Performance of Different Utility Functions in Pricing Algorithms for Interference Alignment

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***Resumo*— A ideia básica do alinhamento de interferência consiste em pré-codificar os sinais transmitidos de tal maneira que eles estejam alinhados nos receptores em que constituem interferência, e ao mesmo tempo desalinhados do sinal desejado. Dessa forma o sinal desejado e a interferência são facilmente separados em cada receptor. Este artigo fornece uma análise da abordagem baseada no *interference price* através do uso de duas diferentes funções utilidade no algoritmo de *Pricing distribuído* proposto na literatura. Resultados numéricos são apresentados, os quais exibem uma comparação das métricas *Sum Rate* e CDF para as funções utilidade mencionadas.**

***Palavras-Chave*— Alinhamento de Interferência, Algoritmos de *Pricing*, Funções Utilidade, Canais Interferentes MIMO.**

***Abstract*— The basic idea of interference alignment consists in precoding the transmitted signals such that they are aligned at the receiver where they constitute interference, while at the same time disjointed from the desired signal. Thus, the desired signal and the interference are easily separated at each receiver. This paper provides an analysis of an interference align technique based on the *pricing* approach through the use of two different utility functions in the distributed pricing algorithm proposed in the literature. Numerical results are presented, which depict a comparison of the Sum Rate and CDF metrics for both utility functions.**

***Keywords*— Interference Alignment, Pricing Algorithms, Utility Functions, MIMO Interfering Broadcast Channels.**

I. INTRODUCTION

It is well known that the capacity of the single user point-to-point Multiple-Input Multiple-Output (MIMO) channel with M transmit antennas and N receive antennas increases linearly with $\min(M, N)$ [1], [2] in the high Signal-to-Noise Ratio (SNR) regime. This linear growth, addressed as Degrees of Freedom (DoF) or *capacity pre-log factor*, commonly known in the single user case literature as *multiplexing gain*, is defined as [3]–[5]

$$\eta \triangleq \lim_{\mathrm{SNR}\to\infty} \frac{C(\mathrm{SNR})}{\log(\mathrm{SNR})}, \tag{1}$$

where C is the sum rate capacity.

Similarly, in the multi-user case it is useful to characterize the DoF of the network (related to the sum rate capacity of the network). To give it a simple intuition, it is worth to note that:

1) The degrees of freedom of a network may be interpreted as the number of resolvable (interference-free) signal space dimensions and its determination can be considered as a preliminary characterization of the capacity for a network;
2) It provides a good indicative of capacity behavior in the high SNR regime.

The interference alignment in signal vector space was initially introduced by Maddah-ali *et al.* (2006) [6], where iterative schemes were formulated for optimizing transmitters and receivers in conjunction with dirty paper coding and successive decoding schemes. Initially, research focused on the determination [7] of the achievable degrees of freedom for different scenarios [4], [5], [8]–[10] and finding techniques to obtain them, i.e., the maximization of the number of interference-free dimensions in the system was the main objective.

On the other hand, for low to moderate SNR values the complete alignment does not generally maximize the sum utility and there is interest in finding precoders that relax the perfect alignment constraint, where now the objective is maximizing the sum rate performance [7]. Specifically, each transmitter will face a trade-off between finding a precoder that minimizes the interference that its own receiver sees ("egoistic" or "help yourself" approach) and minimizing the interference that it causes at the non-intended receivers ("altruistic" or "do no harm" approach).

In [11]–[15], the idea of interference price was proposed where each transmitter's beams were treated separately and associated with an interference price, which corresponds to a metric of how much the utility of a user will decrease per marginal increase of the interference caused by the other users after the appropriate receive filter. The algorithm was introduced initially to Single-Input Single-Output (SISO) links by Huang *et al.* in [16]. Then, it was extended to Multi-Input Single-Output (MISO) and MIMO channels, respectively in [11] and [12].

The trade-off between the egoistic and altruistic objectives is clear in the pricing algorithm, where the objective includes a term corresponding to the egoistic objective and another one for the altruistic objective, the former weighted by the price. The trade-off is then controlled by the choice of the utility function and, consequently, of the pricing.

In this paper we compare the rate utility and the "α-fair" utility [16] functions in the distributed pricing algorithm

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through analysis of the sum rate for a set of SNR values in the MIMO interference channel scenario. The intention of this work is to analyze if the "α-fair" utility function can increase the sum rate and/or if it improves the fairness in the allocation of resources between the users by choosing an appropriate value of α.

