(x^d+\cdots+x^k)(M-1)x)^{-1}; \\ &= \tfrac{1-x}{1-x+(M-1)x^{2+k}-(M-1)x^{1+d}}. \end{aligned} \tag{8}$$

The reciprocal denominator polynomial of the generating series $F_{\mathcal{S}_{M,d,k}}(x)$ is:

$$D^\star(x) = x^{k+2} - x^{k+1} - (M-1)x^{k-d+1} + M - 1.$$

The main concern of the next section is to address the the problem of calculating the probability of subsets of error sequences, where each sequence is generated by an FSC model. The probability of error events then follows by acting on the generating series with a mapping, extended as a homomorphism to the whole of the ring of all formal power series. This approach allows us to obtain analytic expressions for several burst error statistics of interest.

III. Deriving Burst Statistics for Finite State Channels

An FSC model is characterized by an underlying non-observable Markov chain [12]. The discrete output symbol to the channel at the k^{th} time interval, y_k, is a function of the input symbol x_k and the state of the Markov chain s_k. The channel is described statistically by the conditional probability $P(y_k, s_k \mid x_k, s_{k-1})$. For example, the Gilbert-Elliott channel [13] is a two-state Markov chain composed of a good state, state 0, where errors occur with small probability, and a bad state, state 1, where errors occur with higher probability. When the chain is in the good state the error bit e_k is zero (correct) with probability $1-g$, or one (error) with probability g. Otherwise, when it is in a bad state, the error bit is zero with probability $1-b$, or one with probability b. The matrices $\mathbf{P}$ and $\mathbf{\Pi}$ for the Gilbert-Elliott channel are:

$$\mathbf{P} = \begin{bmatrix} 1-Q & Q \\ q & 1-q \end{bmatrix} \tag{9}$$

$$\mathbf{\Pi} = [\pi_0\ \pi_1]^T = \left[\frac{q}{q+Q}\ \ \frac{Q}{q+Q}\right]^T. \tag{10}$$

Define two $N \times N$ matrices, $\mathbf{P}(0)$ and $\mathbf{P}(1)$, where the $(i,j)^{th}$ entry of the matrix $\mathbf{P}(e_k)$, $e_k \in \{0,1\}$, is $P(E_k = e_k, S_k = s_k \mid S_{k-1} = s_{k-1})$, which is the probability that the output symbol is e_k when the chain makes a transition from state s_{k-1} to s_k. The probability of an error sequence of length n, $\mathbf{e}_n \stackrel{\triangle}{=} e_1 e_2 \ldots e_n$, may be expressed in a matrix form as:

$$P(\boldsymbol{e}_n) = \mathbf{\Pi}^T \left(\prod_{k=1}^{n} \mathbf{P}(e_k) \right) \mathbf{1}.$$

The matrices $\mathbf{P}(0)$, $\mathbf{P}(1)$ for the Gilbert-Elliott channel are given by:

$$\mathbf{P}(0) = \begin{bmatrix} (1-Q)(1-g) & Q(1-b) \\ q(1-g) & (1-q)(1-b) \end{bmatrix} \tag{11}$$

$$\mathbf{P}(1) = \begin{bmatrix} (1-Q)\,g & Q\,b \\ q\,g & (1-q)\,b \end{bmatrix}. \tag{12}$$

It is well known that the probability of an error sequence is not preserved under commutation of its symbols. This prompts us to define the generating series in non-commuting indeterminates in order to keep all the information about the original sequence. So denote the generating series for $\mathcal{E}_n$ (error event) as:

$$F_{\mathcal{E}_n} = \sum_{\boldsymbol{e}_n \in \mathcal{E}_n} x_{e_1} x_{e_2} \ldots x_{e_n}, \qquad x_{e_i} \in \{x_0, x_1\} \tag{13}$$

which is in $R \ll x_0, x_1 \gg$, the ring of all power series in the non-commuting indeterminates x_0, x_1 with coefficients taken from the field of real numbers R. The indeterminates x_0 and x_1 mark an error bit equal to 0 or 1, respectively. Notice that $P(\mathcal{E}_n)$ may be obtained from the generating series $F_{\mathcal{E}_n}$ simply by replacing x_{e_i} by $\mathbf{P}(e_i)$ and wrapping the vector $\mathbf{\Pi}^T$ around the front and $\mathbf{1}$ around the back. We can formalize this concept by defining the mapping:

