von Mises Tapering: A Circular Data Windowing

H. M. de Oliveira and F. Chaves

***Abstract*— Continuous standard windowing is revisited and a new taper shape is introduced, which is based on the normal circular distribution by von Mises. Continuous-time windows are considered and their spectra obtained. A brief comparison with classical window families is performed in terms of their spectral properties. These windows can be used as an alternative in spectral analysis.**

***Keywords*— von Mises, tapering function, windows, circular distributions, apodization.**

I. Introduction

Due to the cyclic or quasi-periodic nature of various types of signals, the signal processing techniques developed for real variables in the real line may not be appropriate. For circular data [2], [9], it makes no sense to use the sample mean, usually adopted for the data line as a measure of centrality. Circular measurements occur in many areas [22], such as biology (or Chronobiology) [26], economy [8], [4], geography [5], medical (Circadian therapy [28], epidemiology [13]...), geology [47], [41], [34], meteorology [3], [6], [42], acoustic scatter [23] and particularly in signals with some cyclic structure (GPS navigation [31], characterization of oriented textures [7], [38], discrete-time signal processing and over finite fields). Even in political analysis [16]. Probability distributions can be successfully used as a support tool for several purposes: for example, the beta distribution was used in wavelet construction [11]. In this paper, the von Mises distribution (VM) is used in the design of tapers. Tools such as rose diagram [32], [21] allow rich graphical interpretation. For random signals, the focus is on circular distributions of probability. The uniform distribution of an angle ϕ, circular in the range $[0, 2\pi]$, is given by

$$f_{\Phi;1}(\phi) := \frac{1}{2\pi}\mathbb{I}_{[0,2\pi]}(\phi), \tag{1}$$

where $\mathbb{I}_A(.)$ is the indicator function of the interval $A \subset \mathbb{R}$. It is denoted by $\phi \sim U(0, 2\pi)$.

Another very relevant circular distribution is the normal circular distribution, introduced in 1918 by von Mises [46], defined in the interval $[0, 2\pi]$ and denoted by $\phi \sim VM(\phi_0, \beta)$,

$$f_{\Phi;2}(\phi) := \frac{1}{2\pi I_0(\beta)} e^{\beta \cos(\phi-\phi_0)}, \tag{2}$$

where $\beta \geq 0$ and $I_0(.)$ is the zero-order modified Bessel function of the first kind [1], [20] (not to be confused with the indicator function), i.e.

$$I_0(z) := \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta} d\theta = \sum_{n=0}^{+\infty} \frac{(z/2)^{2n}}{n!^2}. \tag{3}$$

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This probability density dominates in current analysis of

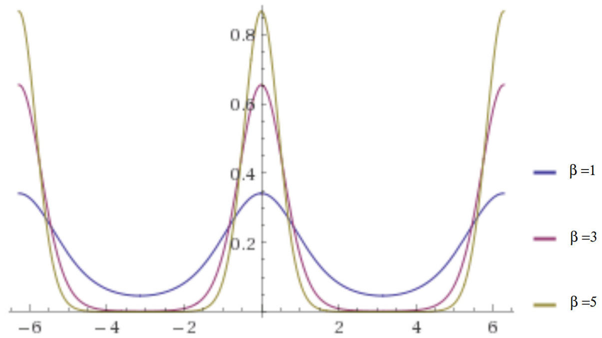


Fig. 1: Periodic extension of the von Mises distribution with zero-mean for several parameter values: β = 1,3,5. Note that the support of the density is confined to $[-\pi, \pi]$.

circular data because it is flexible with regard to the effect of parameters and easy to interpret. In a standard notation,

$$f(x|\mu,\kappa) := \frac{e^{\kappa\cos(x-\mu)}}{2\pi I_0(\kappa)}\mathbb{I}_{[-\pi,\pi]}. \tag{4}$$

Standardized distribution support is $[-\pi, \pi]$ and the mean, mode and median values are equal to μ. The parameter κ plays a role connected to the variance, namely $\sigma^2 \approx 1/\kappa$.

$$\mathbb{E}(X) = \mu \text{ and } \mathbb{V}ar(X) = 1 - \frac{I_1(\kappa)}{I_0(\kappa)}.$$