In the next section, we present the system model. Section III illustrates how the interference alignment works, while Section IV shows the distributed pricing algorithm and our modification of the utility function. Simulation results are presented in Section V and conclusions are given in Section VI.

II. System Model

We consider a time-invariant interference wireless network consisting of K transmitter-receiver pairs, as shown in Fig. 1. Each pair has M transmit/receive antennas, where each transmitter sends useful information only to its own receiver, while causing interference at the other receivers. This is the so called "K-user MIMO Interference Channel" [4], [10], [17] and we refer to each transmitter-receiver pair as a *user*.

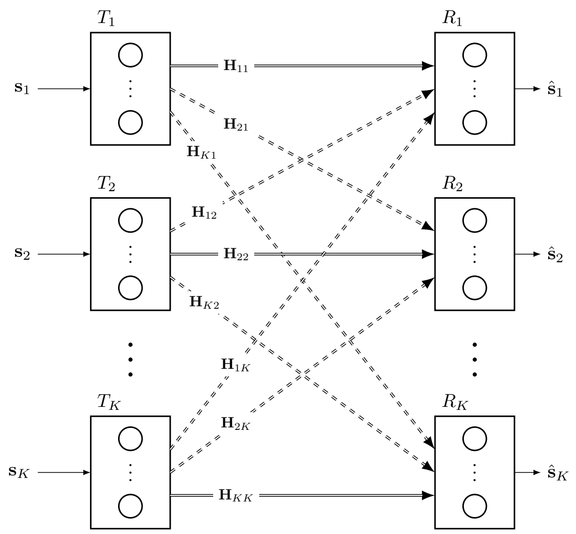

Fig. 1
MIMO Interference Channel with K users.

The k-th receiver decodes the signal from the k-th transmitter while considering the interference from all other transmitters as noise. The decoded symbol at receiver k in a given time interval is then

$$\hat{s}_k = \underbrace{\mathbf{g}_k^H \mathbf{H}_{kk} \mathbf{v}_k s_k}_{\text{desired signal}} + \underbrace{\sum_{\substack{j=1 \\ j \neq k}}^{K} \mathbf{g}_k^H \mathbf{H}_{kj} \mathbf{v}_j s_j}_{\text{interference}} + \underbrace{\mathbf{g}_k^H \mathbf{n}_k}_{\text{noise}}, \tag{2}$$

where $\mathbf{g}_k$ is the linear receiver filter of *user* k, $\mathbf{H}_{kk}$ is the direct channel matrix, $\mathbf{H}_{kj}$ is the cross-channel matrix between transmitter j and receiver k, s_k is the transmitted symbol from transmitter k, $\mathbf{v}_k$ is the precoding vector at transmitter k and $\mathbf{n}_k$ is a zero mean additive complex Gaussian noise vector with covariance matrix $\mathbb{E}\{\mathbf{n}_k \mathbf{n}_k^H\} = \sigma^2 \mathbf{I}$.

From (2), assuming the transmit symbol has unit variance for all users, we can see that the Signal-to-Interference-plus-Noise Ratio (SINR) at the k-th receiver is given by

$$\gamma_k = \frac{|\mathbf{g}_k^H \mathbf{H}_{kk} \mathbf{v}_k|^2}{\sum_{j \neq k} |\mathbf{g}_k^H \mathbf{H}_{kj} \mathbf{v}_j|^2 + ||\mathbf{g}_k||_2^2 \sigma^2} = \frac{S_k}{I_k + N_k}. \tag{3}$$

The overall system objective is a function of the SINR at each receiver and corresponds to maximizing the sum-utility across users. That is, the optimization function is given by

$$\max_{\substack{\mathbf{v}_1 \ldots \mathbf{v}_k \\ \mathbf{g}_1 \ldots \mathbf{g}_k}} \sum_{k=1}^{K} u_k(\gamma_k) \quad \text{s.t.:} \quad ||\mathbf{v}_k||_2^2 \leq P_k^{\max} \quad \forall k \in \{1, \ldots, K\}, \tag{4}$$

where $P_k^{\max}$ denotes the power constraint of user k.

The properties of this optimization problem depend on the utility functions employed, since due to interference it may have multiple locally optimal solutions [15]. However, for a wide class of utility functions the constraint set is convex yielding a unique local optimum, which is also the global optimum and can be solved using standard optimization techniques.

III. Interference Alignment

The principle of interference alignment consists in the basic idea of constructing transmit signals in such a way that the interference they cause at all unintended receivers overlaps onto the same subspace, while they still remain separable at the intended receivers [5].