$$\Delta : R \ll x_0, x_1 \gg \longrightarrow \boldsymbol{M}_N(R) : x_k \mapsto \mathbf{P}(k),$$

acting as a homomorphism to the whole of the ring. $\boldsymbol{M}_N(R)$ is the ring of all $N \times N$ matrices with entries taken from R (the field of real numbers). The probability of the set $\mathcal{E}_n$ may be expressed very compactly as:

$$P(\mathcal{E}_n) = \mathbf{\Pi}^T (\Delta\, F_{\mathcal{E}_n})\, \mathbf{1}. \tag{14}$$

The main step to find $P(\mathcal{E}_n)$ is to determine the generating series $F_{\mathcal{E}_n}$. The key point to solve problems of this type is to find a bijection that expresses the set $\mathcal{E}_n$ in terms of concatenation products of binary strings.

In this section we will derive expressions for two important statistics:

- The error weight probability, $P(m,n)$, the probability of exactly m errors occurring in a block of n bits. This measure is important to determining the performance of block codes and interleaving on FSC models.
- The multigap distribution, $M(r,l)$, is the probability of r consecutive gaps in a sequence of length l. The multigap distribution has been used as a test of non-renewalness of the error process.

In all cases, we first obtain an expression for an N-state FSC models in terms of the matrices $\mathbf{\Pi}$, $\mathbf{P}(0)$, $\mathbf{P}(1)$, and $\mathbf{P} = \mathbf{P}(0) + \mathbf{P}(1)$.

A. The Error Weight Distribution $P(m,n)$

We wish to determine the probability of the set $\mathcal{E}_n$ composed of sequences of Hamming weight m and length n. The generating series for $\mathcal{E}_n$ is obtained directly from the generating series for the set of all binary sequences, $\{0,1\}^\star$, by defining the indeterminate z to mark the length of the sequence and s to mark the number of 1's. Since $F_{\{0,1\}^\star} = (1 - x_0 - x_1)^{-1}$, it follows that $F_{\mathcal{E}_n}$ is:

$$F_{\mathcal{E}_n} = [s^m\, z^n]\,(1 - x_0 z - x_1 s z)^{-1} \quad \in R \ll x_0, x_1 \gg . \tag{15}$$

From Equations (14) and (15) the error weight distribution $P(m,n)$ is given by:

$$\begin{aligned} \mathbf{P}(m,n) &= [s^m\, z^n]\ \mathbf{\Pi}^T(\boldsymbol{I} - \mathbf{P}(0)z - \mathbf{P}(1)sz)^{-1}\mathbf{1}; \\ &= [s^m\, z^n]\, H_P(s,z), \end{aligned} \tag{16}$$

where

$$\begin{aligned} H_P(s,z) &= \sum_{m,n} P(m,n)\, s^m\, z^n \qquad \in\ R[s][[z]]; \\ &= \mathbf{\Pi}^T(\, \boldsymbol{I} - \mathbf{P}(0)z - \mathbf{P}(1)sz)^{-1}\ \mathbf{1}\,. \end{aligned} \tag{17}$$

The generating series $H_P(s,z)$ is a polynomial in s. Then, $H_P(s,z)$ is a formal power series in z with a coefficient ring $R[s]$. An expression for $H_P(s,z)$ for the Gilbert-Elliott channel can be obtained upon substitution of Equations (11)-(10) into (17). An explicit formula for $P(m,n)$ can be found by carrying out the partial fraction technique to extract the coefficient of Equation (16). Alternatively, it is simple go from generating series to recurrence formulas, which provides a rapid computational scheme for the problem. From Equation (16) we can prove that $P(m,n)$ for the Gilbert-Elliott channel satisfies a 6-term recurrence formula:

$$\begin{aligned} P(m,n) \ = \\ & -[Q(1-g) + q(1-b) - (2-g-b)]\ P(m,n-1) \\ & +[b(1-q) + g(1-Q)]\ P(m-1,n-1) \\ & -[(1-b)(1-g)(1-q-Q)]\ P(m,n-2) \\ & -[(1-q-Q)(b+g-2gb)]\ P(m-1,n-2) \\ & -[(1-q-Q)gb]\ P(m-2,n-2) \end{aligned} \tag{18}$$