Two limiting behaviors can be observed:

- $$\lim_{\kappa\to 0} f(x|\mu,\kappa) = \frac{1}{2\pi}\text{rect}\left(\frac{x}{2\pi}\right), \tag{5}$$
  where $\text{rect}(x) := \begin{cases} 1, & \text{if } |x| \leq 1/2 \\ 0, & \text{otherwise.}\end{cases}$ is the normalized gate function, and therefore
  $$\lim_{\kappa\to 0} VM(\mu,\kappa) \sim U(-\pi,\pi). \tag{6}$$
- $$\lim_{\kappa\to +\infty} f(x|\mu,\kappa) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \tag{7}$$
  where $\sigma^2 := 1/\kappa$, and therefore
  $$\lim_{\kappa\to +\infty} VM(\mu,\kappa) \sim N(\mu, 1/\kappa). \tag{8}$$

Hence the reason why this distribution is also known as the *circular normal distribution*. The von Mises distribution (VM) is considered to be a circular distribution having two parameters and is the natural analogue on the circle of the Normal distribution on the real line. Maximum entropy distributions are outstanding probability distributions, because maximizing entropy minimizes the amount of prior information built into the distribution. Furthermore, many physical systems tend to move towards maximal entropy configurations over time. This encompasses distributions such as uniform, normal, exponential, beta...

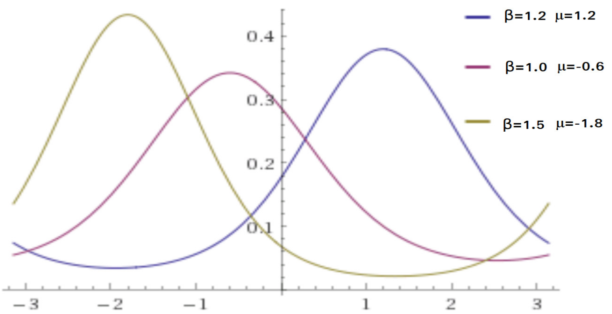


Fig. 2: Circular behavior of the von Mises distribution plotted for different mean values (1.2, -0.6 and -1.8). The cyclical feature of the distribution is explained, in this case, within $[-\pi, \pi]$.

The von Mises distribution is the maximum entropy distribution for circular data when the first circular moment is specified [22]. The corresponding cumulative distribution function (CDF) is expressed by

$$F_X(x|\mu,\kappa) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\frac{I_{|n|}(\kappa)}{I_0(\kappa)}\,(x-|n|)\,\mathrm{Sa}\left(n(x-\mu)\right), \tag{9}$$

where $\mathrm{Sa}(x) := \sin(x)/x$ is the sample function [29].
Through a simple transformation of random variable, the distribution support can be modified to an interval defined between two integers:

$$f_{X_1}(x) := \frac{e^{\beta\cos\left(\frac{2\pi}{N}x\right)}}{NI_0(\beta)},\ \text{circular in } 0 \leq x \leq N. \tag{10}$$

Another closely related continuous distribution (with a minimal - but relevant difference) is

$$f_{X_2}(x) := \frac{e^{\beta\cos\left(\frac{\pi}{N}x\right)}}{NI_0(\beta)},\ \text{circular in } 0 \leq x \leq N. \tag{11}$$

This distribution has a circular pattern as best illustrated in Figure 2. Decaying pulses for constraining the signal support play a key role in a large number of domains, including: tapers [12] [37], linear networks (filtering [19], inter-symbolic interference control [29], modulation), wavelets [10], time series, Fourier transform spectroscopy [35] ...

II. Tapering: Standard Windows

In this section we review some of the continuous windows (also known as a apodization function) used in signal processing (spectrum analysis [36], [12]), antenna array design [45], characterization of oriented textures [38], image warping ([19]). Although discrete windows are more common, some studies address continuous windows [44], [15], besides their application in short-time Fourier transforms. Among the most used windows, it is worth mentioning: rectangular, Bartlett, cosine-tip, Hamming, Hanning, Blackman, Lanczos, Kaiser, modified Kaiser, de la Vallé-Pousin, Poisson, Saramäki [43], *Dolph-Chebyshev...* (non-exhaustive list [40]). Tutorials on the subject are available [14], [40], [18], [17]. Figure 3 describes the approaches to the standard rectangular window in two cases: 1) continuous non-causal, 2) continuous causal.
It is worth revisiting the spectra of each of these windows.