Fig. 2 depicts the perfect alignment of the interference caused by transmitters i and j at the receiver k. Note that the

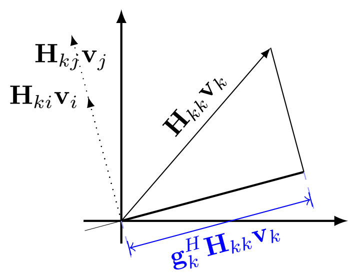

Fig. 2
Perfect alignment at receiver k of the interference caused by transmitters i and j.

precoders $\mathbf{v}_j$ and $\mathbf{v}_i$ align the cross-channels $\mathbf{H}_{kj}$ and $\mathbf{H}_{ki}$, respectively, in the same subspace, represented by the dotted line. On the other hand, the precoder $\mathbf{v}_k$ does not project the direct channel $\mathbf{H}_{kk}$ in any particular direction[1] and, due to the randomness of the channel, the equivalent channel $\mathbf{H}_{kk}\mathbf{v}_k$ is likely to span the whole space. At the receiver k the zero-force filter $\mathbf{g}_k$ will simply project the received signal (2) in the

[1]The precoder $\mathbf{v}_k$ is intended to only align the interference at receivers j and i.

space orthogonal to the interference space, thus eliminating all the interference. Mathematically, the interference alignment conditions are given by [18]:

$$\mathbf{g}_k^H \mathbf{H}_{kj} \mathbf{v}_j = 0, \, \forall j \neq k \tag{5}$$

$$\mathbf{g}_k^H \mathbf{H}_{kk} \mathbf{v}_k \neq 0, \tag{6}$$

resulting in the equivalent channel

$$\overline{H}_{kk} \triangleq \mathbf{g}_k^H \mathbf{H}_{kk} \mathbf{v}_k. \tag{7}$$

The DoF for the MIMO Interference Channel (MIMO-IC) with $K = 3$ users and M transmit/receive antennas is given by [10]

$$\eta_{\text{MIMO-IC}} = \frac{3M}{2}. \tag{8}$$

In pure interference alignment the goal is to completely align the interference in order to obtain the maximum DoF the system can provide. However, by doing so each transmitter is trying to improve the performance of the unintended receivers while neglecting its own. By relaxing the perfect alignment constraint one can obtain better results, for instance, in terms of the sum rate capacity.

IV. Distributed Pricing Algorithm

In the *distributed interference pricing* algorithm [11]–[15], the optimization problem in (4) is broken down into optimizing the precoders $\mathbf{v}_k$ and optimizing the receive filters $\mathbf{g}_k$. Therefore, the optimization of the precoders is formulated in a distributed approach as

$$\max_{\mathbf{v}_k} u_k(\gamma_k) - \sum_{j \neq k} \pi_j |\mathbf{g}_j^H \mathbf{H}_{jk} \mathbf{v}_k|^2 \quad \text{s.t.} \quad ||\mathbf{v}_k||_2^2 \leq P_k^{\max}, \tag{9}$$

where the utility function u_k is a function of the SNR γ_k and measures the quality of service of the user k. The second term in (9) corresponds to the cost for the transmitter k to cause interference at the unintended receivers and it depends on the values π_j ($j \neq k$) announced by them. The interference price for a user j is calculated by

$$\pi_j = -\frac{\partial u_j(\gamma_j)}{\partial I_j}. \tag{10}$$

The utility function u_k is assumed to be a monotonically increasing, concave and twice differentiable function of γ_k.

Traditionally the rate utility, $u_k = \log(1 + \gamma_k)$, is utilized because it corresponds to the user's maximum achievable rate assuming Gaussian codebooks [19]. It is also employed in the original interference pricing algorithm. Thus, the interference price π_k results in

$$\pi_{k_{\text{rate}}} = \frac{I_k + N_k}{I_k + N_k + S_k} \cdot \frac{S_k}{(I_k + N_k)^2}. \tag{11}$$

There are several utility functions which can be used to replace the rate utility and it is interesting to note that the prices π_j yields a balance between maximizing the utility function $u_k(\gamma_k)$ (egoistic objective) and minimizing the interference caused at the unintended receivers (altruistic objective). Furthermore, other aspects such as the fairness in the distributed resources can be analysed, but no comparison has been made so far between different utility functions (and therefore their corresponding interference prices) in the pricing algorithm regarding the obtained system performance. In this paper we analyse the impact of using the "α-fair" utility function shown below [16]

$$u(\gamma_k) = \frac{\gamma_k^\alpha}{\alpha}, \quad 0 < \alpha < 1, \tag{12}$$

for which the corresponding interference price can then be derived as

$$\pi_{k_{\alpha\text{-fair}}} = \left(\frac{S_k}{I_k + N_k} \right)^{\alpha - 1} \frac{S_k}{(I_k + N_k)^2}. \tag{13}$$