with initial conditions

$$\begin{aligned} P(m,n) &= 0, \text{ for } m,n < 0,\ \ m > n; \\ P(0,0) &= 1; \\ P(0,1) &= \tfrac{q}{q+Q}(1-g) + \tfrac{Q}{q+Q}(1-b); \\ P(1,1) &= \tfrac{q}{q+Q}g + \tfrac{Q}{q+Q}b. \end{aligned} \tag{19}$$

Let the random variable E^n be the number of errors in a block of length n. It is obvious that $P(E^n = m) = P(m,n)$. Moments of the random variable E^n of any order, $\mathbf{E}\{(E^n)^k\}$, can be obtained from the k^{th} derivative of the matrix $(\,\boldsymbol{I} - \mathbf{P}(0)z - \mathbf{P}(1)sz)^{-1}$, since:

$$\begin{aligned} &\mathbf{E}\{E^n\,(E^n-1)\,\ldots\,(E^n-k+1)\} = \\ &= [z^n]\ \left\{\tfrac{\partial^k}{\partial s^k}\, H_P(s,z)\right\}_{s=1}; \\ &= [z^n]\ \mathbf{\Pi}^T \left\{\tfrac{\partial^k}{\partial s^k}\,(\boldsymbol{I} - \mathbf{P}(0)z - \mathbf{P}(1)sz)^{-1}\right\}_{s=1} \mathbf{1}. \end{aligned}$$

An exact formula for $\frac{\partial^k}{\partial s^k}(\,\boldsymbol{I} - \mathbf{P}(0)z - \mathbf{P}(1)sz)^{-1}$ will be stated without proof in the next lemma.

Lemma III.1. The k^{th} partial derivative of the matrix $\boldsymbol{A}(s,z) \triangleq (\,\boldsymbol{I} - \mathbf{P}(0)z - \mathbf{P}(1)sz)^{-1}$ is:

$$\tfrac{\partial^k}{\partial s^k}\ \boldsymbol{A}(s,z) \ = \ k!\,(\ \boldsymbol{A}(s,z)P(1)z)^k\ \ \boldsymbol{A}(s,z).$$

Using the result of the lemma, we are able to prove the following result:

$$\begin{aligned} &\mathbf{E}\{E^n\,(E^n-1)\,\ldots\,(E^n-k+1)\} = \\ &= k!\,[z^n]\ \ \mathbf{\Pi}^T((\ \boldsymbol{I} - \mathbf{P}z)^{-1}\mathbf{P}(1)z)^k\,(\ \boldsymbol{I} - \mathbf{P}z)^{-1}\,\mathbf{1}; \\ &= k!\,[z^n]\ \ \mathbf{\Pi}^T((\ \boldsymbol{I} - \mathbf{P}z)^{-1}\mathbf{P}(1)z)^k\,\tfrac{\mathbf{1}}{1-z}; \\ &= k!\,[z^{n-k}]\ \ \mathbf{\Pi}^T\,((\ \boldsymbol{I} - \mathbf{P}z)^{-1}\mathbf{P}(1))^k\,\tfrac{\mathbf{1}}{1-z}; \\ &= k!\sum_{j=0}^{n-k}[z^j]\ \ \mathbf{\Pi}^T\,((\ \boldsymbol{I} - \mathbf{P}z)^{-1}\mathbf{P}(1))^k\ \mathbf{1}. \end{aligned} \tag{20}$$

It is easy to see from Equation (20) that $\mathbf{E}\{E^n\} = n\,P(1)$ where $P(1)$ is the probability the error bit is a 1. Moreover, the variance of E^n, for a general FSC model can be found directly from (20) by setting $n = 2$. The final expression is:

$$\begin{aligned} Var(E^n) &= \mathbf{E}\{(E^n)^2\} - (n\,p(1))^2; \\ &= 2\sum_{j=0}^{n-2}(n-j-1)\ \ \mathbf{\Pi}^T\mathbf{P}(1)\mathbf{P}^j\mathbf{P}(1)\ \mathbf{1} \\ &\quad + n\,P(1)\,(1 - nP(1)). \end{aligned} \tag{21}$$