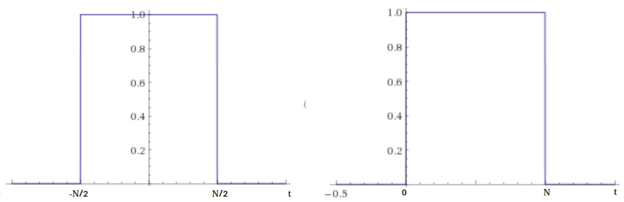


Fig. 3: rectangular windows with length N. a) continuous, b) continuous causal.

$$w_{REC;1}(t) = \mathrm{rect}\left(\frac{t}{N}\right), \tag{12a}$$

$$w_{REC;2}(t) = \mathrm{rect}\left(\frac{t-N/2}{N}\right). \tag{12b}$$

In the continuous case, $w_1(t)$ has spectrum given by:

$$W(w) := \mathscr{F}[w(t)] = \int_{-\infty}^{+\infty} w(t)e^{-jwt}dt. \tag{13}$$

Indeed $W_{REC;1}(w) = N\mathrm{Sa}\left(\frac{wN}{2}\right)$. Now the spectrum of $w_{REC;2}(t) = \mathrm{rect}\left(\frac{t-N/2}{N}\right)$ can be evaluated using the time-shift theorem [29], $w(t-t_0) \leftrightarrow W(w)e^{-jwt_0}$, resulting in $W_{REC;2}(w) = N\mathrm{Sa}\left(\frac{Nw}{2}\right)e^{-jNw/2}$.

Several of the windows of interest can be encompassed taking into account the following definition

$$w_{\alpha;1}(t) := \left\{\alpha + (1-\alpha)\cos\left(\frac{2\pi}{N}t\right)\right\}\mathrm{rect}\left(\frac{t}{N}\right), \tag{14}$$

The Hanning (Raised Cosine) window corresponds to $\alpha = 0.5$, whereas the standard Hamming window corresponds to $\alpha = 0.54$ [39]. In the case of a cosine-tip continuous window ($\alpha = 0$), the corresponding window and spectrum are [15]

$$w_{\alpha=0;1} := \cos\left(\frac{2\pi}{N}t\right)\mathrm{rect}\left(\frac{t}{N}\right), \tag{15}$$

and from the convolution theorem in the frequency [29])

$$W_{\alpha=0;1}(w) = \frac{N}{2}\mathrm{Sa}\left(\frac{Nw}{2}-\pi\right) + \frac{N}{2}\mathrm{Sa}\left(\frac{Nw}{2}+\pi\right). \tag{16}$$

The Kaiser window in continuous variable is defined by (non-causal window centered on the origin, and its corresponding causal version)

$$w_{KAI;1}(t) := \frac{I_0\left(\beta\sqrt{1-\left(\frac{t}{N/2}\right)^2}\right)}{I_0(\beta))}\mathrm{rect}\left(\frac{t}{N}\right), \tag{17a}$$

$$w_{KAI;2}(t) := \frac{I_0\left(\beta\sqrt{1-\left(\frac{t-N/2}{N/2}\right)^2}\right)}{I_0(\beta))}\mathrm{rect}\left(\frac{t-N/2}{N}\right). \tag{17b}$$

The corresponding spectrum is given by:

$$W_{KAI;1}(w) = \frac{N}{I_0(\beta)} \text{Sa}\left(\sqrt{(\frac{Nw}{2})^2 - \beta^2}\right). \quad (18)$$

III. Introducing a New Window: the Circular Normal Window

The proposal is to use a window (support length N) with shape related to the function (This type of windowing is superficially mentioned for spatial smoothing of spherical data [27])