In [13] a numerical algorithm for solving the nonlinear optimization problem in (9) is presented, where the Karush-Kuhn-Tucker (KKT) conditions for user k are given by

$$\underbrace{\left[a_k(\mathbf{v}_k) \mathbf{H}_{kk}^H \mathbf{g}_k \mathbf{g}_k^H \mathbf{H}_{kk} - \sum_{i \neq k} \pi_i \mathbf{H}_{ik}^H \mathbf{g}_i \mathbf{g}_i^H \mathbf{H}_{ik} \right]}_{\mathbf{X}_k} \mathbf{v}_k = \lambda_k \mathbf{v}_k, \tag{14}$$

with

$$a_k(\mathbf{v}_k) = \frac{u_k^{'}(\gamma_k)}{|\mathbf{g}_k^H \mathbf{n}_k|^2 + \sum_{i \neq k} |\mathbf{g}_k^H \mathbf{H}_{ki} \mathbf{v}_i|}, \tag{15}$$

where λ_k is the Lagrange multiplier associated with the power constraint, and $u_k^{'}$ is the derivative of the utility function $u_k(\gamma_k)$ with respect to γ_k.

Equation (14) has the form of an eigenvector equation. If all eigenvalues of $\mathbf{X}_k$ are negative, then the updated precoder $\mathbf{v}_k$ is the zero vector, otherwise, it is the eigenvector associated with the largest eigenvalue of $\mathbf{X}_k$ with an appropriate scale factor.

Considering the MMSE criterion, the optimal receive filter for user k is given by [13]

$$\mathbf{g}_{\text{MMSE},k} = \left(\sum_{j=1}^{K} \mathbf{H}_{kj} \mathbf{v}_j \mathbf{v}_j^H \mathbf{H}_{kj}^H + \sigma^2 \mathbf{I} \right)^{-1} \mathbf{H}_{kk} \mathbf{v}_k. \tag{16}$$

The iterative algorithm is summarized by Algorithm 1. In the next section, we show the simulation results obtained.

Algorithm 1 Distributed Interference Pricing

1) Initialize randomly a beamforming vector $\mathbf{v}_k$ for each user k respecting the power constraint;
2) Then, optimize the receive filter $\mathbf{g}_k$ according to (16);
3) Each receiver k calculates the interference price π_k and announces it to other users;
4) Next, one random user is chosen to solve (9) and optimize his beamforming vector according to (14);
5) The remaining users update their receive filters using (16);
6) Repeat from step 2 until convergence (proved in [19])

V. Simulation Results

In this section, we show the simulation results for both utility functions. We consider a three-user system with two transmit and two receive antennas for each user. The direct channel and cross channel matrices have i.i.d complex Gaussian entries with unit variance, that is, the direct and interference channel matrices have elements with Rayleigh fading.

In Fig. 3, we show the sum rate performance versus SNR, averaged over 500 channel realizations, for different values of α. We observe the impact of α in the trade-off between the egoistic (first term in (9)) and altruistic objectives (second term in (9)) yielding different sum rate values.

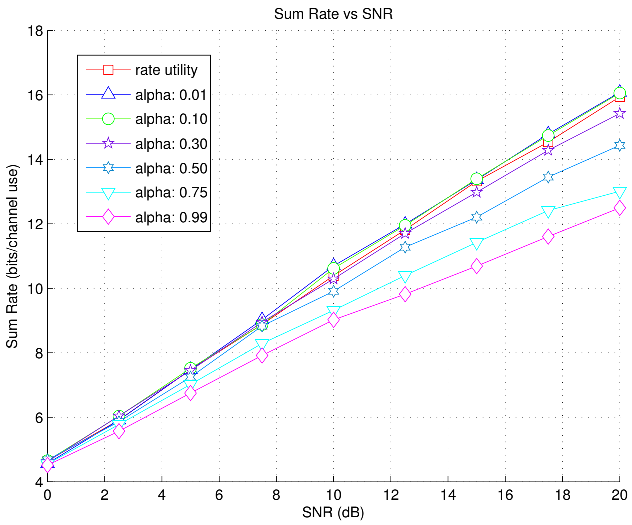

Fig. 3
Illustration of the performance of rate utility and "α-fair" utility functions in the distributed pricing algorithm.