For the Gilbert-Elliott channel (GEC), the matrix $\mathbf{P}^j$ may be expressed as:

$$\mathbf{P}^j \ = \left[\begin{array}{cc} 1 & -\frac{Q}{q} \\ 1 & 1 \end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ 0 & (1-q-Q)^j \end{array}\right]\left[\begin{array}{cc} \frac{q}{q+Q} & \frac{Q}{q+Q} \\ -\frac{q}{q+Q} & \frac{q}{q+Q} \end{array}\right]. \tag{22}$$

Substitution of Equation (22) into (21) yields the following expression for the variance of E^n for the Gilbert-Elliott channel:

$$\begin{aligned} &Var_{GEC}(E^n) = \\ &2\tfrac{(b-g)^2\,Qq(1-q-Q)}{(q+Q)^4}((1-q-Q)^n + n(q+Q) - 1) \\ &+ n\,P(1)(1 - P(1)). \end{aligned} \tag{23}$$

Notice that the term $n\,P(1)(1-P(1))$ in Equation (23) is the variance of E^n for the memoryless BSC channel with crossover probability $P(1)$. Equation (23) shows that the asymptotic behavior of $Var_{GEC}(E^n)$ grows linearly with n, for $0 < (1-q-Q) < 1$.

B. The Multigap Distribution $M(r,l)$.

The length of a gap is the number of zeros between two errors plus one (the last error is included). The error process $\{E_k\}_{k=1}^{\infty}$ can be regarded as a sequence of gaps $\{G_k\}_{k=1}^{\infty}$, where G_k is the length of the k^{th} gap. The gap process is a convenient representation for the error sequence, since a large number of consecutive 0's is expected to occur on channels with low bit error probability. Let the random variable $G^r = \sum_{i=k}^{k+r-1} G_k$ be the sum of r consecutive gap lengths G_k. The multigap length distribution, denoted as $M(r,l)$, is defined as $M(r,l) = P(G^r = l)$. If the error process were renewal, this means that $\{G_k\}_{k=1}^{\infty}$ are independent random variables, then the variance of G^r is $Var(G^r) = r\, Var(G_1)$.

The problem of finding $M(r,l)$ may be formulated as follows: Find the probability of the set $\mathcal{E}_l$, composed of binary sequences of length l such that the r^{th} error will occur at the l^{th} time interval. Note that the set of all sequences that ends with a 1 may be expressed as $\{0^\star 1\}^\star$. Let the indeterminate z mark the length of the sequence and let s mark the occurrence of a 1. The generating series for the set $\mathcal{E}_l$ may be obtained from $\{0^\star 1\}^\star$ by replacing:

$$\begin{array}{lll} 0^\star & \text{by} & 1 + x_0 z + x_0^2 z^2 + \ldots = (1 - x_0 z)^{-1}; \\ 1 & \text{by} & x_1 s z. \end{array}$$

It follows that $F_{\mathcal{E}_l}$ is:

$$F_{\mathcal{E}_l} \;=\; [s^r\, z^l]\, (1 - (1 - x_0 z)^{-1} x_1 s z)^{-1} \;\in\; R \ll x_0, x_1 \gg .$$

The multigap distribution $M(r,l)$ is the probability of the set $\mathcal{E}_l$, conditioned on $E_0 = 1$. Then

$$M(r,l) \;=\; [s^r\, z^l]\, H_M(s,z), \tag{24}$$

where the generating series $H_M(s,z)$ is:

$$\begin{aligned} H_M(s,z) \;&= \sum_{r,l} M(r,l)\, s^r\, z^l \qquad\qquad \in\; R[s][[z]]; \\ &= \frac{1}{P(1)}\, \mathbf{\Pi}^T \mathbf{P}(1)(\boldsymbol{I} - (\boldsymbol{I} - \mathbf{P}(0) z)^{-1} \mathbf{P}(1) s z)^{-1} \mathbf{1}. \end{aligned} \tag{25}$$

It is interesting to notice that the computational effort to calculate $M(r,l)$ and $P(m,n)$ is very closely related, as will be stated in the following proposition.

References

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