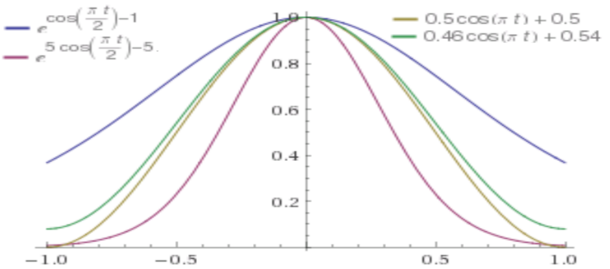
$$w(t) = K\frac{e^{\beta \cos\left(\frac{\pi}{N}t\right)}}{I_0(\beta)} \text{rect}\left(\frac{t}{N}\right). \quad (19)$$

The value of the constant K can be set so that, as in the other classic windows, $w(0) = 1$. Thus, for continuous case (noncausal and causal, respectively), one has

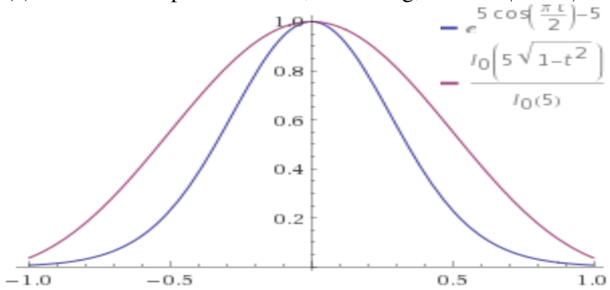
$$w_{CIR;1}(t) = \frac{e^{\beta \cos\left(\frac{\pi}{N}t\right)}}{e^{\beta}} \text{rect}\left(\frac{t}{N}\right), \quad (20a)$$

$$w_{CIR;2}(t) = \frac{e^{\beta \cos\left(\frac{\pi}{N}t\right)}}{e^{\beta}} \text{rect}\left(\frac{t - N/2}{N}\right). \quad (20b)$$

A straightforward comparison among different tapers in displayed in Figure 4.



(a) Windows shape: von Hann, Hamming, circular $\beta = 1,\ 5$.



(b) Windows shape: Kaiser vs normal circular windows, $\beta = 5$.

Fig. 4: Shape comparison of different normalized windows for support $[-1, 1]$.

IV. Spectrum calculation of the Normal Circular Window: the Continuous Case

In order to evaluate the spectrum of the continuous window introduced in the previous section, we use Eq. (13),

$$W_{CIR;1}(w) = \int_{-N/2}^{N/2} e^{\beta\left[\cos\left(\frac{\pi}{N}t\right)-1\right]} e^{-jwt} dt. \quad (21)$$

The interest function involved in defining the window is $\cos\left(\frac{\pi}{N}t\right)$, with period $2N$, sketched below in $[-N, N]$. The rectangular term included in the window is responsible for cutting the window, confining it in the range $[-N/2, N/2]$ as viewed in Figure 5. MacLaurin's series development of

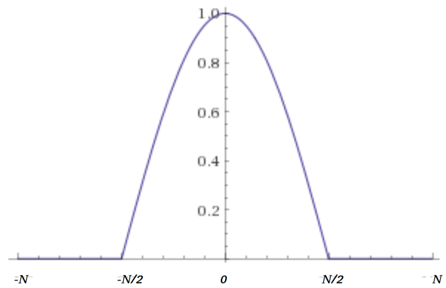


Fig. 5: Normalized cosine exponent of the exponential function in von Mises window: the (entire) cosine $\cos(\pi t/N)$ is periodic in $[-N, N]$, but the support is confined within $[-N/2, N/2]$ due to the rectangular pulse.