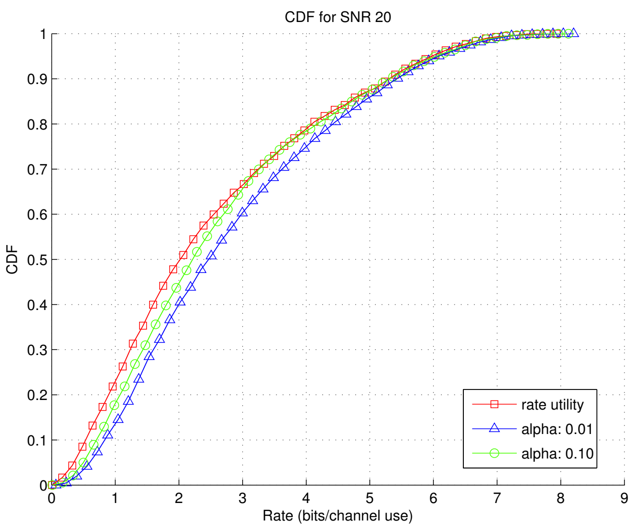

Fig. 4
Illustration of CDF of the worst rate.

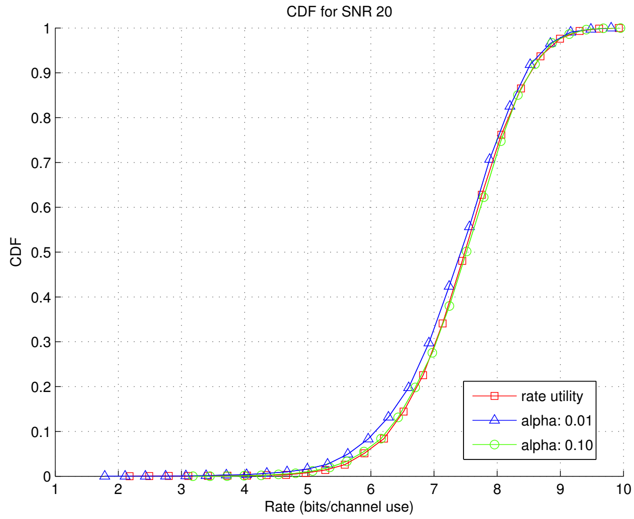

Fig. 5
Illustration of CDF of the best rate.

In order to draw some insights about the impact of changing the α parameter in the pricing algorithm, lets consider the egoistic and altruistic parts in (9). The egoistic part can be written as

$$u_k(\gamma_k) = \frac{\gamma_k^\alpha}{\alpha} = \frac{{S_k}^\alpha}{\alpha(I_k+N_k)^\alpha} \tag{17}$$

and the altruistic part can be written as

$$\sum_{j\neq k}\pi_j|\mathbf{g}_j^H\mathbf{H}_{jk}\mathbf{v}_k|^2 = \sum_{j\neq k}\pi_j C_{jk} = \sum_{j\neq k}\frac{{S_j}^\alpha}{(I_j+N_j)^{\alpha+1}}C_{jk}, \tag{18}$$

where C_{jk} is a constant (we are interested in the variation with alpha). Because both terms vary differently with alpha, then changing α will change the relation between the egoistic and altruistic objectives. However, determining the optimal value of α is a difficult problem. An empirical value of α can be obtained from Fig. 3 by choosing the value that yields the highest sum rate. The numerical results show that for low values of α (0.10 or lower) the α-fair function achieves better performance than the rate utility function. In order to analyse how this gain is spread among the users, the Cumulative Distribution Functions (CDFs) of the worst and best obtained rates are shown, respectively, in Figures 4 and 5.

In Fig. 4 we see that the worst obtained rate is better comparing to the obtained rate when using the original rate utility function. On the other hand, Fig. 5 shows that the best obtained rate has no gain (or loss), resulting in approximately the same performance as the original rate utility function. That is, although not large, the gain when using the α-fair utility function comparing to the original rate utility goes to the user who needs it the most.

VI. Conclusions

Several papers in the literature analyse the distributed pricing algorithm performance in different scenarios, as well as its convergence. However, none of them investigates the performance impact regarding the choice of the utility function.

In this work we have presented a comparative analysis of the rate utility and "α-fair" utility functions with the distributed pricing algorithm. We have empirically determined an interval of α values for the "α-fair" utility function that provides a higher sum rate than the rate utility.

Another interesting investigation aspect is the definition of the interference price π_k. Although equation (10) is intuitive (the more sensitive the utility of a user is to interference, the greater its interference price should be), no derivation of an optimal price is available in the literature. The derivation and analysis of different interference prices and their impact on the convergence of the algorithm will be left for future work.

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