$e^{\beta\cdot\left[\cos\left(\frac{\pi}{N}t\right)\right]}$ gives the following Fourier series:

$$e^{\beta\left[\cos\left(\frac{\pi}{N}t\right)\right]} = \sum_{n=-\infty}^{+\infty} I_{|n|}(\beta) \cos\left(\frac{n\pi}{N}t\right). \quad (22)$$

Thus, one obtains:

$$W_{CIR;1}(w) = e^{-\beta} \sum_{n=-\infty}^{+\infty} I_{|n|}(\beta) \mathscr{F}\left(\cos\left(\frac{n\pi}{N}t\right) \text{rect}\left(\frac{t}{N}\right)\right). \quad (23)$$

From the convolution theorem in the frequency [29]), the spectrum sought (a bit detailed here) is

$$W_{CIR;1}(w) =$$

$$= \frac{1}{2\pi} e^{-\beta} \sum_{n=-\infty}^{+\infty} I_{|n|}(\beta). \mathscr{F}\left(\cos\left(\frac{n\pi}{N}t\right)\right) * \mathscr{F}\left(\text{rect}\left(\frac{t}{N}\right)\right). \quad (24)$$

By evaluating the internal terms in the summation, we obtain

$$\frac{N}{2\pi}\pi\left\{\delta\left(w - \frac{n\pi}{N}\right) + \delta\left(w + \frac{n\pi}{N}\right)\right\} * \text{Sa}\left(\frac{wN}{2}\right), \quad (25)$$

where $\delta(.)$ is the Dirac impulse [29], so that

$$W_{CIR;1}(w) = \frac{N}{2} e^{-\beta}.$$

$$\sum_{n=-\infty}^{\infty} I_{|n|}(\beta)\left\{\text{Sa}\frac{N}{2}\left(w - \frac{n\pi}{N}\right) + \text{Sa}\frac{N}{2}\left(w + \frac{n\pi}{N}\right)\right\}, \quad (26)$$

or

$$W_{CIR;1}(w) = Ne^{-\beta} \sum_{n=-\infty}^{\infty} I_{|n|}(\beta)\left\{\text{Sa}\left(\frac{Nw}{2} - \frac{n\pi}{2}\right)\right\}. \quad (27)$$

This expression is like a series of reconstitution (with coefficients c_n) of the type

$$\sum_{n=-\infty}^{+\infty} c_n \ \text{Sa}\left(\frac{Nw}{2} - \frac{n\pi}{2}\right).$$

Let us apply the Shannon-Nyquist-Koteln'kov sampling theorem in the frequency domain, for time-limited signals ([30], [24], `http://ict.open.ac.uk/classics`.
Since

$$F(w) = \frac{w_s t_m}{\pi} \sum_{n=-\infty}^{+\infty} F(nw_s) \mathrm{Sa}\left(wt_m - nt_m w_s\right). \qquad (28)$$

The rate w_s must comply with the restriction $w_s \leq \pi/t_m$, and the choice made is $w_s = \pi/2t_m$, so that the previous equation is

$$F(w) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} F(\frac{n\pi}{2t_m}) \mathrm{Sa}\left(wt_m - \frac{n\pi}{2}\right). \qquad (29)$$

Now let us choose the duration t_m to be $t_m := N/2$ (Figure 5), which leads to

$$F(w) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} F(\frac{n\pi}{N}) \mathrm{Sa}\left(\frac{Nw}{2} - \frac{n\pi}{2}\right). \qquad (30)$$

This is a variation of the cardinal Whittaker-Shannon series [33]. Observing the series described in Eq. (27), it is seen that the signal corresponds to a continuous signal defined by samples such that $F\left(\frac{n\pi}{N}\right) = 2I_{|n|}(\beta)$ and

$$F(w) = 2I_{\left|\frac{Nw}{\pi}\right|}(\beta), \qquad (31)$$

and the spectrum is given by

$$W_{CIR;1}(w) = \frac{2NI_{\left|\frac{Nw}{\pi}\right|}(\beta)}{e^{\beta}}. \qquad (32)$$

In the case of the causal window, $w_{CIR;1}(t)$, the application of the time-shift theorem provides the spectrum

$$W_{CIR;2}(w) = \frac{2NI_{\frac{N}{\pi}|w|}(\beta)}{e^{\beta}} e^{-jw\frac{N}{2}}. \qquad (33)$$

It is worth remembering that the ν argument of the $I_\nu(z)$ function is a real number in this case [1].

V. Conclusions

The closeness to the normal distribution and the fact that they are associated with a shape linked to the maximum entropy for circular data suggests interesting properties to be explored in later investigations. Windowing circular data with von Mises circular window can possibly improve spectral evaluation in these cases. Discrete data windows of this kind is currently under investigation.

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